

Structure of an internal wave

Examples

⇒ Gravity waves that oscillate on internal density differences play an important role in redistribution of energy in the ocean.

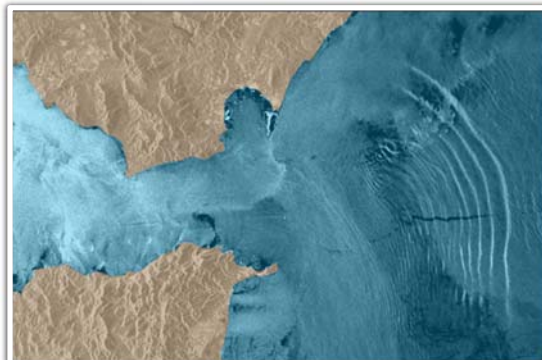
⇒ They are the major contributor of vertical mixing in the open ocean, especially when they break.

📷 An aerial photo of the coast of Somalia, at the very tip Horn of Africa, shows some ripples at the surface of the ocean. They are an indicator of internal waves at greater depth.



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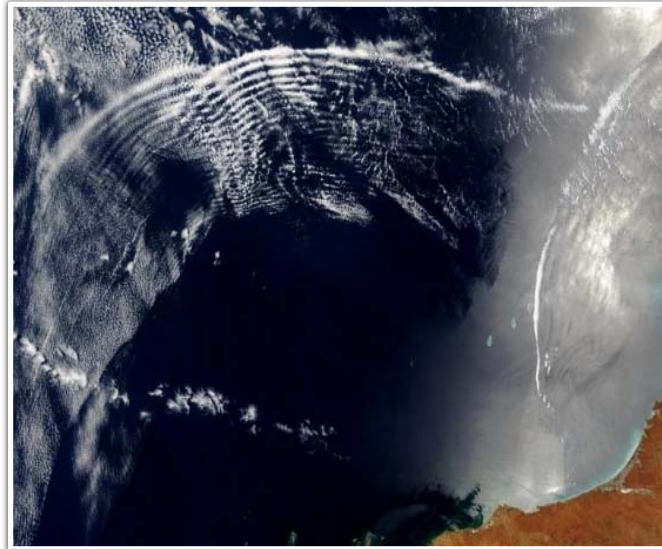


📷 The Strait of Gibraltar, between the southern coast of Spain and the northern coast of Morocco where water from the Atlantic Ocean mixes with salty water from the Mediterranean Sea

📷 The Bay of Biscay is a place where there is a very rapid change in topography.

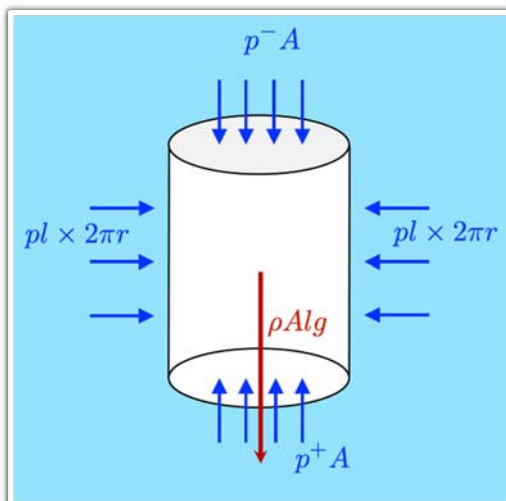
Examples

⇒ Gravity waves that oscillate on internal density differences play an important role in redistribution of energy in the ocean



🌀 Internal waves in the **atmosphere** on the north coast of Australia. The ripples in the cloud structure are associated with vertical undulations of the stratification of the atmosphere.

Archimedes principle



⇒ Net vertical force =

$$(p^+ - p^-)A - \rho A l g$$

↪ For equilibrium

$$(p^+ - p^-) = \rho l g \quad \frac{\Delta p}{\Delta z} = -\rho g$$

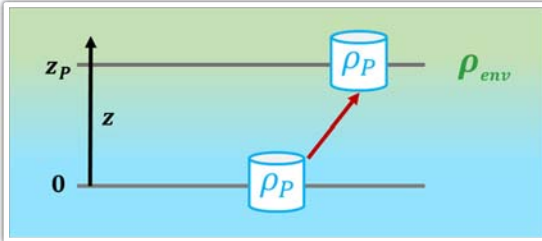
⇒ This is the **hydrostatic relation**

⇒ Now consider parcel of different density (gold): upward pressure force is unchanged !

↪ It depends only on the volume displaced. It is equal to the weight of fluid displaced by the **parcel**.

Parcel displacements and stability

⇒ Consider a displaced **parcel**:



⇒ The vertical momentum equation for the **parcel** is:

$$\rho \frac{dw}{dt} = \rho_p \frac{d^2 z_p}{dt^2} = -\frac{\partial p}{\partial z} - \rho_p g$$

⇒ We assume **parcel** density conserved

⇒ Need to pay attention to which variables apply to the **parcel** and which variables apply to the **environment**.

▪ The environment is in **hydrostatic balance**:

$$0 = -\frac{\partial p}{\partial z} - \rho_{env} g$$

$$\Rightarrow \rho_p \frac{d^2 z_p}{dt^2} = (\rho_{env} - \rho_p) g$$

reduced gravity (g')

$$\frac{d^2 z_p}{dt^2} = \frac{(\rho_{env} - \rho_p)}{\rho_p} g$$

↪ vertical acceleration is produced by (buoyancy-gravity)

Vertical oscillations, Brunt-Väisälä frequency

⇒ Now we argue that if the parcel conserves its density during the displacement, then we can identify the vertical variation of density with the difference between the **parcel** and its new **environment** :

$$\frac{\rho_p - \rho_{env}}{z_p} = -\left. \frac{\partial \rho}{\partial z} \right|_{env}$$

↪ This leads to: $\frac{d^2 z_p}{dt^2} = \frac{g}{\rho} \frac{\partial \rho}{\partial z} z_p = -N^2 z_p$ N is the Brunt Väisälä frequency

⇒ The solution of this equation depends on the sign of N^2 :

- If N^2 is positive (stable stratification) ⇒ buoyancy oscillations / gravity waves.

$$z_p = A \sin Nt + B \cos Nt$$

- If N^2 is negative ⇒ growing / decaying solutions, instability, convection.

$$z_p = A e^{|N|t} + B e^{-|N|t}$$

Dispersion relation

⇒ Linear dynamics without rotation.
Allow vertical acceleration.

$$\rho = \rho_0 + \bar{\rho}(z) + \rho'(x, y, z, t)$$

(Boussinesq if $\bar{\rho}(z) \ll \rho_0$)

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \\ \frac{\partial v}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p'}{\partial y} \\ \frac{\partial w}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} - \frac{1}{\rho_0} g \rho' \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\ \frac{\partial \rho'}{\partial t} + w \frac{\partial \bar{\rho}}{\partial z} &= 0\end{aligned}$$

⇒ now if $N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}$

⇒ and we look for solutions of the form

$$e^{i(lx + my + nz - \omega t)}$$

5 x 5 matrix of coefficients
the determinant is zero if

$$\omega^2 = N^2 \frac{l^2 + m^2}{l^2 + m^2 + n^2}$$

Internal wave properties

1) $\omega \leq N$

$$\omega^2 = N^2 \frac{l^2 + m^2}{l^2 + m^2 + n^2}$$

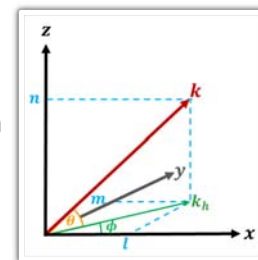
- 2) if $l^2 + m^2 \ll n^2 \Rightarrow$ large horizontal scales, low frequency
if $l^2 + m^2 \gg n^2 \Rightarrow$ buoyancy oscillations, $\omega \rightarrow N$

$l^2 + m^2$ in denominator comes from non-hydrostatic term

- 3) if $l = k \cos \theta \cos \phi$ θ = angle with horizontal
 $m = k \cos \theta \sin \phi$ ϕ = angle with x axis
 $n = k \sin \theta$

$$\Rightarrow \omega = \pm N \cos \theta$$

The frequency depends on the vertical angle
Excitation leads to angled rays.



- 4) **Group velocity** (2D analysis: define $k_h = (l, m)$ then drop subscript $k_h \rightarrow k$)

$$c_{gk} = \frac{\partial \omega}{\partial k} = \frac{\omega n^2}{k(k^2 + n^2)}$$

$$c_{gn} = \frac{\partial \omega}{\partial n} = -\frac{\omega n}{(k^2 + n^2)}$$

$$c_g = \frac{\omega n}{k(k^2 + n^2)}(n, -k)$$

Group speed is perpendicular
to the phase speed and
directed downwards.

Derivation for group velocity

⇒ in k direction:

$$\omega^2 = N^2 \frac{k^2}{k^2 + n^2}$$

$$\frac{2\omega}{N^2} \frac{\partial \omega}{\partial k} = \frac{2k}{k^2 + n^2} - \frac{k^2 \times 2k}{(k^2 + n^2)^2}$$

$$\frac{\omega^2}{N^2} \frac{\partial \omega}{\partial k} = \frac{\omega k}{k^2 + n^2} - \frac{\omega k \times k^2}{(k^2 + n^2)^2}$$

$$\frac{\partial \omega}{\partial k} = \frac{\omega}{k} - \frac{\omega k}{k^2 + n^2} = \omega \left(\frac{k^2 + n^2 - k^2}{k(k^2 + n^2)} \right) = \frac{\omega n^2}{k(k^2 + n^2)}$$

⇒ in n direction:

$$\omega^2 = N^2 \frac{k^2}{k^2 + n^2}$$

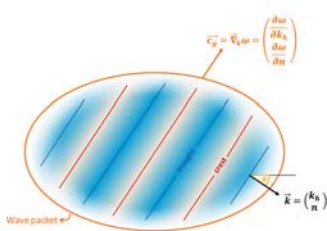
$$\frac{2\omega}{N^2} \frac{\partial \omega}{\partial n} = -\frac{k^2 \times 2n}{(k^2 + n^2)^2}$$

$$\frac{\omega^2}{N^2} \frac{\partial \omega}{\partial n} = -\frac{k^2 \omega n}{(k^2 + n^2)^2}$$

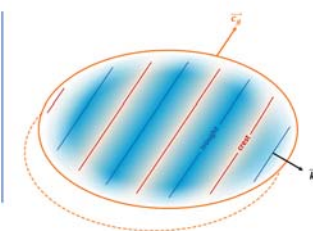
$$\frac{\partial \omega}{\partial n} = -\frac{\omega n}{k^2 + n^2}$$

Internal wave properties

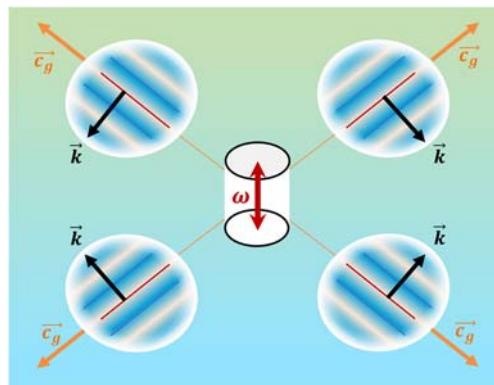
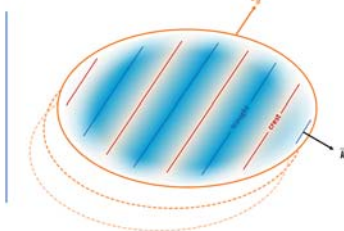
Internal wave propagation at $t = t_0$



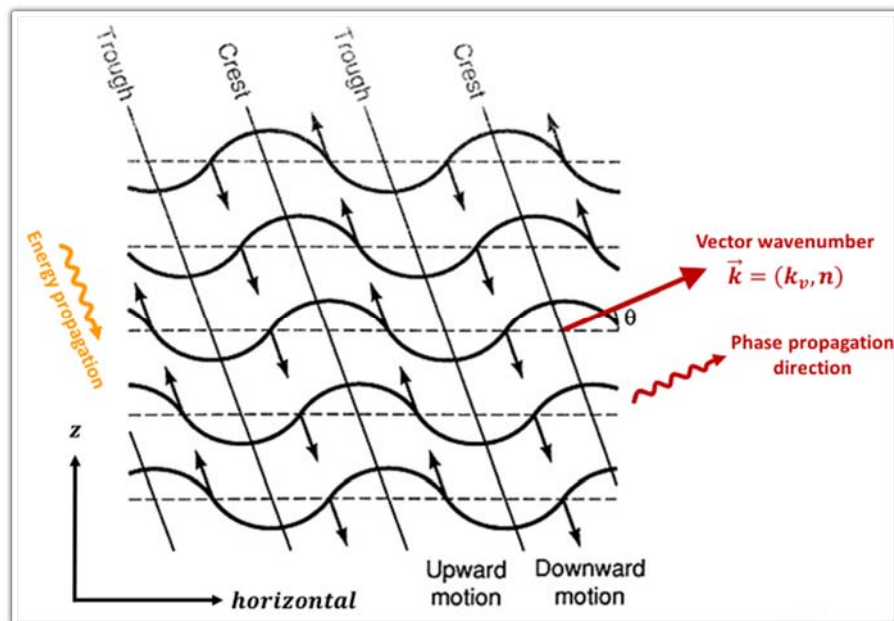
Internal wave propagation at $t = t_0 + \Delta t$



Internal wave propagation at $t = t_0 + 2\Delta t$

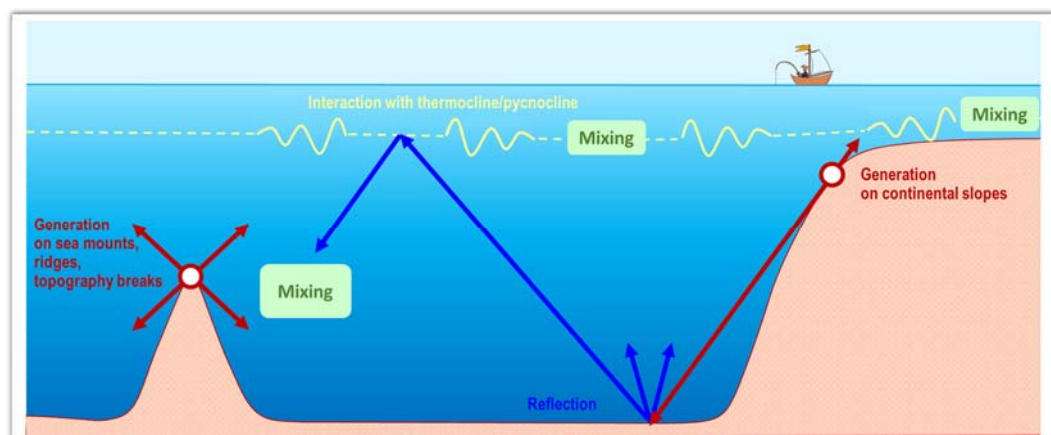


Vertical structure

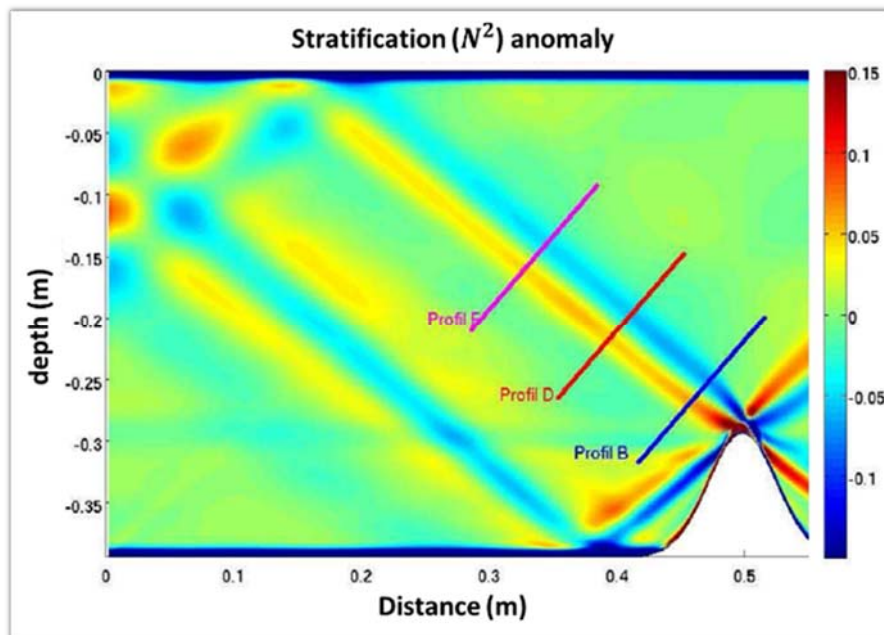


The internal tide

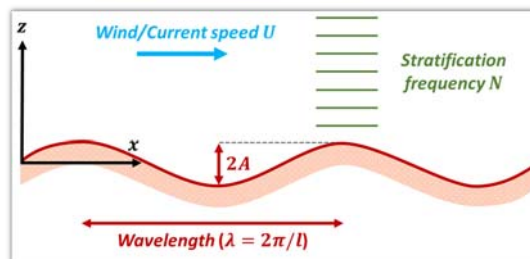
Tides interacting with shelf-breaks or sea mounts can generate internal waves. As we have seen these internal waves propagate in a specific direction. Can lead to preferred locations for mixing.



Numerical model simulation



Lee waves



⇒ to simulate a mean flow we translate the x -axis at speed U

$$z = A \cos(lx - \omega t) = A \cos(l(x + Ut)) \quad (\omega = -lU)$$

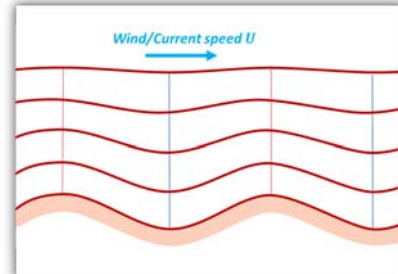
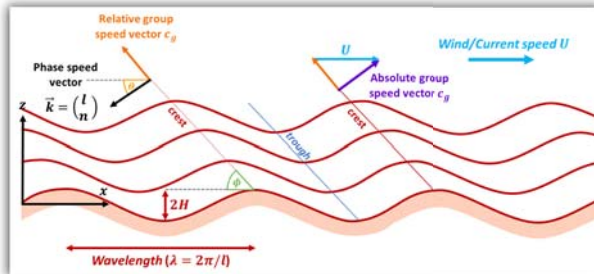
⇒ if the wave frequency matches forcing frequency and $m = 0$

$$\omega^2 = N^2 \frac{l^2 + m^2}{l^2 + m^2 + n^2} = \frac{N^2 l^2}{l^2 + n^2} = l^2 U^2$$

$$\Rightarrow n^2 = \frac{N^2}{U^2} - l^2$$

Two cases

$$n^2 = \frac{N^2}{U^2} - l^2$$



- 1) $\frac{N}{U} > l \Rightarrow n$ real: strong stratification / large scale topography / slow current
forcing frequency lower than the buoyancy frequency - can excite a travelling wave. Wave can cause drag on mean flow.
- 2) $\frac{N}{U} < l \Rightarrow n$ imaginary: weak stratification / short scale topography / fast current
excitation frequency > buoyancy frequency - exponential solutions - trapped waves decaying upwards but not propagating - same wavelength as terrain - no drag.