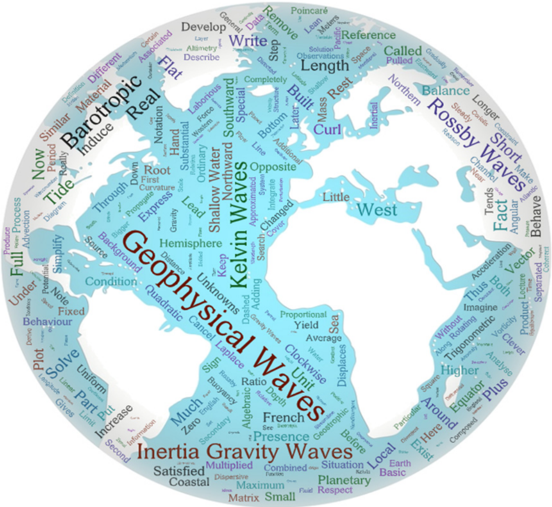
Chapter3: Some geophysical waves

- Gravity waves from the shallow-water equations
- Inertia gravity waves
- Boundary Kelvin waves
- Rossby waves



Full equations of motion

For a single shallow layer of water in a rotating system

x momentum equation

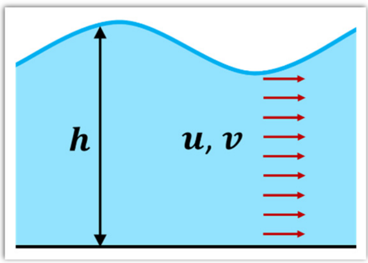
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

y momentum equation

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

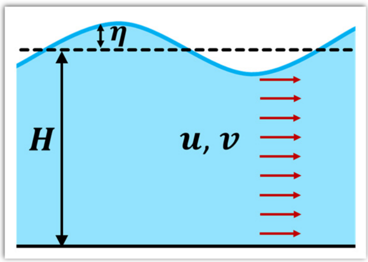
mass conservation equation

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$



Gravity waves in shallow water

Let's start with something simple: a one-dimensional non-rotating linear system.
Which terms shall we cross out ?



$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} - \cancel{fv} = -g\frac{\partial h}{\partial x}$$ x-momentum

$$\cancel{\frac{\partial v}{\partial t} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{fu} = -g\frac{\partial h}{\partial y}}$$ y-momentum

$$\frac{\partial h}{\partial t} + \cancel{u\frac{\partial h}{\partial x}} + \cancel{v\frac{\partial h}{\partial y}} = -\overset{H}{h}\left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}\right)$$ continuity $(h = H + \eta)$

This leaves us with

$$\frac{du}{dt} = -g\frac{dh}{dx}, \quad \frac{dh}{dt} = -H\frac{du}{dx} \quad \Rightarrow \quad \frac{d^2u}{dt^2} = gH\frac{d^2u}{dx^2}$$

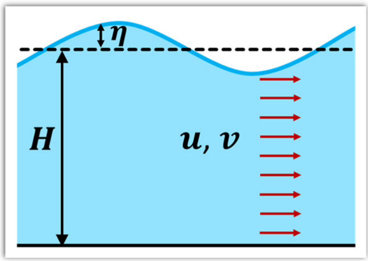
and the solution of this equation is $u = Re\,\tilde{u}\,e^{i(lx-\omega t)}$

a gravity wave with a simple dispersion relation $\omega^2 = gHl^2, \quad c = \frac{\omega}{l} = \pm\sqrt{gH}\left(= \frac{d\omega}{dl}\right)$

Adding rotation

Flat bottom, f-plane, linear perturbations *u, v, η*

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$



Equation for surface height development

Linear shallow water equations on an f-plane

$$u_t - fv + g\eta_x = 0 \quad (1)$$
$$v_t + fu + g\eta_y = 0 \quad (2)$$
$$\eta_t + H(u_x + v_y) = 0 \quad (3)$$

vorticity equation: $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1) \rightarrow \quad \frac{\partial}{\partial t}(v_x - u_y) + f(u_x + v_y) = 0 \quad (V)$

divergence equation: $\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) \rightarrow \quad \frac{\partial}{\partial t}(u_x + v_y) - f(v_x - u_y) + g\nabla^2\eta = 0 \quad (D)$

substitute (V) into (3) $\eta_t - \frac{H}{f}\frac{\partial}{\partial t}(v_x - u_y) = 0$

Equation for surface height development

subsitute (D) into $\frac{\partial}{\partial t}$ (3)

$$\eta_{tt} + fH(v_x - u_y) - gH\nabla^2\eta = 0 \quad \frac{\partial}{\partial t} \rightarrow$$

$$\eta_{ttt} + fH\frac{\partial}{\partial t}(v_x - u_y) - gH\nabla^2\eta_t = 0$$

substitute from $\eta_t - \frac{H}{f}\frac{\partial}{\partial t}(v_x - u_y) = 0$ gives $\eta_{ttt} + f^2\eta_t - gH\nabla^2\eta_t = 0$

With appropriate initial condition at *t = 0,* the departure from geostrophic disequilibrium follows:

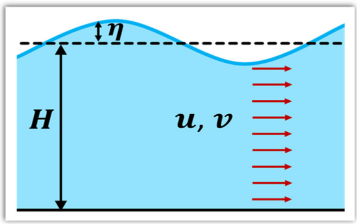
$\eta_{tt} - gH\nabla^2\eta + f^2\eta = 0$ substitute solution $\eta = \tilde{\eta}e^{i(\mathbf{k}.\mathbf{x} - \omega t)}$

leads to dispersion relation

$$\boxed{\omega = \pm\sqrt{f^2 + gHk^2}}$$

General method for finding wave solutions

⇨ flat bottom, f-plane, linear perturbations u, v, η



$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Wave solution: $(u, v, \eta) = (\tilde{u}, \tilde{v}, \tilde{\eta})e^{i(lx+my-\omega t)}$

With: $\frac{\partial}{\partial x} \to il\times \quad \frac{\partial}{\partial y} \to im\times \quad \frac{\partial}{\partial t} \to -i\omega\times$

⇨ substitute wave solution: differential equations become linear algebraic equations

$$-i\omega\tilde{u} - f\tilde{v} = -igl\tilde{\eta}$$
$$-i\omega\tilde{v} + f\tilde{u} = -igm\tilde{\eta}$$
$$-i\omega\tilde{\eta} + H(il\tilde{u} + im\tilde{v}) = 0$$

The unknowns are the wave amplitudes $\tilde{u}, \tilde{v}, \tilde{\eta}$

The parameters are the wave properties l, m, ω and the geophysical constants f, g, H

Inertia-gravity (Poincaré) waves

We need to solve algebraic system

$$\begin{pmatrix} -i\omega & -f & igl \\ f & -i\omega & igm \\ ilH & imH & -i\omega \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

⇨ trivial solution $\tilde{u} = \tilde{v} = \tilde{\eta} = 0$ (no flow)

⇨ The condition for having non-trivial solutions is hat the determinant of the matrix is zero.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \bullet \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \bullet \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \bullet \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

This leads to $\omega\left[\omega^2 - f^2 - gH(l^2 + m^2)\right] = 0$

this is a more complicated dispersion relation !

three solutions: $\omega = 0$ steady geostrophic flow

$\omega = \pm\sqrt{f^2 + gHk^2}$ inertia-gravity waves

Limit behaviour

<u>short wave limit:</u> $k^2 \gg f^2/gH$

i.e. the wavelength $\lambda^2 \ll 2\pi L_d^2$
(L_d is the radius of deformation)

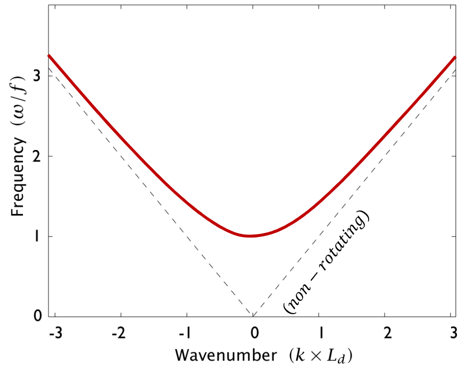
waves behave as in non-rotating case (provided shallow water condition is not violated)

<u>long wave limit:</u> $k^2 \ll f^2/gH$

dispersion relation reduced to

$$\omega = f$$

free inertial oscillations



Boundary Kelvin waves

Add a lateral boundary to the problem, cross out terms involving flow perpendicular to the boundary

$$\cancel{\frac{\partial u}{\partial t}} \boxed{- fv = -g\frac{\partial \eta}{\partial x}}$$ Geostrophic balance

$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y}$$

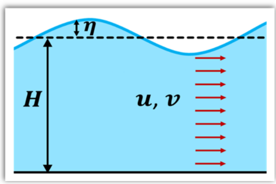
$$\frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0$$

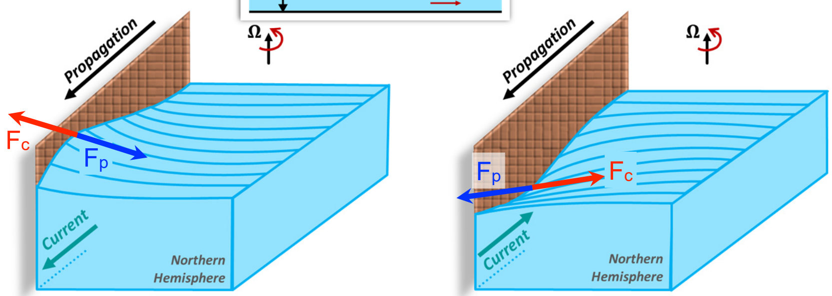
non-dispersive waves

In the x direction we have geostrophic balance, with pressure and Coriolis forces alternating in direction as crests and troughs propagate meridionally.

In the y direction we have non-dispersive gravity waves with a fixed phase speed independent of horizontal scale

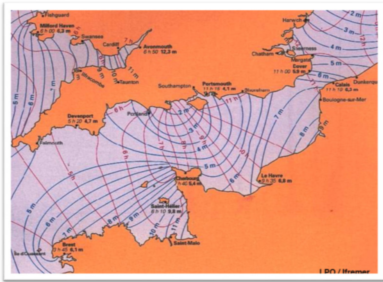
$$|c| = \sqrt{gH}$$



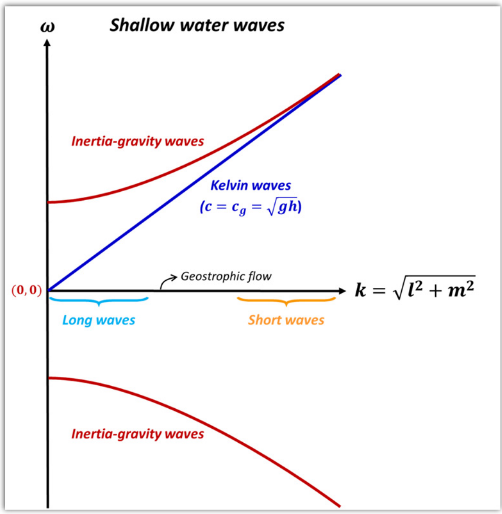


Properties of Kelvin waves

Since the only admissible solution is V_1, we conclude that for a system bounded on the west (x positive) the wave propagates in the negative y direction, i.e. to the south (in the northern hemisphere). If x is negative this reverses. Kelvin waves thus propagate south on the western boundary and north on the eastern boundary. They circuit ocean basins in an anticlockwise (cyclonic) direction. In the southern hemisphere the direction is clockwise (but still cyclonic).



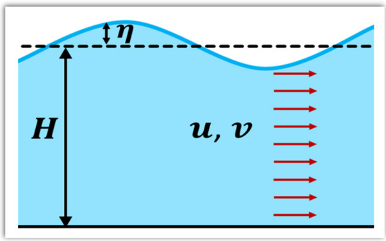
Tides are higher on the French side because the signal propagates in from the west



Parcel displacements in a vorticity gradient

Cross differentiating the horizontal momentum equations in a barotropic fluid eliminates the pressure terms

$$-\frac{\partial}{\partial y}\{u_t + uu_x + vu_y - fv = -p_x/\rho\}$$
$$+\frac{\partial}{\partial x}\{v_t + uv_x + vv_y + fu = -p_y/\rho\}$$



Leads to the barotropic vorticity equation

$$\frac{D}{Dt}(f + \xi) = -(f + \xi)\nabla.\mathbf{v}$$

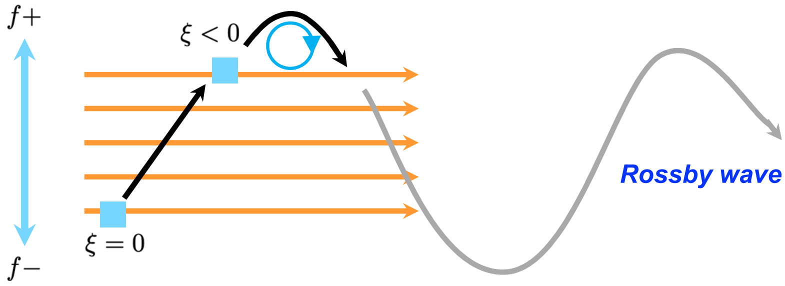
where $\xi = v_x - u_y$ is the "relative vorticity", $f = 2\Omega \sin\phi$ is the "planetary vorticity"

and $f + \xi$ is called the "absolute vorticity".

- Divergence can be viewed as a source of absolute vorticity
- In nondivergent barotropic flow, absolute vorticity is conserved

Parcel displacements in a vorticity gradient

⇨ Consider a parcel of fluid that **conserves** its **absolute vorticity** in a westerly current $f + \xi = \text{const}$



External (Barotropic) Rossby waves

To describe midlatitude wave motions where the geometry of the earth is important, we start with a barotropic system, using the beta approximation for the variation of the Coriolis parameter with latitude: $f = f_0 + \beta y$

The conservation relation is $\frac{D}{Dt}(\xi + \beta y) = 0$ $\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + (U + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$

assume a uniform background zonal flow U represented by streamfunction $\Psi = -Uy$, and add a perturbation streamfunction ψ'.

$$\psi = \Psi + \psi'(x, y, t)$$

our linear conservation law is now

$$\frac{\partial}{\partial t}\nabla^2\psi' + U\frac{\partial \nabla^2 \psi'}{\partial x} + \beta \frac{\partial \psi'}{\partial x} = 0$$

and we look for wave solutions of the form $\psi' = Re\ \tilde{\psi} e^{i(lx + my - \omega t)}$

Rossby wave dispersion

Substituting this wave form into the conservation equation give the dispersion relation (for $\tilde{\psi} \neq 0$):

$$\boxed{\omega = Ul - \frac{\beta l}{l^2 + m^2}}$$

The phase speed and group speed in the *x* direction are given by

$$c = \frac{\omega}{l} = U - \frac{\beta}{l^2 + m^2} \qquad c_g = \frac{\partial \omega}{\partial l} = U + \frac{\beta(l^2 - m^2)}{(l^2 + m^2)^2}$$

The phase speed is westwards against the mean flow, which Doppler shifts the waves.

- Longer waves (smaller k^2) travel faster.
- Waves closer to the equator (bigger β) travel faster.
- Group velocity depends on ratio of zonal to meridional scales (for bigger meridional scales/smaller zonal scales, the group velocity is eastwards).

derivation

$$\frac{\partial}{\partial t}\nabla^2\psi + U\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0 \qquad \frac{\partial}{\partial t} \to -i\omega \quad \frac{\partial}{x} \to il \quad \nabla^2 \to -(l^2 + m^2)$$

$$i\omega(l^2 + m^2) - il(l^2 + m^2)U + il\beta = 0$$

$$\omega(l^2 + m^2) = l(l^2 + m^2)U - l\beta$$

$$\omega = lU - \frac{l\beta}{(l^2 + m^2)}$$

$$\frac{\omega}{l} = c_x = U - \frac{\beta}{(l^2 + m^2)}$$

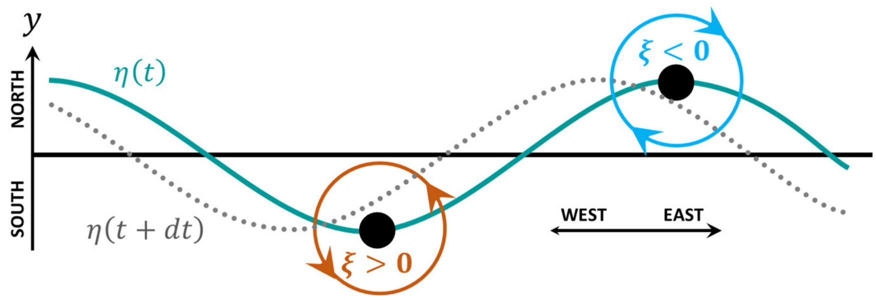
$$\frac{\partial \omega}{\partial l} = U - \beta \underline{\frac{\partial}{\partial l}(l(l^2 + m^2)^{-1})}$$

$$\frac{1}{(l^2 + m^2)} + l(-(l^2 + m^2)^{-1} \times 2l) = \frac{(l^2 + m^2) - 2l^2}{(l^2 + m^2)^2} = -\frac{l^2 - m^2}{(l^2 + m^2)^2}$$

$$\frac{\partial \omega}{\partial l} = U + \beta\frac{l^2 - m^2}{(l^2 + m^2)^2}$$

Rossby wave propagation mechanism

Can be understood in terms of the conservation of absolute vorticity. When a parcel of fluid changes latitude, to compensate for its changing planetary vorticity, it must acquire either positive or negative relative vorticity. This induces a circulation that leads to the westward propagation of the disturbance.



Observations

Evidence of Rossby wave propagation in satellite altimetry of the sea surface ???