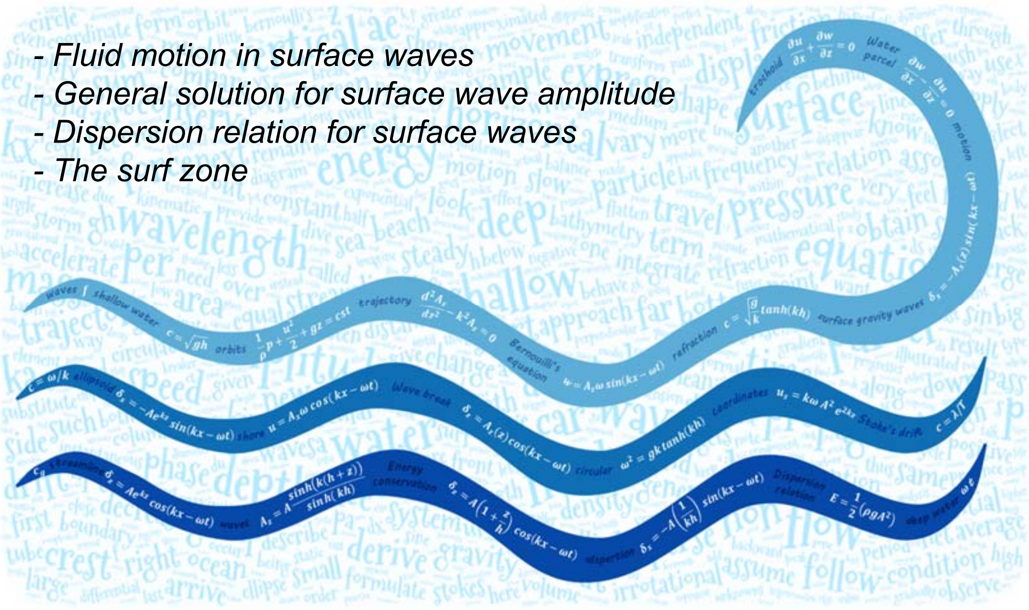
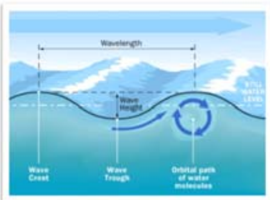
Chapter 2: Deep and shallow water waves

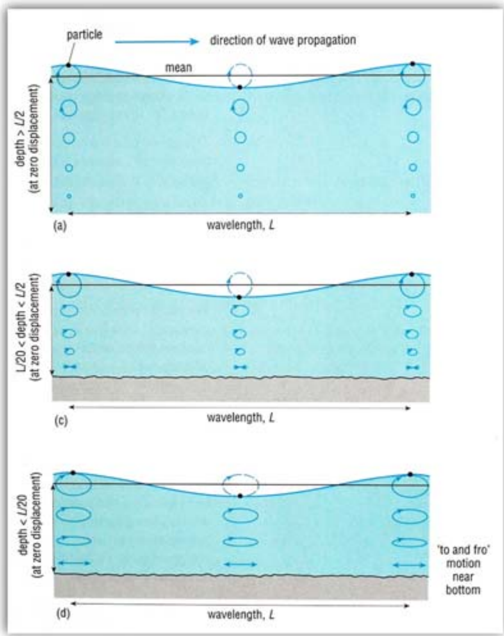
- Fluid motion in surface waves
- General solution for surface wave amplitude
- Dispersion relation for surface waves
- The surf zone



Movement of fluid in a wave

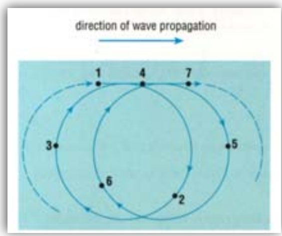
How do actual parcels of fluid move within the wave ?





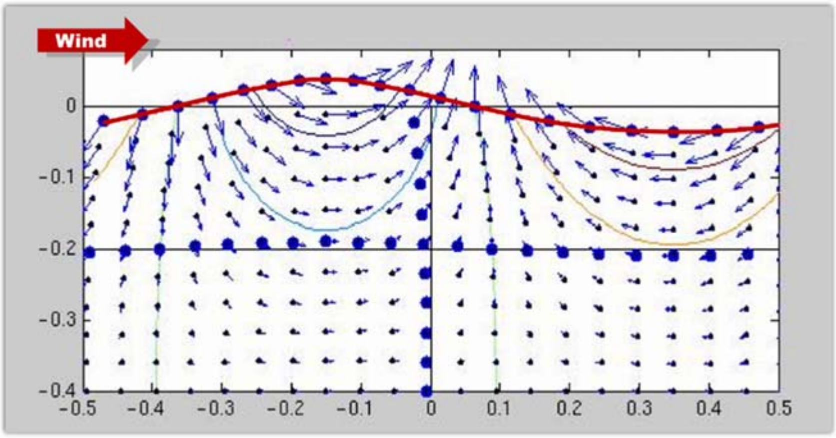
⇨ They execute elliptical trajectories.

⇨ Only the wave propagates. The water itself can drift very slowly in the direction of propagation.

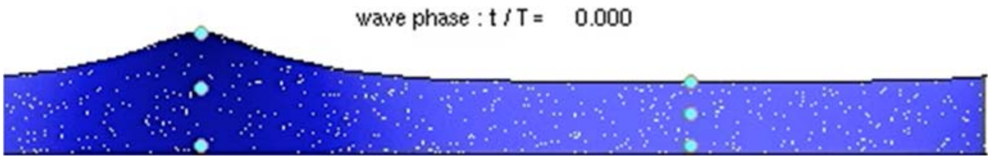


Streamlines of flow in a propagating wave

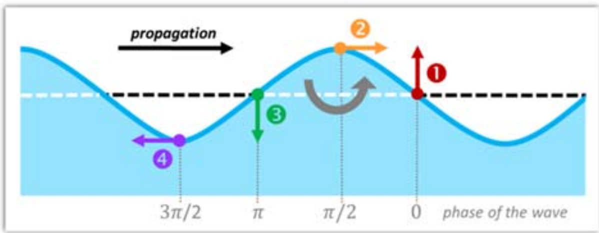
- ⇨ Note that the vertical velocity at the surface = the vertical velocity of the surface.
- ⇨ Note the confluent flow where the surface is rising, diffluent where it is falling.



Water parcel orbits

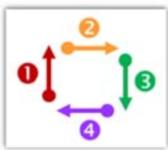


Instantaneous circulation = Streamline



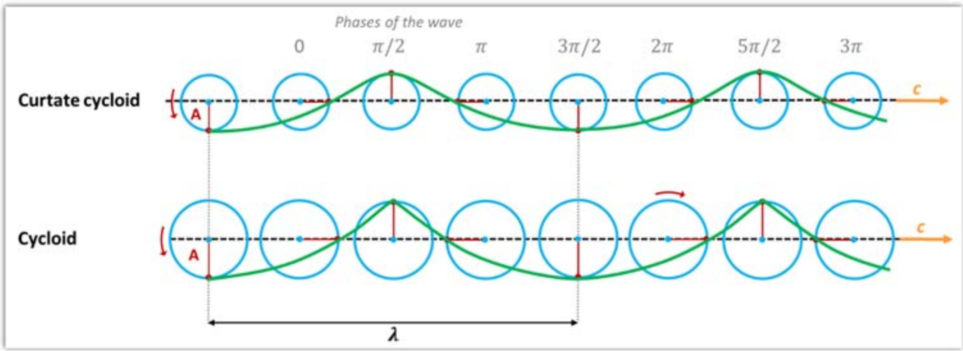
$\Longleftrightarrow$

Particle trajectory



Trochoidal shape

The sinusoidal function we chose to represent the surface wave is an approximation.
It can be more accurately represented by a trochoid.
This is the locus of a point on a moving wheel.



Analysis of fluid motion in 2d

Denote displacement of a particle of fluid compared to its equilibrium position x,z as δx, δz.

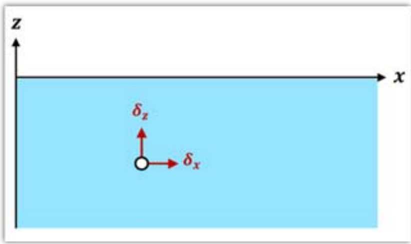
Corresponding fluid velocity components are u,w.

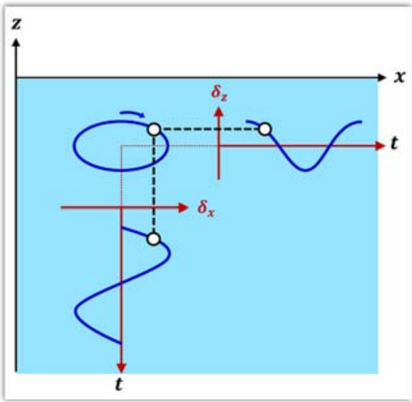
Assume δx, δz are in phase quadrature, so fluid particle traces out an ellipse.

Arrange phase so flow is to the right on wave crests to ensure conservation of mass.

thus we choose:

$$\begin{cases} \delta_z = A_z(z)\cos(kx - \omega t) \\ \delta_x = -A_x(z)\,sin(kx - \omega t) \end{cases}$$



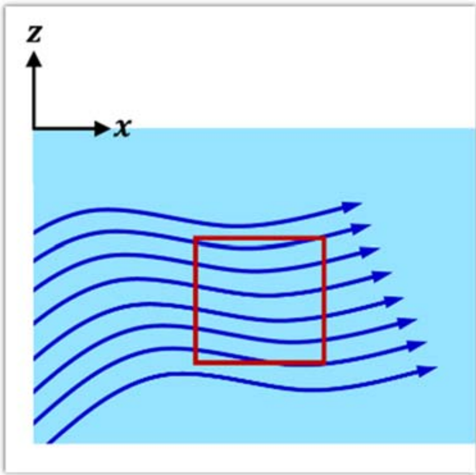


Nondivergent irrotational flow

Calculate velocity

$$w = \frac{d}{dt}(\delta z) = A_z \omega \sin(kx - \omega t)$$
$$u = \frac{d}{dt}(\delta x) = A_x \omega \cos(kx - \omega t)$$

$$\delta z = A_z(z)\cos(kx - \omega t)$$
$$\delta x = -A_x(z)\sin(kx - \omega t)$$



Condition that velocity field is nondivergent and irrotational comes from conservation of mass and absence of viscocity

Nondivergent: $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Irrotational: $$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0$$

Solution for depth dependence

substitute in

$$\frac{du}{dx} + \frac{dw}{dz} = -A_x k\omega \sin(kx - \omega t) + \frac{d}{dz}A_z \omega \sin(kx - \omega t) = 0$$

$$-kA_x + \frac{d}{dz}A_z = 0$$

$$\frac{dw}{dx} - \frac{du}{dz} = A_z k\omega \cos(kx - \omega t) - \frac{d}{dz}A_x \omega \cos(kx - \omega t) = 0$$

$$kA_z - \frac{d}{dz}A_x = 0$$

so

$$\frac{d^2 A_z}{dz^2} - k^2 A_z = 0$$

$$A_z = \alpha e^{kz} + \beta e^{-kz}$$

Boundary conditions

At the surface, the vertical displacement δz must correspond to a travelling wave

$$\delta z = A\cos(kx - \omega t) \qquad z = 0 \implies A_z = \alpha + \beta = A$$

At the bottom no vertical displacement allowed

$$\delta z(-h) = 0 = \alpha e^{-kh} + \beta e^{kh}$$

$$A = \alpha - \frac{\alpha e^{-kh}}{e^{kh}} = \alpha\left(\frac{e^{kh} - e^{-kh}}{e^{kh}}\right) \quad \text{likewise} \quad A = \beta - \frac{\beta e^{kh}}{e^{-kh}} = \beta\left(\frac{e^{-kh} - e^{kh}}{e^{-kh}}\right)$$

or

$$\alpha = A\frac{e^{kh}}{e^{kh} - e^{-kh}} \qquad \beta = -A\frac{e^{-kh}}{e^{kh} - e^{-kh}}$$

so

$$A_z = \alpha e^{kz} + \beta e^{-kz} = \frac{A}{e^{kh} - e^{-kh}}(e^{kh}e^{kz} - e^{-kh}e^{-kz}) = A\frac{e^{k(h+z)} - e^{-k(h+z)}}{e^{kh} - e^{-kh}}$$

$$A_z = A\frac{\sinh(k(h+z))}{\sinh(kh)}$$

Depth dependence of coefficients

To find A_x substitute solution for A_z back into

$$-kA_x + \frac{d}{dz}A_z = 0$$

$$-kA_x + \frac{d}{dz}\left(A\frac{e^{k(h+z)} - e^{-k(h+z)}}{e^{kh} - e^{-kh}}\right) = 0$$

$$kA_x = Ak\left(\frac{e^{k(h+z)} + e^{-k(h+z)}}{e^{kh} - e^{-kh}}\right)$$

so

$$A_z = A\frac{\sinh(k(z+h))}{\sinh(kh)}, \qquad A_x = A\frac{\cosh(k(z+h))}{\sinh(kh)}$$

Water parcel orbits

Note that A_z and A_x are both positive, and for a wave travelling to the right k is positive.

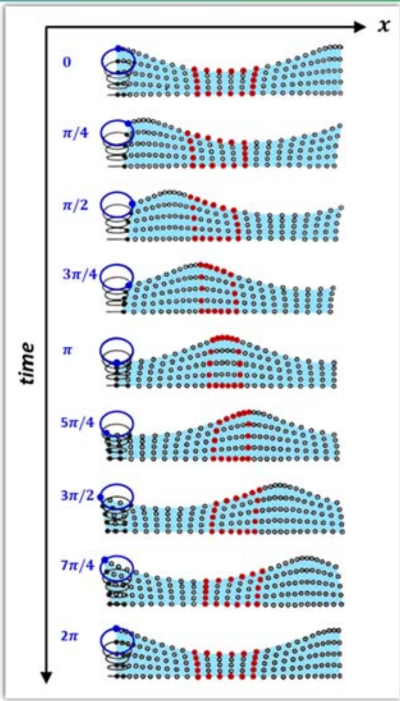
At $x=0$ we have $\delta z \propto \cos(\omega t)$

and $\delta x \propto \sin(\omega t)$

So parcels of water describe clockwise rotations for a wave travelling to the right (positive x direction) and anticlockwise rotations for a wave travelling to the left.

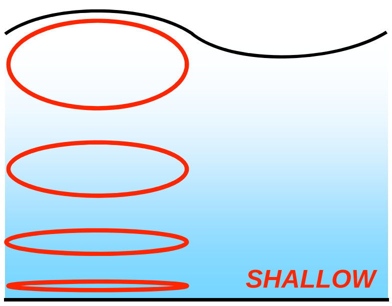
Note that parcel ellipses flatted as we get nearer to the bottom.

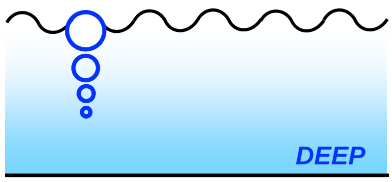
Note that because the amplitude is exaggerated in the diagram, the typical shape of sharpened wave peaks is apparent.



Deep and shallow water

Which is shallow and which is deep ?





Motion in shallow water

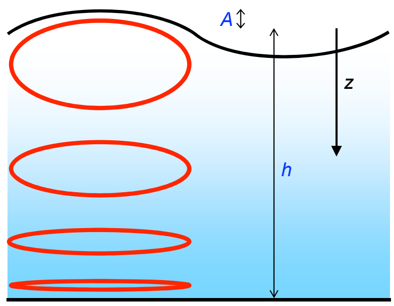
$$A_z = A\frac{\sinh(k(z+h))}{\sinh(kh)}, \qquad A_x = A\frac{\cosh(k(z+h))}{\sinh(kh)}$$

⇨ Depth of water much less than wavelength
kh << 1

$$\sinh(kh) \approx kh \qquad \sinh(k(z+h)) \approx k(z+h) \qquad \cosh(k(z+h)) \approx 1$$

$$\delta z = A\left(1+\frac{z}{h}\right)\cos(kx-\omega t) \qquad \delta x = -A\left(\frac{1}{kh}\right)\sin(kx-\omega t)$$

- Horizontal amplitude ~ A/kh, independent of depth and >> surface wave amplitude.
- Elliptical orbits becoming pure horizontal motion at the bottom.
- Shallow water waves are essentially longitudinal waves !



Motion in deep water

$$A_z = A\frac{\sinh(k(z+h))}{\sinh(kh)}, \qquad A_x = A\frac{\cosh(k(z+h))}{\sinh(kh)}$$

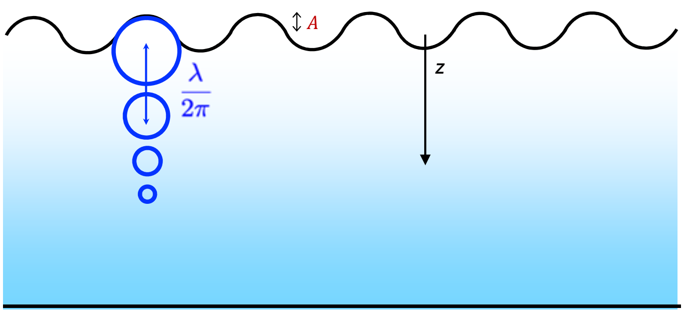
⇨ Depth of water much greater than wavelength
$kh \;>>\; 1$

⇨ Waves are confined to near the surface z << h

$$\sinh(kh) \approx \frac{e^{kh}}{2} \qquad \sinh(k(z+h)) \approx \cosh(k(z+h)) \approx \frac{e^{k(z+h)}}{2}$$

$$\delta z = Ae^{kz}\cos(kx-\omega t) \qquad \delta x = -Ae^{kz}\sin(kx-\omega t)$$

- circular orbits
- decay in the vertical on the order of one wavelength



Stokes' drift

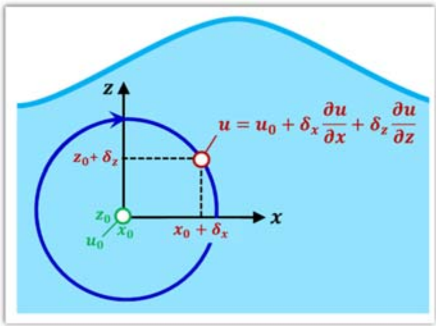
⇨ The average displacement of a fluid parcel (or a piece of driftwood) can be calculated from its average velocity within a single period, recall for deep water:

$$\delta x = -Ae^{kz}\sin(kx-\omega t)$$
$$\delta z = Ae^{kz}\cos(kx-\omega t)$$

⇨ Express Stokes' horizontal velocity u_s as average horizontal velocity in one cycle based on Taylor expansion about displacements δx and δz:

$$u_s = \delta x\frac{\partial u_x}{\partial x} + \delta z\frac{\partial u_x}{\partial z} \approx \delta x\frac{\partial^2 \delta x}{\partial x\partial t} + \delta z\frac{\partial^2 \delta x}{\partial z\partial t}$$

$$u_s = -Ae^{kz}\sin(kx-\omega t)\times -Ae^{kz}k\omega\sin(kx-\omega t)$$
$$+Ae^{kz}\cos(kx-\omega t)\times Ae^{kz}k\omega\cos(kx-\omega t)$$



↳ It yields:

$$u_s = k\omega A^2 e^{2kz}$$

Dispersion relation

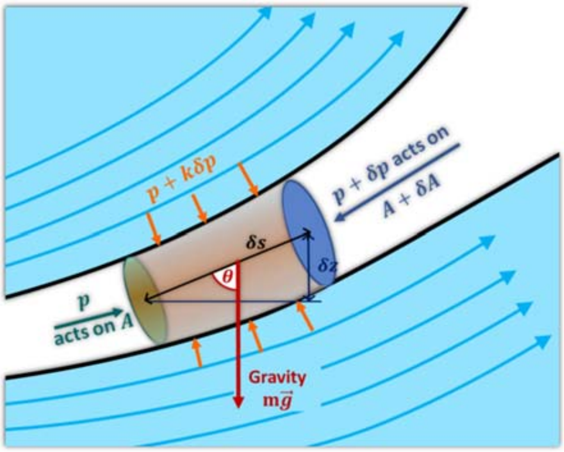
The dispersion relation gives the relationship between frequency and wavelength, and we use it to diagnose the wave speeds in a given medium.

The physical properties of the medium are important, so we must move from basic kinematics to include dynamics, i.e. the study of forces and accelerations in the fluid.

We must therefore apply Newton's second law, which for fluid motion under certain conditions can be expressed in terms of Euler's equation (for forces) and Bernoulli's Equation (for energy).

Aside: Bernoulli's equation

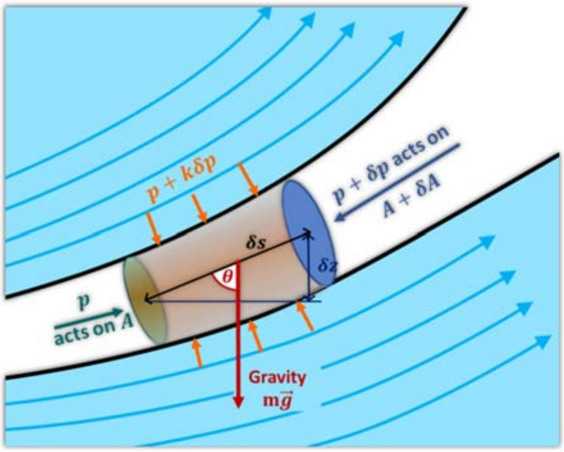
The force that produces acceleration in the direction of movement is provided by pressure forces and gravity.



The force on an element in the direction of the flow is

Aside: Bernoulli's equation

The force that produces acceleration in the direction of movement is provided by pressure forces and gravity.



The force on an element in the direction of the flow is

$$pA - (p + \delta p)(A + \delta A) + (p + k\delta p)\delta A - \rho g A \delta s \cos\theta$$

Aside: Bernoulli's equation

Neglect second order terms

$$\rightarrow \quad -A\delta p - \rho g A \delta z$$

This force must be equal to the acceleration of the parcel times its mass $\rho A \delta s \dfrac{du}{dt}$

so $\quad A\delta p + \rho A \delta s \dfrac{du}{dt} + \rho g A \delta z = 0$

$$\frac{1}{\rho}\frac{dp}{ds} + \frac{du}{dt} + g\frac{dz}{ds} = 0$$

recall $\quad \dfrac{du}{dt} = u\dfrac{\partial u}{\partial s} + \cancel{\dfrac{\partial u}{\partial t}}$ (steady flow)

Aside: Bernoulli's equation

which gives Euler's equation for force and acceleration

$$\underbrace{\frac{1}{\rho}\frac{dp}{ds}}_{\text{pressure gradient}} + \underbrace{u\frac{du}{ds}}_{\text{steady acceleration}} + \underbrace{g\frac{dz}{ds}}_{\text{gravity}} = 0$$

Integrate Euler's equation for a fluid of constant density to give Bernoulli's equation for energy conservation

$$\underbrace{\frac{p}{\rho}}_{\text{pressure work}} + \underbrace{\frac{u^2}{2}}_{\text{kinetic energy}} + \underbrace{gz}_{\text{gravitational potential energy}} = \text{const}$$

Aside: Bernoulli's equation

The conditions for Bernoulli's equation to be applicable are:

1) Steady flow (time independent)
2) Inviscid ("perfect fluid")
3) Constant density
4) Following a single streamline
5) Body forces derive from a potential function.

We are OK with all except the first one !

Coordinate transformation

To express our surface wave as a time-independent equation we transform

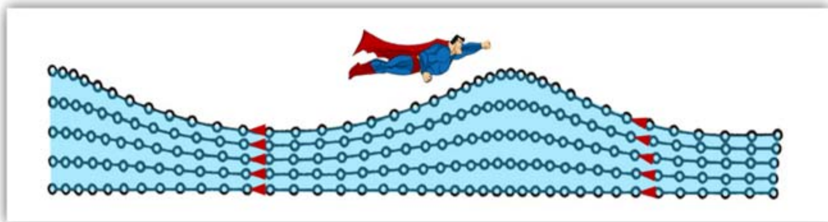
$$x' = x - \omega t/k$$

so

$$w' = A_z \omega \sin(kx')$$

$$u' = u - \omega/k = A_x \omega cos(kx') - \omega/k$$

independent of time



Bernoulli's equation

Now apply Bernoulli's equation at the surface $E = \frac{p_a}{\rho} + \frac{v^2}{2} + g\delta z$

(neglect surface tension in energy equation - so we will not consider capillary waves)

$$v^2 = u'^2 + w'^2 = A^2\omega^2 \left(\frac{\cosh^2(k(z+h))}{\sinh^2(kh)}\right)\cos^2(kx') + \frac{\omega^2}{k^2}$$
$$-2A\frac{\omega^2}{k}\left(\frac{\cosh(k(z+h))}{\sinh(kh)}\right)\cos(kx')$$
$$+A^2\omega^2\left(\frac{\sinh^2(k(z+h))}{\sinh^2(kh)}\right)\sin^2(kx')$$

with the linear approxmation $z + h \approx h$ we can write

$$v^2 = A^2\omega^2(\sin^2(kx') + \coth^2(kh)\cos^2(kx')) + \frac{\omega^2}{k^2} - 2A\frac{\omega^2}{k}\coth(kh)\cos(kx')$$

and neglect the term in blue because for small amplitude waves

$$A << 1/k, \quad kA << 1, \quad (kA)^2 << kA$$

Bernoulli's equation

So the Bernoulli relation becomes

$$E = \frac{p_a}{\rho} + \frac{1}{2}\frac{\omega^2}{k^2} - A\frac{\omega^2}{k}\coth(kh)\cos(kx') + gA\cos(kx')$$

This must be independent of *x'* for the property to be conserved, so the coefficients of cos*(kx')* must sum to zero.

$$-A\frac{\omega^2}{k}\coth(kh) + gA = 0$$

so

$$\omega^2 = gk\tanh(kh)$$

This is the dispersion relation for linear surface waves on a homogeneous fluid.

Shallow water wave speeds

For shallow water

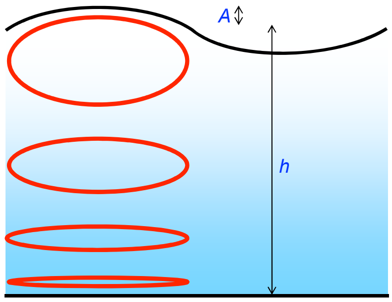
$\tanh(kh) \approx kh \qquad \omega^2 = ghk^2 \qquad \omega = k\sqrt{gh}$

$$\omega^2 = gk\tanh(kh)$$

phase speed $\quad c_\phi = \frac{\omega}{k} = \sqrt{gh}$

group speed $\quad c_g = \frac{d\omega}{dk} = \sqrt{gh}$

Group speed = phase speed.
All wavelengths travel at same speed.
Wave is non-dispersive.



Deep water wave speeds

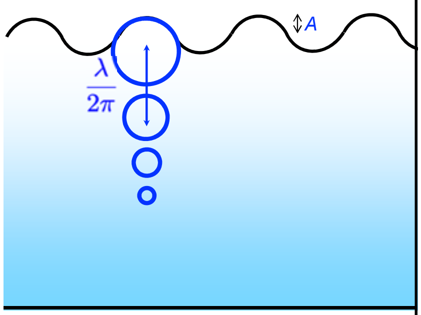
For deep water

$kh >> 1 \qquad \tanh(kh) \approx 1 \qquad \omega = \sqrt{gk}$

$$\omega^2 = gk\tanh(kh)$$

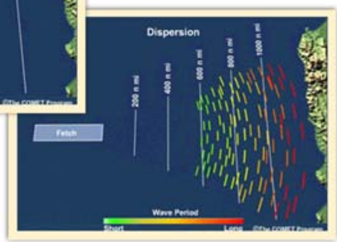
phase speed $\quad c_\phi = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$

group speed $\quad c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c_\phi$



Group speed = half phase speed
Wave shows “normal dispersion”

Dispersal of waves from a storm

⇨ Waves **spread out laterally** but also along the direction of propagation as **different periods and wavelengths separate**.

Measuring distance to a storm from the coast

At a certain distance from the origin, due to dispersion, wave period will decrease with time, t.
We can use this to deduce the distance, L, to the origin.

$$c = \frac{\boldsymbol{L}}{\boldsymbol{t}} \qquad c = \frac{\lambda}{T} = \frac{2\pi}{kT} \quad \text{and} \quad c = \sqrt{\frac{g}{k}} \Leftrightarrow k = \frac{g}{c^2}$$

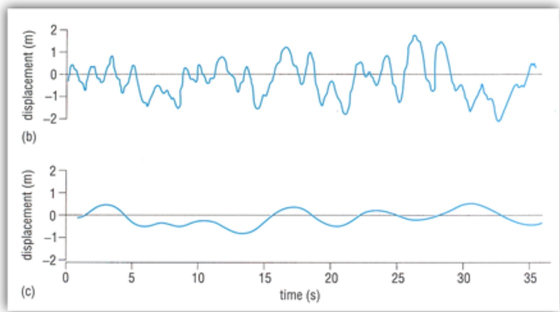
Deep-water waves

$$c = \frac{2\pi c^2}{gT} \Leftrightarrow c = \frac{gT}{2\pi}$$

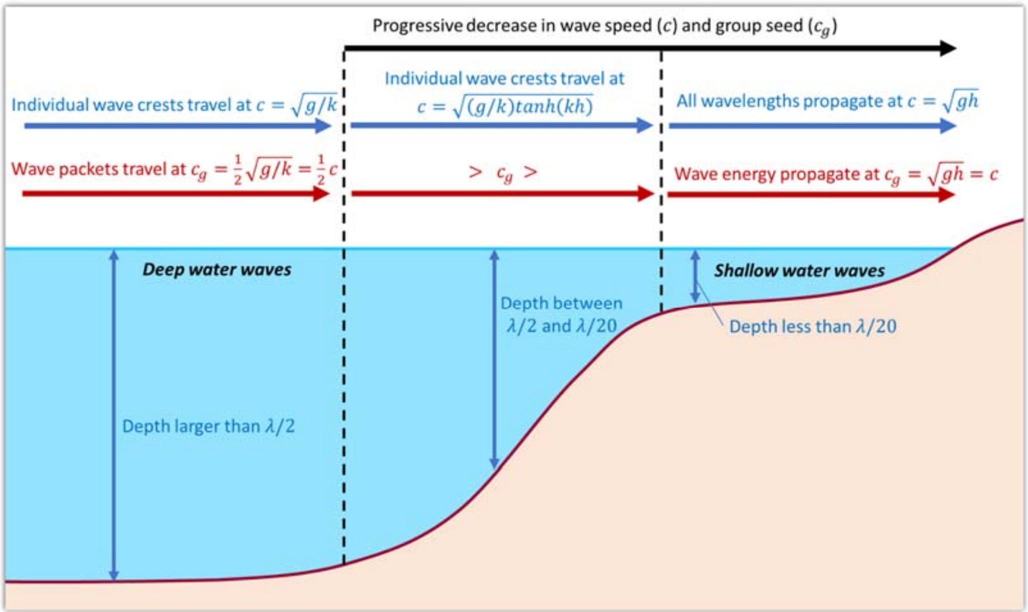
$$\frac{\boldsymbol{L}}{\boldsymbol{t}} = \frac{gT}{2\pi} \Leftrightarrow \boxed{T = \frac{2\pi \boldsymbol{L}}{g\boldsymbol{t}}}$$

One equation, two unknowns !

But by timing wave periods more than once over a sufficient interval we can measure the distance to the storm

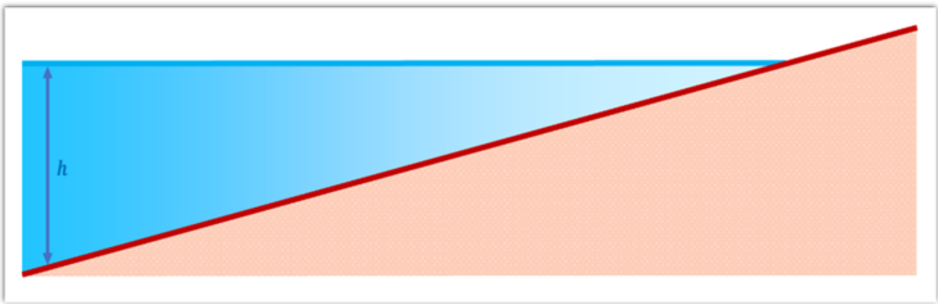


Waves change properties approaching the coast



Approaching the shore in shallow water

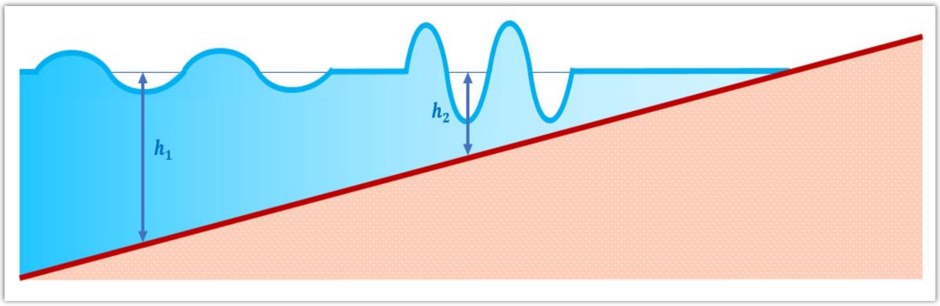
Water gets shallower so waves must slow down
c = ω/k = fλ, so does the frequency change, or the wavelength ?



Approaching the shore in shallow water

Water gets shallower so waves must slow down
c = ω/k = fλ, so does the frequency change, or the wavelength ?

$$\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{h_1}{h_2}}$$



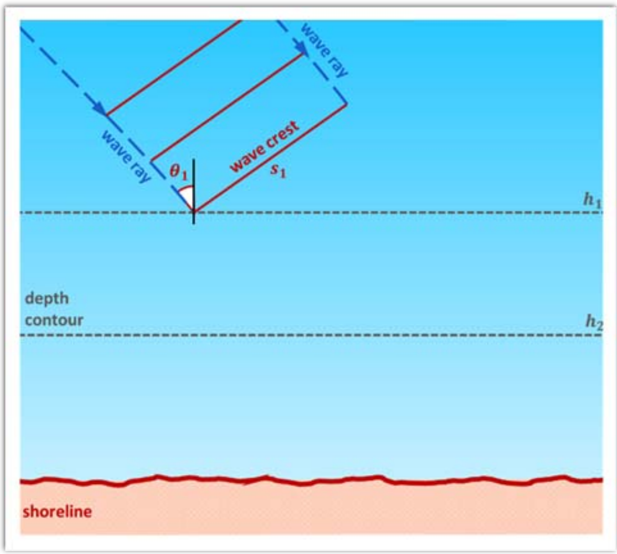
$$A_1^2\lambda_1 = A_2^2\lambda_2$$

Wave refraction

For shallow water waves the wave speed depends on the depth.

In a shallow region with variable depth this leads to curved ray paths.

Rays will converge towards shallower regions.



Wave refraction

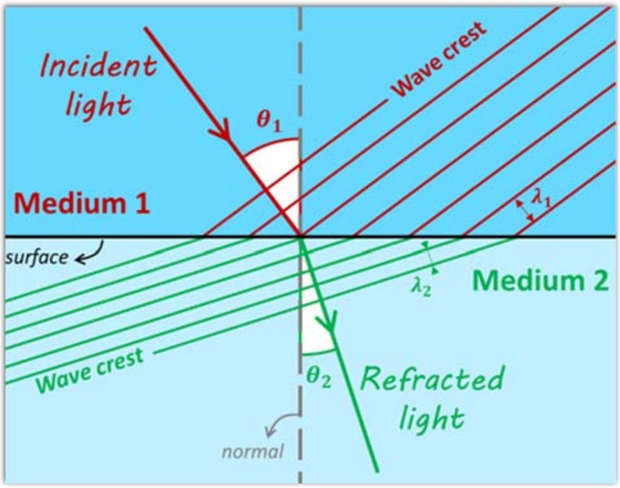
For shallow water waves the wave speed depends on the depth.

In a shallow region with variable depth this leads to curved ray paths.

Rays will converge towards shallower regions.

Over a sudden depth change Snell's law can be applied

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2} = \frac{\sqrt{gh_1}}{\sqrt{gh_2}} = \sqrt{\frac{h_1}{h_2}}$$



Wave refraction

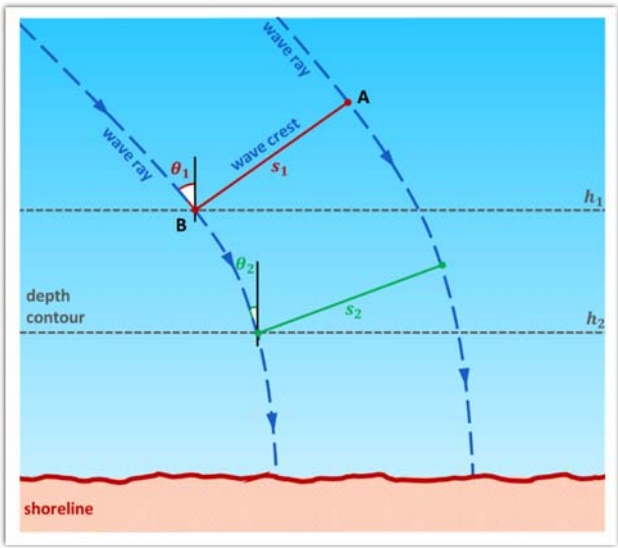
For shallow water waves the wave speed depends on the depth.

In a shallow region with variable depth this leads to curved ray paths.

Rays will converge towards shallower regions.

Over a sudden depth change Snell's law can be applied

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2} = \frac{\sqrt{gh_1}}{\sqrt{gh_2}} = \sqrt{\frac{h_1}{h_2}}$$



Wave amplification

Approaching the shore wavelengths decrease to accommodate slowing waves at constant frequency.

Rays converge towards capes.

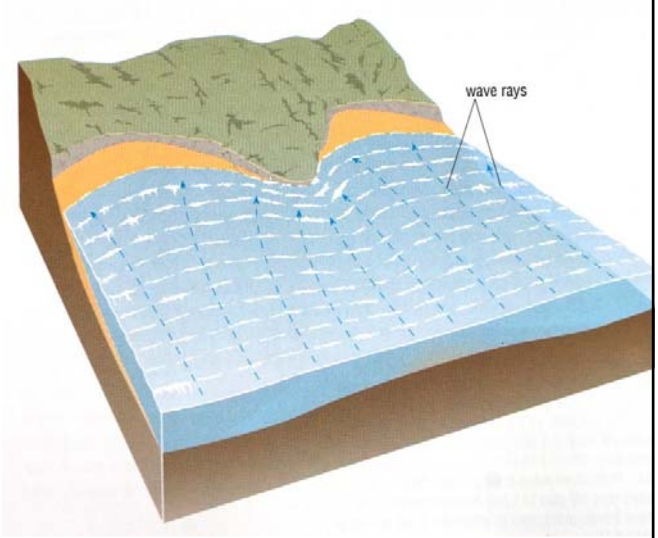
But energy must still be conserved.

Consider a surface area *s* with wave amplitude A. The energy is proportional to A^2s.

Energy conservation

$$A_1^2 s_1 = A_2^2 s_2$$

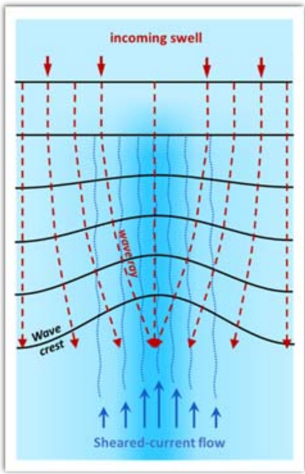
leads to larger wave amplitudes approaching the shore, particularly at capes.

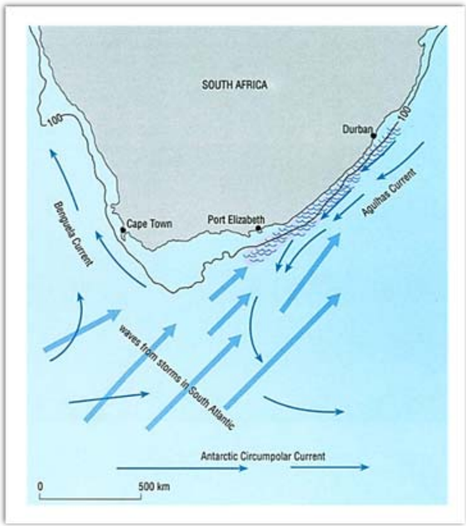


Giant waves

The amplification by refraction effect can also occur in varying currents.

Waves of a given frequency must adjust their wavelength to accommodate the added - or subtracted current speed.

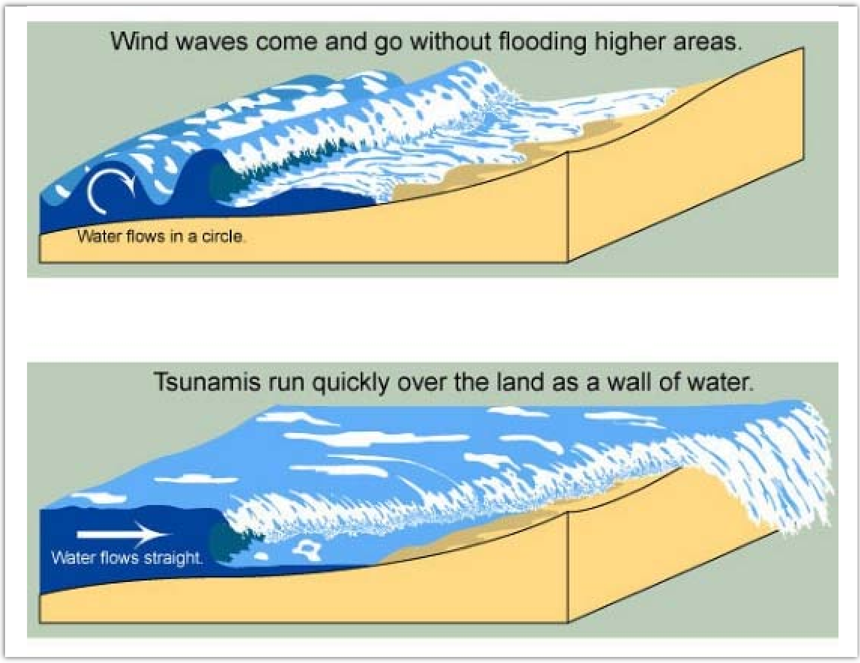




Horizontal current shear can lead to convergence of ray paths.

This leads to very large amplitude waves.

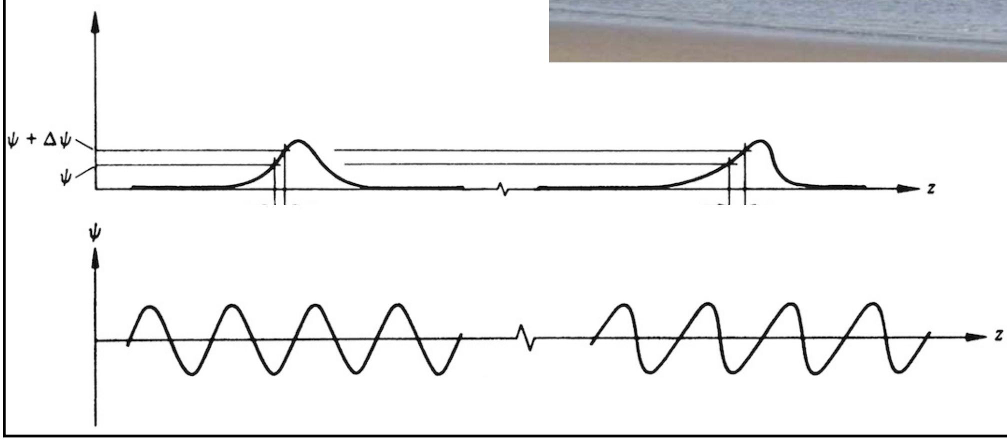
Tsunamis are shallow water waves



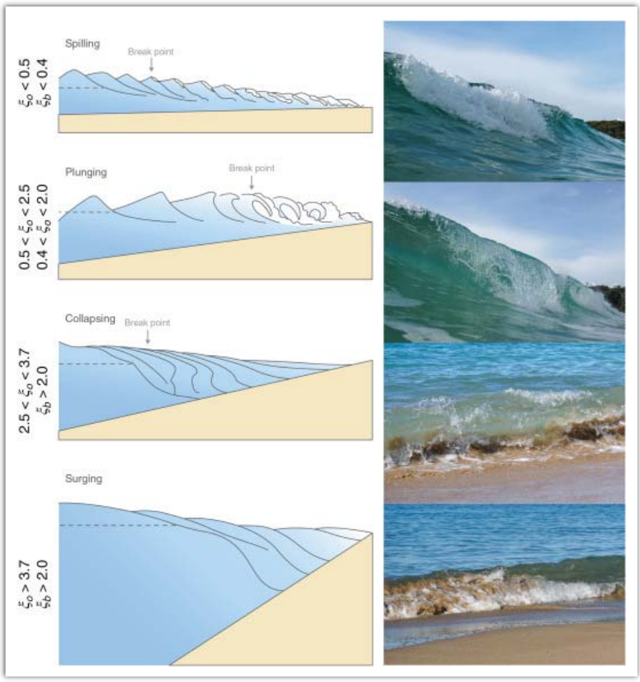
Finite aplitude waves

Nonlinear effects can modify the dispersive properties of waves and keep the short and long waves travelling together in a wave front called a "soliton".

Tital bores can show this property.
Also useful for transmitting signals.



Wave breaking



Four categories of wave breaking:

1) Spilling breakers

Tidal bores and waves on gently sloping coastlines.
Shape-preserving solitons.

2) Plunging breakers

Waves for surfing. Gently sloping beaches.
Originate from long wave swell.

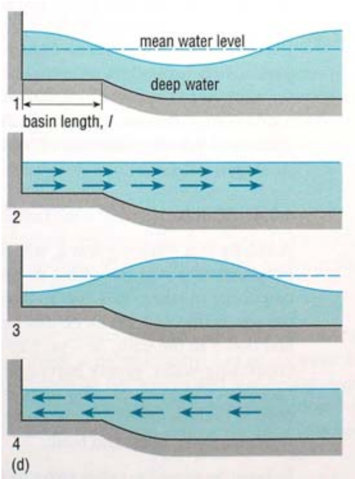
3) Collapsing breakers

Steeper beaches.
Associated with moderate wind conditions.

4) Surging breakers

Steepest beaches.
Wave slides up beach almost without breaking.

Standing waves



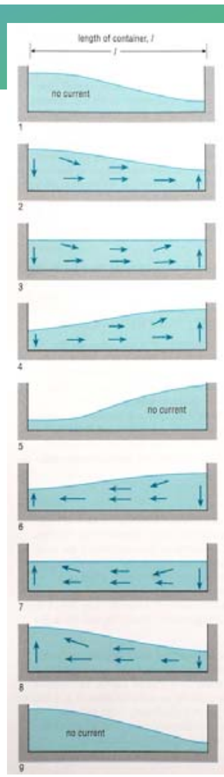
Also known as seiches.

Wavelength is a simple fraction of the width of the bay or of the width of the shelf.

Shallow water dispersion relation gives associated frequency.

Driving at this frequency can lead to a resonant response.

For example large amplitude waves forced by tides.



Standing wave in a small harbour