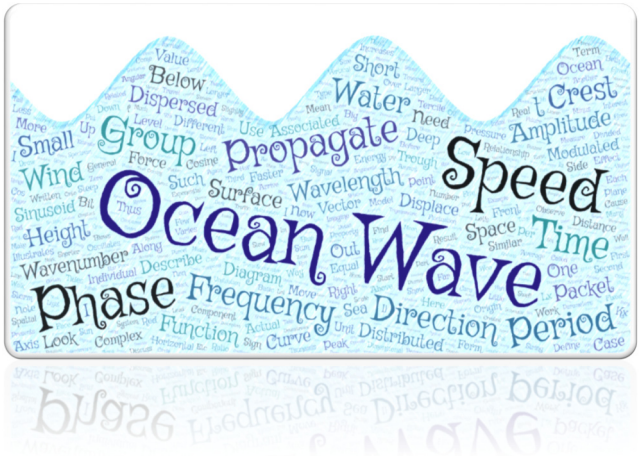
Chapter1: General Properties of the Waves

- *Wind-forced waves*
- *Observed spectrum*
- *Wave kinematics*
- *Wavenumber and frequency*
- *Phase and group speed*
- *Dispersion relation*



Wind-forced waves (1)

⇨ Here is a **hypothesis** to explain how ocean waves might grow:

❶ Let's just imagine the surface of the ocean. It is almost flat with a few undulations to it and a wind is blowing across.

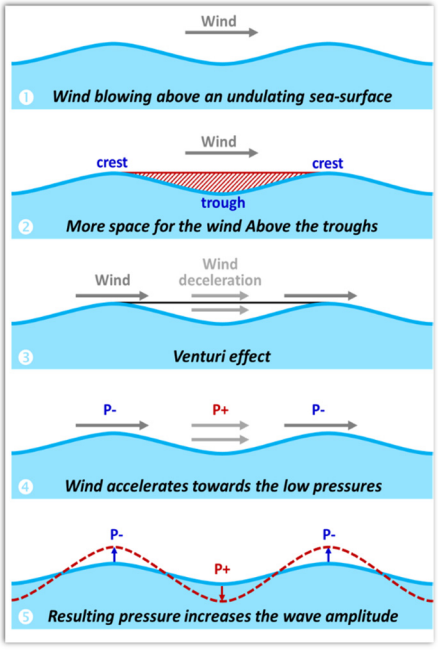
❷ Near the sea-surface, there is a little bit more space above the trough than above the crest (red striped area).

❸ So, the wind is going to occupy more space above the troughs and by conservation of mass, it will slow down. This is the **Venturi effect**.

❹ **Bernoulli's theorem** indicates that changes in the wind speed are associated with a **pressure gradient force**. The wind accelerates from high (**P+**) to low (**P-**) pressures and it decelerates towards higher pressures (**P+**).

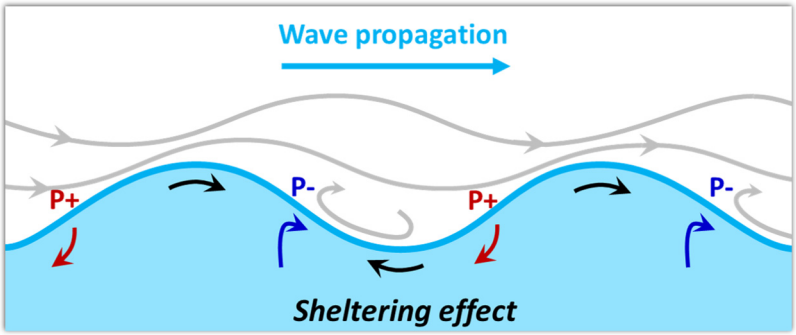
❺ As a consequence, relative to the average pressure, there is slightly less pressure (**P-**) above the crests and slightly more (**P+**) above the troughs. The ocean surface will thus be **pushed up** at the crests and **pushed down** in the troughs, increasing the wave amplitude.

⇩ The wind blowing over a slight undulation makes the undulation get slightly bigger.



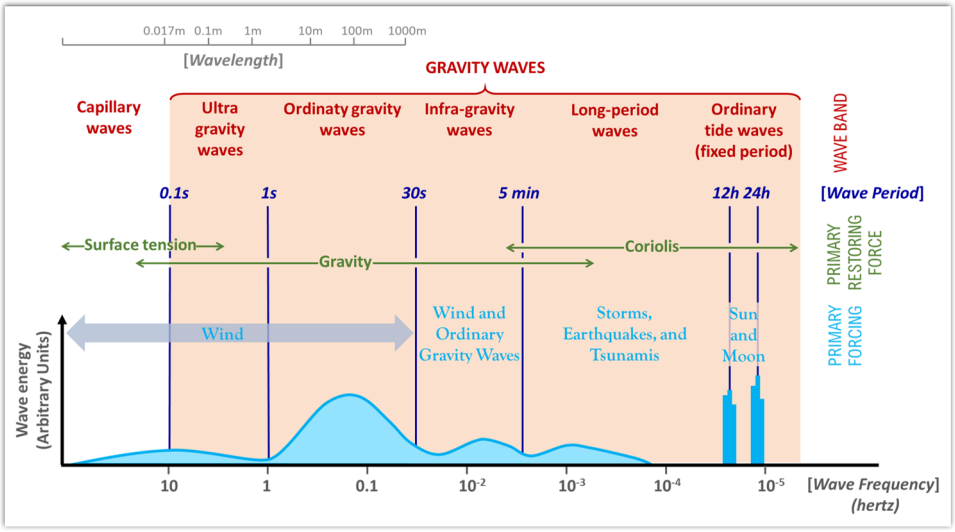
Wind-forced waves (2)

⇨ The "sheltering" model of wave forcing.
The presence of waves modifies the air flow - creates pressure differences that serve to push the wave.
Applies to situations where winds of > 1m/s and faster than wave speed.
Waves also need to be steep enough for the effect to work.

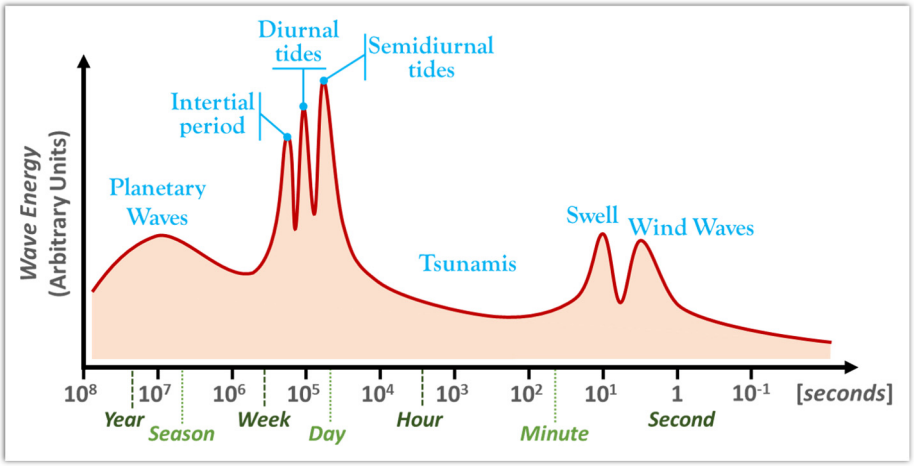


Observed spectrum (1)

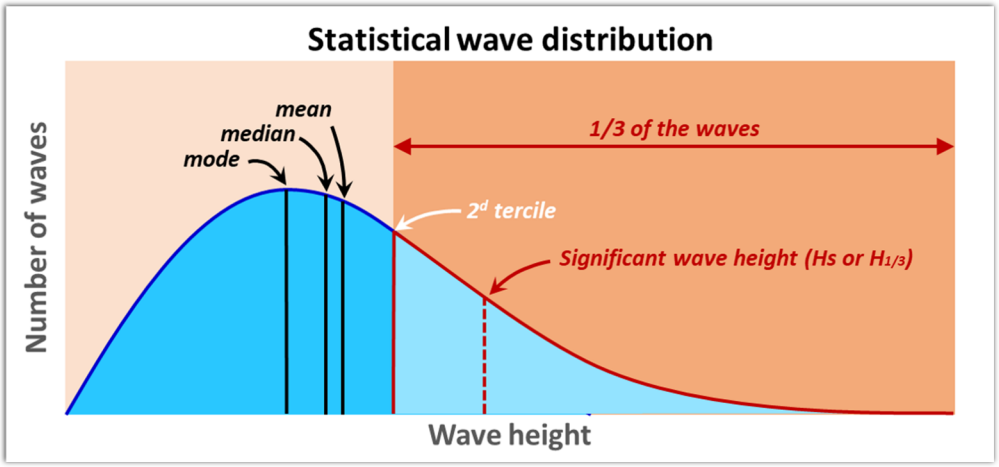
⇨ Distribution of frequency for sea waves, and some of their causes.
(note that for some frequencies the energy may have been transferred from other frequencies)



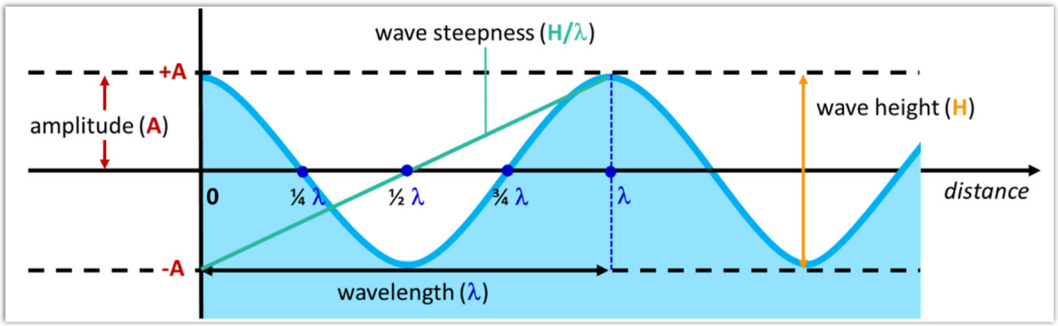
Observed spectrum (2)



Wave height distribution



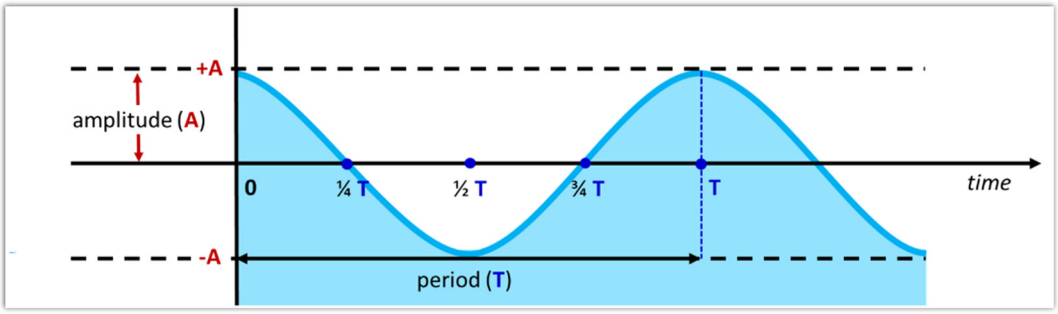
Basic properties in space

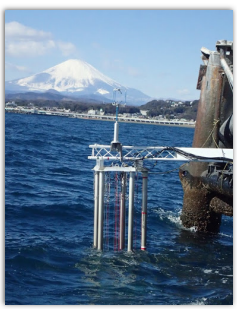


Wave propagating to the right (x increases):

- Amplitude (**A**)
- Wave height (**H**)
- Wavelenght (λ)
- Steepness (H/λ)

Basic properties in time





- Amplitude (**A**)
- Wave height (**H**)
- Period (**T**)

⇨ As well as wavelength and period we also represent waves in terms of frequency and wavenumber

Wave kinematics (1D in space)

⇨ Consider a propagating sinusoidal wave:

$$\eta(x,t) = \boldsymbol{A}\, cos(kx - \omega t + \phi)$$

⇨ Equivalently: $\eta = Re\ \widetilde{\boldsymbol{A}}\, e^{i(kx-\omega t)}$

⇨ Equivalently: $\eta = Re\ \widetilde{\boldsymbol{A}}\, e^{i\theta}$

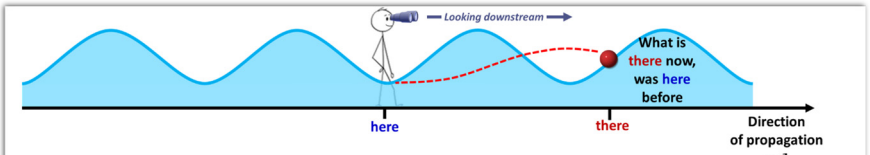
How many parameters do we need to describe this wave?

⇨ Equivalently: $\eta = Re\ \boldsymbol{A}\, e^{i\theta} e^{i\phi}$

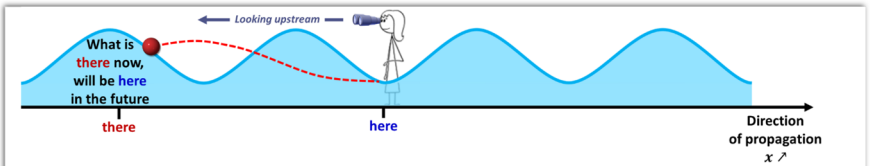
⇨ Equivalently: $\eta = \boldsymbol{A_1}\, cos(kx - \omega t) + \boldsymbol{A_2}\, sin(kx - \omega t)$

Wave kinematics (1D in space)

Why is there a minus in front of ω?



If you look **downstream** ($x > 0$), you will thus be looking into the **past**:
*What is **there** ($x > 0$) now, was **here** before ($t < 0$).*



If you look **upstream** ($x <$), you are looking at oscillations that will arrive at your position in the **future**: *What is over **there** now ($x < 0$), will be **here** in the future ($t > 0$).*

Wave kinematics (2D in space)

⇨ Consider a propagating sinusoidal wave:

$$\eta(x,y,t) = \boldsymbol{A}\, cos(lx + my - \omega t + \phi)$$

How many parameters do we need to describe this 2D wave?

⇨ Equivalently: $\eta = Re\, \widetilde{\boldsymbol{A}}\, e^{i(\boldsymbol{k}\boldsymbol{x} - \omega t)}$

$$\boldsymbol{k} = \begin{pmatrix} l \\ m \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

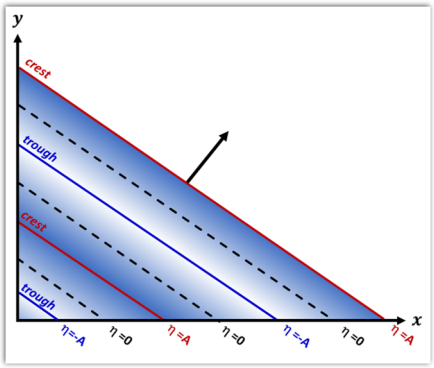
⇨ Equivalently: $\eta = Re\, \widetilde{\boldsymbol{A}}\, e^{i\theta}$

⇨ Equivalently: $\eta = Re\, \boldsymbol{A}\, e^{i\theta} e^{i\phi}$

⇨ Equivalently:

$$\eta = \boldsymbol{A_1}\, cos(kx + my - \omega t) + \boldsymbol{A_2}\, sin(kx + my - \omega t)$$

⇨ We note that the wavenumber has the properties of a vector



Period, wavelength and speed

⇨ Wavelength depends on direction

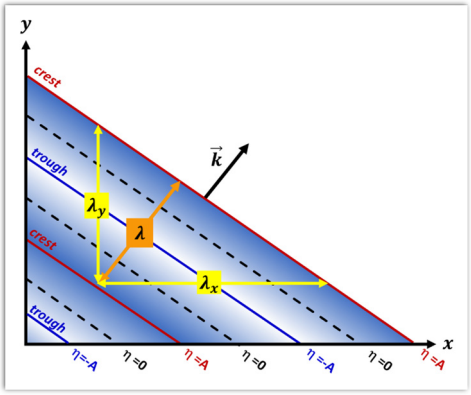
$$\lambda_x = \frac{2\pi}{l} \quad \lambda_y = \frac{2\pi}{m} \quad \lambda = \frac{2\pi}{k}, \quad k^2 = l^2 + m^2, \quad \frac{1}{\lambda^2} = \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2}$$

↳ Wavelength is *not* a vector

• Period $T = \dfrac{2\pi}{\omega}$

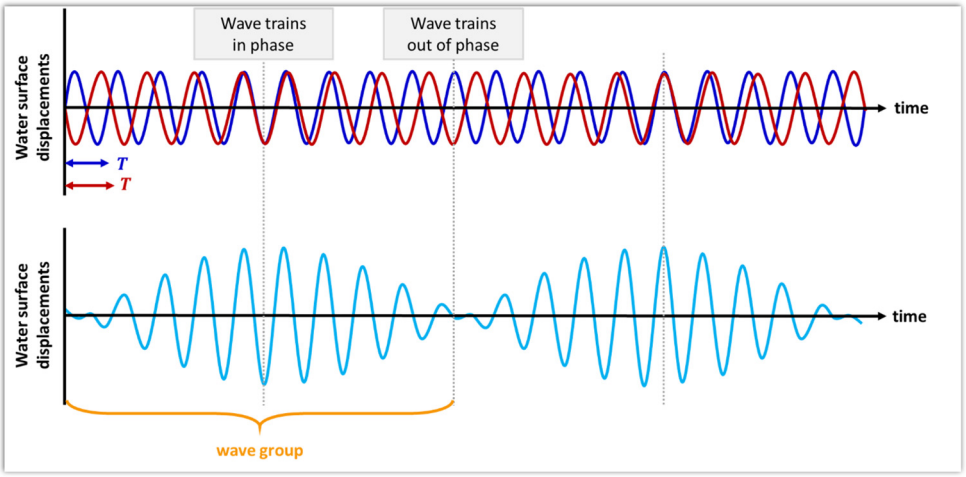
• Phase speed $c_x = \dfrac{\omega}{l} \quad c = \dfrac{\omega}{k}$

(not a vector either)



Interference and modulation

⇨ Interference between frequencies leads to a **modulation envelope** that travels at a different speed to the individual waves.

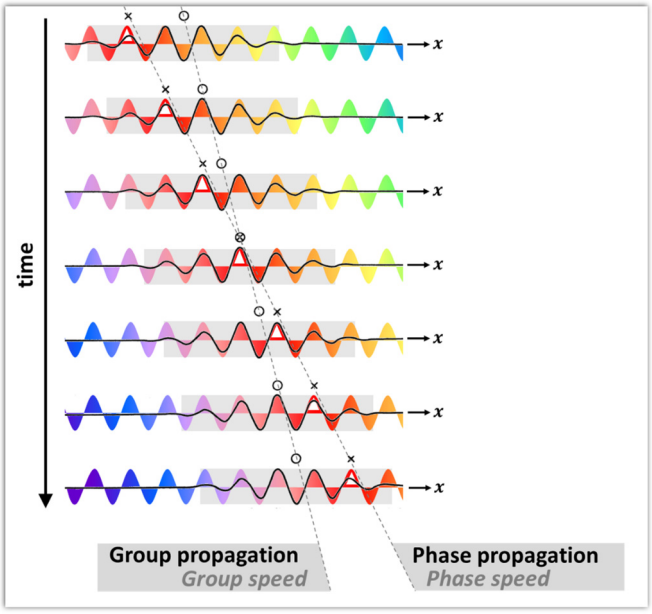


Group propagation

⇨ For normal group propagation, the wave packet moves at half the speed of the phase propagation.

⇨ To study the relative speeds of the packet and the wave crests, we need to find the "dispersion relation"

⇨ For this we need to move beyond kinematics and involve the physical properties of the medium.



Group propagation

⇨ Consider 2 waves of similar frequency and wavenumber

$$\eta = Re\,\widetilde{A}\left[e^{i(k_1x-\omega_1t)} + e^{i(k_2x-\omega_2t)}\right]$$

$$= Re\,\widetilde{A}\,e^{i(kx-\omega t)}\left[e^{i(\Delta kx-\Delta\omega t)} + e^{-i(\Delta kx-\Delta\omega t)}\right]$$

$$\begin{cases} k_1 = k + \Delta k, \;\; k_2 = k - \Delta k \\ \omega_1 = \omega + \Delta\omega, \;\; \omega_2 = \omega - \Delta\omega \end{cases}$$

$$\eta = Re\,\widetilde{A}\,e^{i(kx-\omega t)} \times 2\cos(\Delta kx - \Delta\omega t)$$

Rapidly varying signal — Original signal

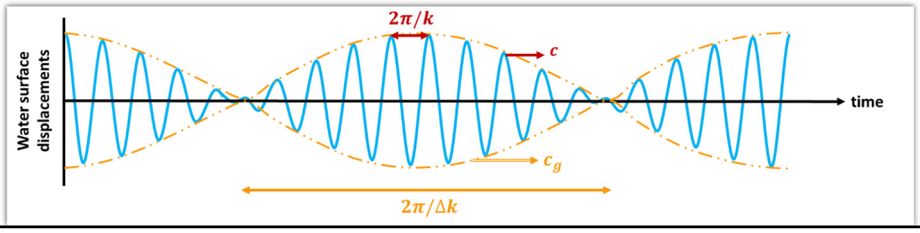
Slowly varying envelope — Modulation

➢ **Envelope** moves with speed

$$\lim_{\Delta\to 0}\frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k}$$

➢ **Group velocity** $\mathbf{c}_g = \nabla_k\omega$

= rate of propagation of information or energy



Dispersion

⇨ However....

$\frac{\omega}{k}$ and $\frac{\partial\omega}{\partial k}$ are not necessarily the same for all values of *k*.

.... so it is possible that the envelope will distort as it displaces.

This is called **dispersion**.

3 cases:

1) Long-waves go faster than short-waves

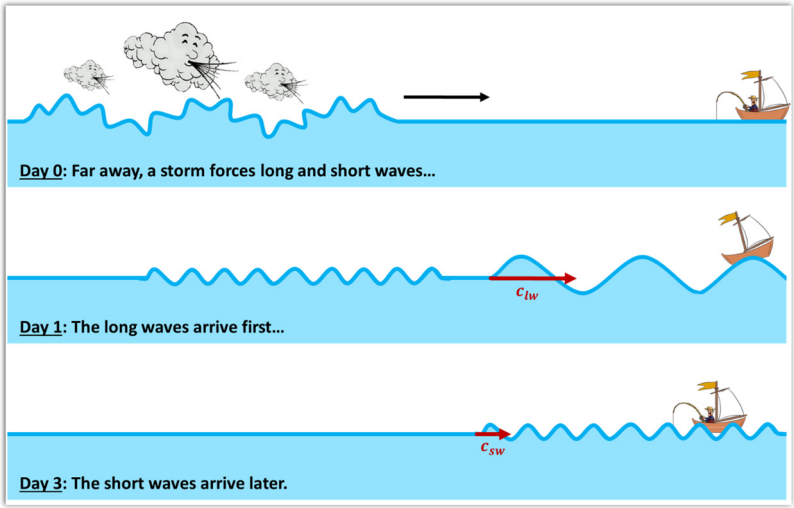
2) Short-waves go faster than long waves

3) Short and lon waves propagate at the same speed

Dispersion

CASE1: Short waves go slow, and long waves go fast.

↳ *What would happen to a wave that is a mixture of short waves and long waves?*



Dispersion relation/ Dispersion diagram

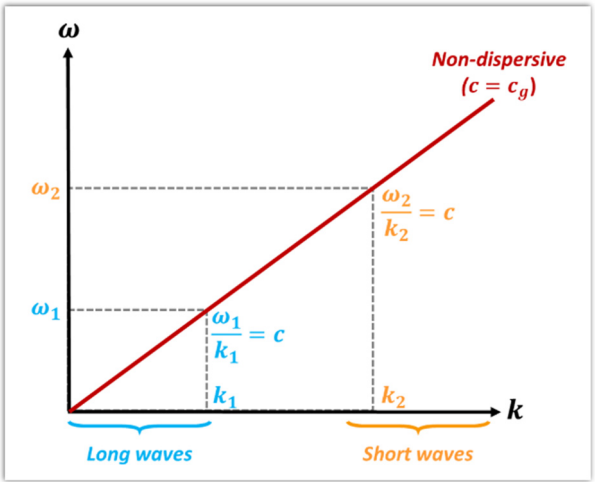
⇨ The relationship between ω and k is called the dispersion relation.

⇨ If this relationship is linear (and of course $\omega=0$ when $k=0$),

$$\frac{\partial \omega}{\partial k} = \frac{\omega}{k}, \quad c_g = c$$

↳ the wave is "non-dispersive"

For non-dispersive waves, all the wavelengths propagate at the same speed. A wave pattern (sum of different wavelength) will not change its shape during its propagation

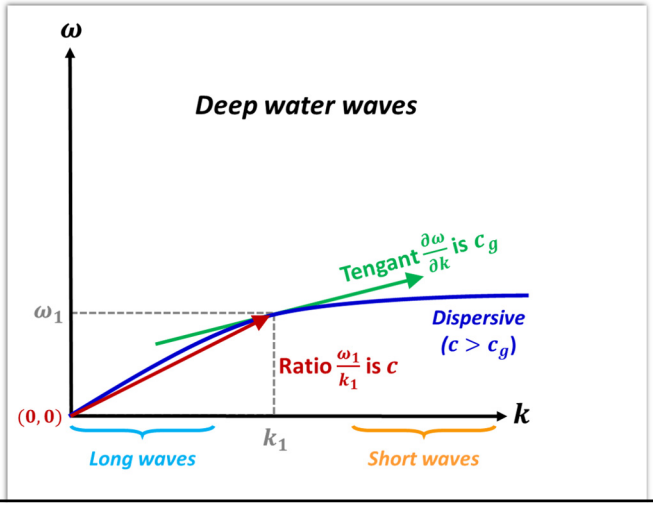


Dispersion diagram

⇨ The relationship between ω and k is called the dispersion relation.

⇢ **the phase speed** (c) is the arrow that points from the origin toward the curve (the ratio $\frac{\omega}{k}$)

⇢ **the group speed** (c_g) is the tangent to the curve $(\frac{\partial \omega}{\partial k})$

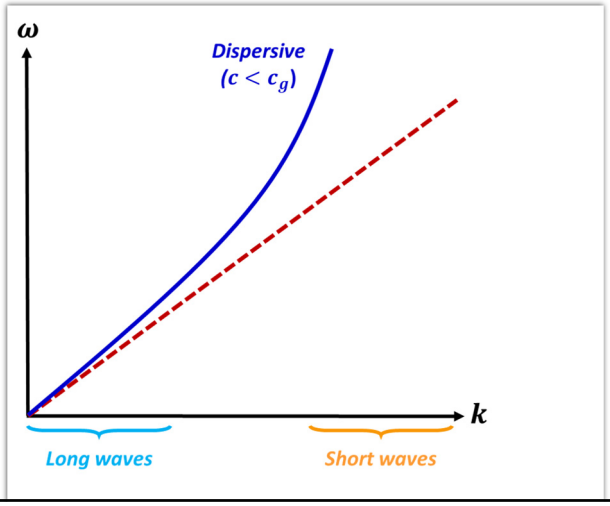


Dispersion diagram

⇨ The relationship between ω and k is called the dispersion relation.

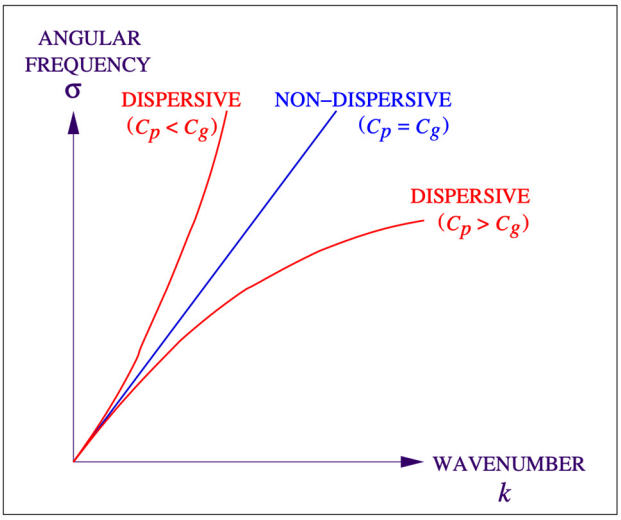
⇢ **the phase speed** (c) is the arrow that points from the origin toward the curve (the ratio $\frac{\omega}{k}$)

⇢ **the group speed** (c_g) is the tangent to the curve $(\frac{\partial \omega}{\partial k})$



Dispersion diagram

Phase Velocity and Group Velocity



Phase velocity, $c_p = \frac{\sigma}{k}$; Group velocity, $c_g = \frac{\partial \sigma}{\partial k}$