CHAPTER 3
Some Geophysical Waves

CHAPTER 3
Some Geophysical Waves

WAVES3 Contents

This chapter focuses on the propagation of waves in a **geophysical fluid**. We will consider barotropic ocean waves on a **rotating planet**.

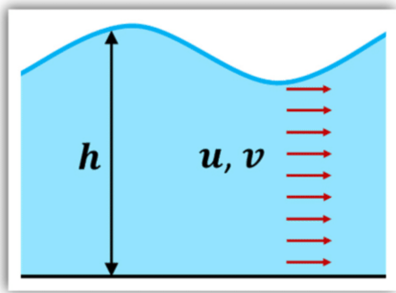
We begin by discussing what happens to **surface gravity waves** when the earth is spinning. From the one-layer shallow-water dynamical equations yielding surface gravity waves (see **#WAVES3.1**), we will **add rotation**, leading to "inertia gravity waves", and we will analyse their properties (**#WAVES3.2**).

Looking for long waves propagating on a rotating planet, we will study the special case of gravity waves that arises when there is a wall against which the waves can lean: **coastal Kelvin waves** (see **#WAVES3.3**).

Both inertia gravity waves and Kelvin waves are gravity waves. They propagate in the horizontal direction and their restoring force, a mixture of gravity and buoyancy, is in the vertical direction. But the Coriolis force, associated with the rotation of our planet, plays a fundamental role in the way they propagate. In **#WAVES3.4**, we move on to **Rossby waves**, for which the restoring force is not gravity. Their existence depends on the strength of the Coriolis force, which varies with latitude on a spherical planet.

WAVES3.1: Gravity Waves from the Shallow Water Equations

3.1.a) Full shallow-water equations of motion



⇨ Let's consider a single shallow layer of fluid on a rotating planet. This layer of fluis of constant density extends from a flat bottom to a free surface. The varying layer thickness is h and the vertically uniform horizontal currents are u and v.

↳ The complete equations of motion describing the dynamics of this single layer of fluid on a rotating planet consist of a set of 3 equations with 3 state variables – 3 unknowns – (u, v, h).

1) First, the **x-momentum equation** for the zonal velocity (u):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

▪ On the left-hand side of the equation, there is the **tendency term** ($\partial u/\partial t$) for the zonal velocity, without which there would be no time evolution and the system would remain stationary.

▪ Then, there are the **zonal** ($u\partial u/\partial x$) and **meridional** ($v\partial u/\partial y$) **advection terms**. Together with the tendency term, they contribute to the acceleration of a parcel of fluid as it moves with the flow and form the material tendency of the zonal velocity ($\frac{Du}{Dt}$).

✋ Note that, the advection terms are non-linear, i.e. they are terms in which there is a product of two state variables ($u \times u$ and $v \times u$). If the fluctuations of u, v, and h are small (relative to a given mean state), then any quadratic term in which there is a product of state variables will be very small.

▪ According to Newton's law – force equals mass times acceleration, or force per unit mass equals acceleration – there are **two forces** acting on this system:

⇢ The **Coriolis force** ($-fv$) is the term associated with the rotation of the planet. It **deflects the flow to its right** in the northern hemisphere. Since this term is not a real force, but only an artifice (fictitious force) of having changed our coordinate system, it is traditionally placed on the left-hand side of the equation.

⇢ The real force, on the right-hand side of the equation, is the pressure gradient force. In a single-layer system it is written in terms of the gradient of the layer thickness $\partial h/\partial x$. In a homogeneous hydrostatic fluid, this term is equal to $\rho_0^{-1}\partial p/\partial x$.

2) The **y-momentum equation** for the meridional velocity (v):

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

↳ This equation is similar to the x-momentum equation. It consists of the material tendency of the meridional velocity, the Coriolis force, and the pressure gradient force.

3) The **equation for the mass conservation**, the continuity equation.

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

↳ It is expressed here as a material tendency of the layer thickness (or surface height) driven by divergence. Convergence or divergence in the fluid results in an increase or decrease in the thickness of the layer.

➢➢ We are going to analyse this set of equations and try to find wave solutions. Before we do that, we will simplify the system to recover the properties of the shallow-water linear waves in a non-rotating system derived in **#Chapter2**. We will then reintroduce rotation.

3.1.b) Gravity waves in non-rotating shallow water

⇨ Let's start with something very simple: a **one-dimensional non-rotating linear system**. We will remove the y-dimension, remove the rotation of the planet, and neglect the non-linear terms to create a linear system that can be solved analytically.

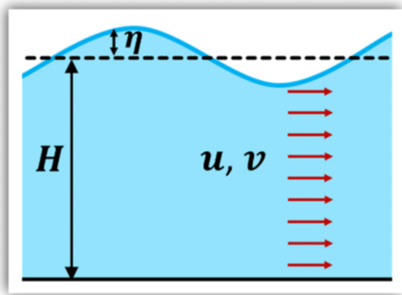
↳ The shallow water system can be simplified as follows:

- A **one-dimensional system** in the x-direction, so we ***cross out the y-momentum equation*** and ***the y-direction terms*** (no v and no variations in y). The flow will be uniform in the y-direction.
- A **non-rotating system**, so we ***remove the terms associated with the Coriolis force***.
- A **linear system**, so we ***eliminate any terms that are quadratic in the state variables*** (all terms with a product between two state variables, i.e., the advection terms and divergence terms).

$$\frac{\partial u}{\partial t}+\cancel{u\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}-\cancel{fv}=-g\frac{\partial h}{\partial x}$$

$$\cancel{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-g\frac{\partial h}{\partial y}}$$

$$\frac{\partial h}{\partial t}+\cancel{u\frac{\partial h}{\partial x}}+\cancel{v\frac{\partial h}{\partial y}}=-h\left(\frac{\partial u}{\partial x}+\cancel{\frac{\partial v}{\partial y}}\right)$$



↳ There is a little subtlety in the continuity equation, as we do not want to completely remove $h\frac{\partial u}{\partial x}$. We want to keep some aspect of divergence otherwise there would not be any variations in surface height.

↪ Unlike to u and v, which are just perturbations, h is the mean depth (H) plus a perturbation (η) around that reference depth, so that we can write $h=H+\eta$, and $\frac{\partial h}{\partial t}=\frac{\partial \eta}{\partial t}$.

↪ We linearize the quadratic term of the continuity equation by considering a **constant** average layer thickness H so that: $h\frac{\partial u}{\partial x}\approx H\frac{\partial u}{\partial x}$.

⇨ This gives us two equations: $\frac{\partial u}{\partial t}=-g\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial t}=-H\frac{\partial u}{\partial x}$

↳ These two equations can be combined to eliminate one variable between the two unknowns. We differentiate the x-momentum equation with respect to x and differentiate the continuity equation with respect to time (t), thus eliminating u. This results in the following **second-order ordinary differential equation** for u:

$$\frac{\partial^2 \eta}{\partial t^2}=gH\frac{\partial^2 \eta}{\partial x^2}$$

⇨ As in #WAVES1.2d, the wave-like solution to this equation can be written with a trigonometric function with an imaginary exponential: $\eta = Re\, \tilde{\eta}\, e^{i(lx-\omega t)}$.

▪ It is the real part of some amplitude coefficient ($\tilde{u}$) times the classical imaginary exponential propagation part:

- l is the **zonal wavenumber** (2π divided by the x-wavelength),
- ω is the **angular frequency** (2π divided by the period).

↪ This is a wave propagating in the positive x-direction when $l > 0$ (see **#WAVES1.2d**).

▪ Taking a derivative of this trigonometric function yields the same function multiplied by some constant coefficients:

$$\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \frac{\partial}{\partial x} \rightarrow il$$

⇨ Substituting this solution and its derivatives into the wave equation gives **a relationship between frequency ω and wavenumber l** (with two geophysical parameters: gravity g and average layer thickness H): $\omega^2 = gHl^2$.

↳ This is the **non-dispersive dispersion relation for gravity waves in shallow water** (see **#WAVES2.3d**), from which we can derive the shallow-water wave phase and group speeds:

$$c = \frac{\omega}{l} = \pm\sqrt{gH} \left(= \frac{\partial \omega}{\partial l} \right)$$

All the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape along its propagation

📖 This derivation is much faster than the procedure in **#WAVES2**. This is because all the assumptions introduced gradually in **#WAVES2**, are built into the shallow-water equation set from the start. We can immediately write the solution and confirm the result for one-dimensional shallow-water waves in a non-rotating linear system. Let's now put rotation back into the system.

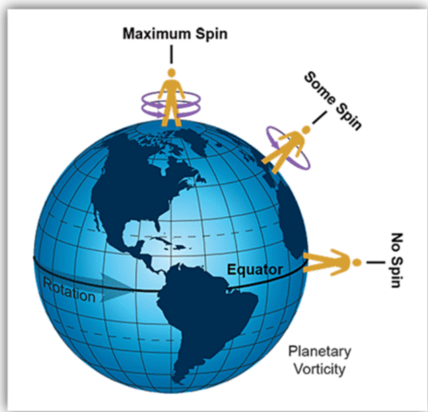
WAVES3.2: Inertia Gravity Waves

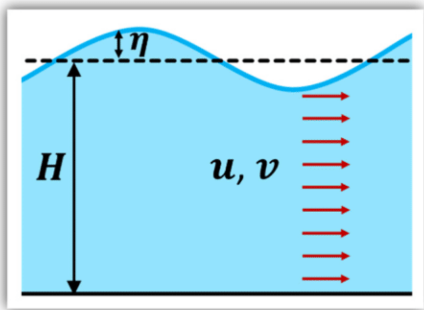
3.2.a) Adding rotation

⇨ The next step is to put constant rotation back into the linear system. We **put the Coriolis terms back into the system** and we maintain linearity.

Since the Coriolis force pushes perpendicular to the direction of motion, we have to go back to a ***two-dimensional situation*** with ***3 equations*** again: the linearized x-momentum, y-momentum, and continuity (with 2D divergence) equations.

These are the single-layer linear shallow-water equations on a flat bottom and an f-plane (f is a constant) with linear perturbations in u, v, and η:





$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

➢➢ There are **two ways** to solve this system for plane-wave solutions. There is **the tedious algebraic way** which we will do first (see **#WAVES3.2b**). Then there is a cleverer way which will be done in **#WAVES3.2c**.

3.2.b) METHOD1: Using differentiation to find the wave solutions

⇨ We have a system of three equations with three unknowns, and three **constant** geophysical parameters (f, g, and H).

↳ To solve this system, we can use the same method we used in **#WAVES3.1b**, i.e., **differentiate** the equations judiciously in order to **eliminate two of these three variables**. This will result in one high-order differential equation for one state variable (u, v, or η).

⇨ We will eliminate u and v and search for an equation in η.

↳ As we manipulate these equations, along the way we will encounter some very important equations that describe **fluid dynamics on a rotating planet**.

⇨ In the following, we will use subscripts notation for derivatives, so that η_x is $\frac{\partial \eta}{\partial x}$. The linear shallow-water equation on an f-plane can be written as:

$$u_t - fv + g\eta_x = 0 \quad (1)$$
$$v_t + fu + g\eta_y = 0 \quad (2)$$
$$\eta_t + H(u_x + v_y) = 0 \quad (3)$$

There are 5 steps to derive a third-order differential equation for η:

1) First, we form the **vorticity equation** from the momentum equations.

↪ The vorticity is the $curl$ of the momentum so that we take the $curl$ of the momentum equations: the x-derivative of the y-momentum equation minus the y-derivative of the x-momentum equation ($\partial(2)/dx - \partial(1)/dy$). We get an equation for the time evolution of the relative vorticity ($\xi = v_x - u_y$):

$$(v_x - u_y)_t + f(u_x + v_y) = 0 \quad (V)$$

f remains constant

↪ The time tendency of the relative vorticity arises from f times the divergence of the flow ($D = u_x + v_y$). Note that the pressure gradient terms cancel out in the process.

2) By taking the divergence of the momentum equations, we can then form the **divergence equation** ($\partial(1)/dx + \partial(2)/dy$), which gives:

$$(u_x + v_y)_t - f(v_x - u_y) + g\nabla^2\eta = 0 \quad (D)$$

↪ The tendency of the divergence arises from the f times the vorticity, plus a Laplacian term in the surface height perturbation (the gravity wave source, see **#WAVES3.1b**).

3) Substituting the vorticity equation (V) into the continuity equation (3) gives an equation for the surface height tendency in terms of the time tendency of the relative vorticity:

$$\eta_t - \frac{H}{f}(v_x - u_y)_t = 0 \quad (*)$$

4) We substitute the divergence (D) into the time derivative of the continuity equation $(3)_t$.

$$\eta_{tt} + fH(v_x - u_y) - gH\nabla^2\eta = 0$$

5) If we take the time derivative of this equation, we can substitute $(v_x - u_y)_t$ using $(*)$:

$$\eta_{ttt} + fH(v_x - u_y)_t - gH\nabla^2\eta_t = 0$$

$$\eta_{ttt} + f^2\eta_t - gH\nabla^2\eta_t = 0$$

↳ We obtain a third-order differential equation for surface height perturbations.

⇨ We can integrate this equation in time and with the appropriate initial condition at $t = 0$, we can set the integration constant to zero. The departure from geostrophic equilibrium follows a second-order differential equation:

$$\eta_{tt} - gH\nabla^2\eta + f^2\eta = 0$$

↳ We search for plane-wave solutions ($\eta = \tilde{\eta}\ e^{i(\boldsymbol{k}.\boldsymbol{x}-\omega t)} = \tilde{\eta}\ e^{i(lx+my-\omega t)}$).

▪ Solutions have the form of an amplitude coefficient times an imaginary exponential:

- l is the **zonal wavenumber** (2π divided by the x-wavelength),
- m is the **meridional wavenumber** (2π divided by the y-wavelength)
- $\boldsymbol{k}$ is the vector wavenumber ($\boldsymbol{k} = (l, m)$), so that $k^2 = l^2 + m^2$
- ω is the **angular frequency** (2π divided by the period).

▪ Spatial and temporal derivatives of this trigonometric function yield the same function multiplied by some constant coefficients:

$$\frac{\partial}{\partial t^2} \rightarrow -\omega^2 \qquad \nabla^2 \rightarrow -k^2 = -(l^2 + m^2)$$

k is the vector wave number: $k^2 = l^2 + m^2$

⇨ This yields the dispersion relation forlinear shallow-water waves on a rotating planet:

$$\omega = \pm\sqrt{f^2 + gHk^2}$$

⇢ If we set $f = 0$ (no rotation), we recover the dispersion relation for shallow-water gravity waves in a non-rotating fluid (see **#WAVES3.1b**), i.e. **non-dispersive waves** with a constant phase and group speeds: $c = c_g = \pm\sqrt{gH}$.

⇢ The extra f^2 under the square root means that the relationship between ω and k is **no longer linear**. These waves are **dispersive**. These are the **inertia-gravity waves**.

➢➢ Before we examine this dispersion relation in detail (see **#WAVES3.2d**), we will derive this dispersion relation all over again, but this time we will use a more general (clever) method to find the wave solutions.

3.2.c) METHOD2: General method for finding wave solutions

⇨ We recall the linear shallow-water equation on an f-plane:

$$u_t - fv + g\eta_x = 0$$
$$v_t + fu + g\eta_y = 0$$
$$\eta_t + H(u_x + v_y) = 0$$

↳ The method is to impose upfront that each state variable behaves as a plane wave, so that:

$$(u, v, \eta) = (\tilde{u}, \tilde{v}, \tilde{\eta})\ e^{i(lx+my-\omega t)}$$

▪ Solutions takethe form of an amplitude coefficient times an imaginary exponential:

- l is the **zonal wavenumber** (2π divided by the x-wavelength)
- m is the **meridional wavenumber** (2π divided by the y-wavelength)
- $\boldsymbol{k}$ is the vector wavenumber ($\boldsymbol{k} = (l, m)$), so that $k^2 = l^2 + m^2$
- ω is the **angular frequency** (2π divided by the period).

▪ The derivatives become coefficients: $\frac{\partial}{\partial x} \rightarrow il \qquad \frac{\partial}{\partial y} \rightarrow im \qquad \frac{\partial}{\partial t} \rightarrow -i\omega$

$$-i\omega\tilde{u} - f\tilde{v} + igl\tilde{\eta} = 0$$
$$-i\omega\tilde{v} + f\tilde{u} + iglm = 0$$
$$-i\omega\tilde{\eta} + H(il\tilde{u} + im\tilde{v}) = 0$$

↳ Substituting the solution and its derivatives into the linear system yields a set of three algebraic equations, in which the three unknowns are the amplitude coefficients ($\tilde{u}, \tilde{v}$, and $\tilde{\eta}$).

↳ The parameters are the wave properties (l, m, and ω) and the geophysical constants (f, g, and H).

⇨ We can write this set of equations in matrix form, resulting in **an algebraic system**:

$$\begin{pmatrix} -i\omega & -f & igl \\ f & -i\omega & igm \\ ilH & imH & -i\omega \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

▪ This equation is trivially satisfied if there is no wave-like perturbation ($\tilde{u} = \tilde{v} = \tilde{\eta} = 0$). But it is a completely uninteresting solution, meaning that the fluid remains at rest.

▪ The condition for the system to be satisfied and for the wave to have some amplitude is that ***the determinant of the matrix must be zero***.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \bullet \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \bullet \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \bullet \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

↳ This gives us the dispersion relation for gravity waves in a rotating fluid:

$$\omega[\omega^2 - f^2 - gH(l^2 + m^2)] = 0$$

⇨ The **solutions** are either:

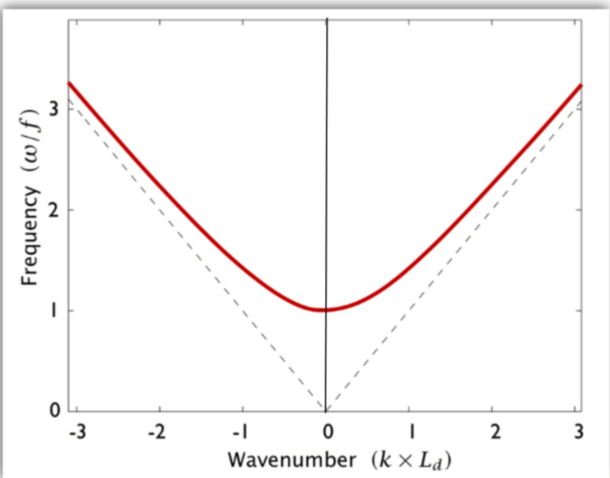
- A steady geostrophic flow ($\omega = 0$, no oscillation in time ≡ a fixed stationary wave).
- A propagating wave that satisfies: $\omega = \pm\sqrt{f^2 + gHk^2}$

k is the vector wave number: $k^2 = l^2 + m^2$

↳ We recover the **dispersion relation for inertia gravity waves** that we laboriously derived in **#WAVES3.2b**.

3.2.d) Limit behaviour

⇨ Here is the **dispersion diagram** for these waves. The frequency ω, expressed in terms of multiples of the Coriolis frequency f, is plotted as a function of the total wavenumber k in dimensionless units ($k^2 = l^2 + m^2$).



▪ When $l > 0$, the wave propagates in the positive x-direction, and when $l < 0$ it propagates in the opposite direction.

▪ The dashed lines are the **shallow-water gravity waves without rotation** ($f = 0$) for which the relation between ω and k is linear. These waves are non-dispersive. The group speed is equal to the phase speed $c = c_g = \pm\sqrt{gH}$ (see **#WAVES3.1b**).

▪ The red curve shows the system with rotation, i.e. adding f^2 under the square root in the dispersion relation. These are ***inertia-gravity or Poincaré waves***.

⇨ There are two interesting limits to consider:

⇢ <u>**For short-waves**</u> (large values of the wavenumber $k \gg \sqrt{f^2/gH}$) rotation does not make much difference to the way the waves propagate. The dispersion relation asymptotes to a straight line in the short-wave limit. In the absence of rotation, these waves behave like ordinary, non-dispersive-shallow water gravity waves ($\omega^2 = gHk^2$).

↪ Thus, if the wave has a sufficiently short length scale, it will not be large enough to feel the effects of the rotation, and the behaviour will approximate that of shallow-water gravity waves. This is the case with **Tsunamis**. Tsunamis are short Poincaré waves, with a wavelength (λ) smaller than $\sqrt{gH}/f$. However, at high latitude, the rotation of the earth can affect their propagation.

⇢ **For larger scales** (wavelengths much longer than $\sqrt{gH/f}$), the **curve flattens out**, and the frequency (ω) has a **lower limit** of f (for $k = 0$). Long waves are **highly dispersive**.

↪ At very small wavenumbers, the wave begins to behave rather strangely:

- As the horizontal scale of the wave becomes larger (k ↘), the phase speed becomes faster. The slope of a line connecting the origin to the curve becomes steeper (see **#WAVES1.4b**).
- For short waves, the group speed (the tangent to the curve, see **#WAVES1.4b**) is equal to the phase speed, and then for larger scales (k ↘), the group speed tends toward zero. $c_g = 0$ would mean that there is **no transmission of information from one position to another**, even though the **oscillations that are separated in space are perfectly coherent**.

↪ For very long waves, there is almost instantaneous communication, but no actual propagation of information. Particles are moving around in phase with a frequency that tends to f, but they are not really propagating as a wave ($c_g = 0$). This is not really a wave anymore. It is **coherent oscillations in space separated by some distance**. In fact, it is just motion in **inertial circles**. The motions are of such large length scales that the effects of rotation are important and dominate over the effects of gravity.

⇨ The ratio $\sqrt{gH/f}$, which defines the boundary between motions dominated by gravity and those dominated by rotation, is an important quantity. It is known as the **Rossby radius of deformation** (L_d). For motions with length scales much larger than L_d, rotation dominates, and for motions with length scales much smaller than L_d, rotation effects are not important and gravity or buoyancy effects dominate. This is why the waves are called inertia-gravity waves or **Poincaré waves**.

➢➢ In conclusion, for large scales, when we add rotation, the waves basically collapse to inertial motion. ***We are left wondering if there is a way to have large-scale propagating geophysical waves on a rotating planet.*** The answer is yes, and they are called Kelvin waves (see **#WAVES3.3**).

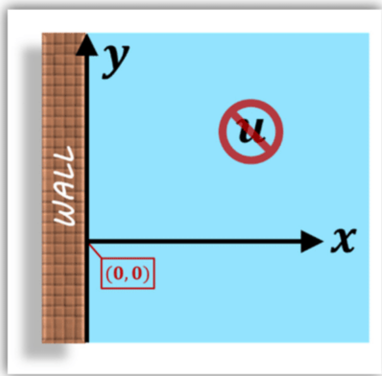
WAVES3.3: Boundary Kelvin Waves

3.3.a) Adding a wall

⇨ Kelvin waves are a special type of wave solution to the shallow water equations in the presence of rotation but also in the presence of a boundary.

⇨ To find ***large-scale propagating geophysical waves on a rotating planet***, we add a lateral boundary to the problem, which introduces a constraint on the shallow-water equation set.

⇨ Consider the case where there is a north-south wall on the western side of the Ocean, at $x = 0$. For inertia-gravity waves, a pressure (or height) gradient in the y-direction would induce a velocity in the x-direction due to the Coriolis force (see **#WAVES3.2a**). However, in the presence of this wall, we have an additional boundary condition: the flow cannot cross the wall: $u = 0$ at $x = 0$.



⇨ It is therefore reasonable to look for solutions that have $u = 0$ everywhere. Therefore, a pressure gradient in the direction of the wall cannot induce a velocity in the x-direction, and different wave dynamics will arise, especially for long waves.

⇨ Consider the linearised shallow-water equations (see **#WAVES3.2a**) but with $u = 0$:

$$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} = -H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right)$$

We put a north-south wall on the western side of the Ocean and the flow cannot cross the wall, i.e. no flow perpendicular to the wall

• This gives the **geostrophic balance** in the x-direction, i.e. the equilibrium between the pressure gradient force $\mathbf{F_P}$ and the Coriolis force $\mathbf{F_C}$.

$$\boxed{fv = g\frac{\partial \eta}{\partial x}}$$

Diagnostic equation
Geostrophic balance

• In the y-direction, we recover the two equations for **non-rotating shallow-water gravity waves** (see **#WAVES3.1b**):

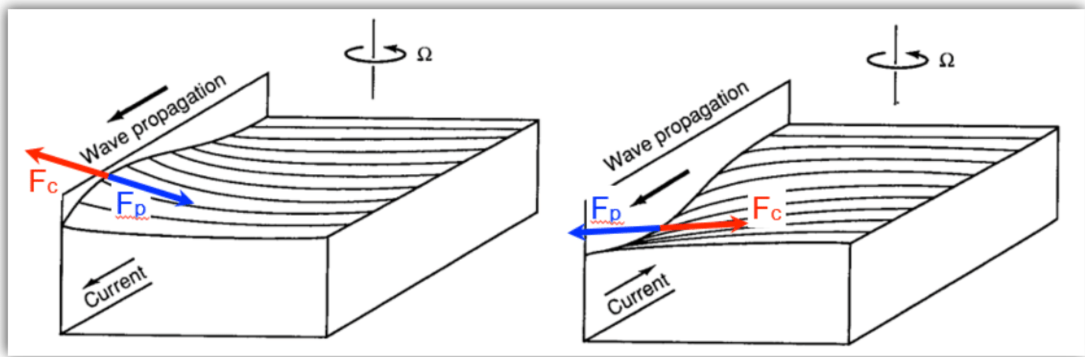
$$\boxed{\begin{aligned}\frac{\partial v}{\partial t} &= -g\frac{\partial \eta}{\partial y}\\ \frac{\partial \eta}{\partial t} &= -H\frac{\partial v}{\partial y}\end{aligned}}$$

Prognostic equations
Non-dispersive waves

↳ They can be combined (see **#WAVES3.1b**) to obtain the following differential equation:

$$\frac{\partial^2 \eta}{\partial t^2} = gH\frac{\partial^2 \eta}{\partial x^2}$$

↳ Thus, in the y-direction, we have **non-dispersive gravity waves** propagating northwards or southwards with a fixed phase speed, independent of horizontal scales ($|c| = \sqrt{gH}$).



▪ When the fluid is piled up against the wall, the pressure force will push out into the fluid. The pressure gradient force and the Coriolis force will balance and we will have a southward flow. (In the northern hemisphere, the Coriolis force is always directed to the right of the current).

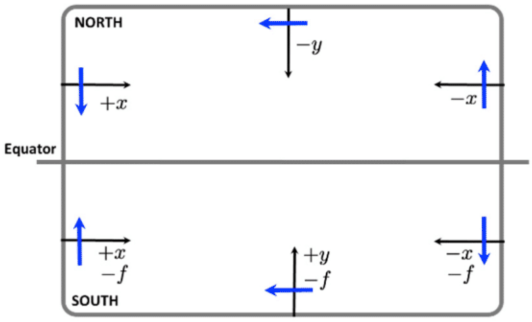
▪ If there is a dip against the wall pressure, the gradient pressure force will push towards the wall and the Coriolis force will balance this, so the flow will be northwards.

↳ We will have **oscillations** between **northward** and **southward** flow alternating with the crests and troughs of the wave, and the whole thing will propagate like a gravity wave along the wall. These waves are ***coastal Kelvin waves***.

⇨ In the northern hemisphere, with a wall on the western side of the domain, this wave propagates southwards. The current oscillates back and forth along the wall, but **the wave propagates with the coast to its right**.

📖 *To prove this, we need to consider the two solutions of the wave equation the propagate in opposite directions ($A(x)\, e^{i(y+ct)}$ and $A(x)\, e^{i(y-ct)}$). One of these solutions is incompatible with the geostrophic balance equation. This is beyond the scope of this lecture (see M2-SOAC GFD lectures).*

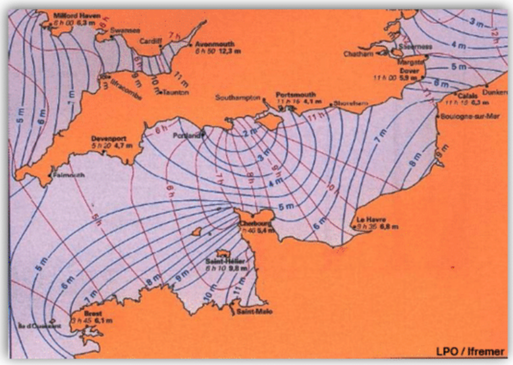
3.3.b) Properties of boundary Kelvin waves



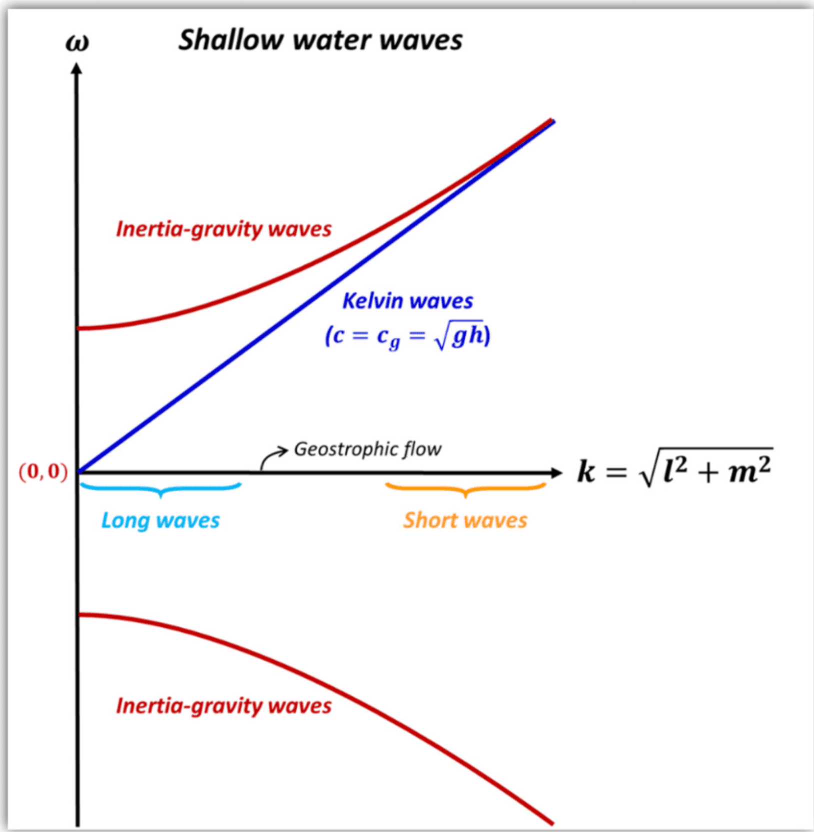
⇨ **In the northern hemisphere, Kelvin waves lean against the coast, with the coast to the right as they propagate.**

⇨ In the **southern hemisphere**, f changes sign, so all of these considerations are reversed, and Kelvin waves propagate with the coast to the left.

⇨ Here is an example of Kelvin waves. This is the English Channel which separates England from the rest of Europe. When tides come in from the Atlantic Ocean, they propagate through this channel as Kelvin waves. They lean against the coast to their right, which is the French coast. The vertical amplitude of the tide as it comes through the channel is greater on the French side than on the English side: up to 11 meters near St Malo, and only about 2 or 3 meters near Southampton. Normandy and Brittany have higher tides than the Southwest England. This explains why a tidal power station has been built on the French side.



↳ **Kelvin waves** are a special case of inertia gravity waves, which can propagate at large scales even when there is a rotation in the system.



WAVES3.4: Rossby Waves

Another important class of planetary waves in the shallow water system are **Rossby waves**. These waves are important in both the ocean and the atmosphere. Their restoring force is not gravity. It is a horizontal displacement arising from the variation of the Coriolis force with latitude.

So far, we have considered shallow-water waves on an f−plane (in **#WAVES3.2** and **#WAVES3.3**), where rotation is assumed to be constant. However, the vertical component of the earth's rotation varies with latitude ($f = 2\Omega sin(\phi)$) and we will see that this has important consequences for large-scale wave dynamics.

3.4.a) Barotropic vorticity equation

⇨ Let's include the variation of the rotation of the earth in the momentum equation and form the **barotropic vorticity equation** (as in **#WAVES3.2b**).

↳ We cross differentiate the **non-linear single-layer** horizontal shallow-water momentum equations ($\partial(2)/dx - \partial(1)/dy$, see **#WAVES3.1a**), but we include the variation of the Coriolis parameter f with latitude (y):

$$\frac{\partial}{\partial x}\{v_t + uv_x + vv_y + fu = -g\eta_y\} - \frac{\partial}{\partial y}\{u_t + uu_x + vu_y - fv = -g\eta_x\}$$

⇢ The pressure gradient terms cancel out by differentiation, so:

$$v_{tx} + u_x v_x + uv_{xx} + v_x v_y + vv_{yx} + fu_x - u_{ty} - u_y u_x - uu_{xy} - v_y u_y - vu_{yy} + fv_y + \boxed{v\frac{\partial f}{\partial y}} = 0$$

⇢ This equation can be rearranged:

$$\left(v_{tx} - u_{ty} + u(v_x - u_y)_x + v(v_x - u_y)_y\right) + f(u_x + v_y) + u_x(v_x - u_y) + v_y(v_x - u_y) + \boxed{v\frac{\partial f}{\partial y}} = 0$$

⇢ By introducing the relative vorticity of the flow, $\xi = v_x - u_y$, it can be written as:

$$\left(\xi_t + u\xi_x + v\xi_y\right) + f(u_x + v_y) + \xi(u_x + v_y) + \boxed{v\frac{\partial f}{\partial y}} = 0$$

⇢ Using the definition divergence of the flow $\nabla.\mathbf{v} = u_x + v_y$, and introducing the material tendency notation: $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$, it follows that:

$$\frac{D\xi}{Dt} + (f + \xi)\nabla.\mathbf{v} + \boxed{v\frac{\partial f}{\partial y}} = 0$$

⇢ Since f varies only with latitude (it does not vary with time or x), it can be included in the substantial derivative.

Divergence : $\nabla.\mathbf{v} = u_x + v_y$
Relative vorticity : $\xi = v_x - u_y$
Planetary vorticity: $f = 2\Omega sin(\phi)$
Absolute vorticity: $\xi_a = f + \xi$

Barotropic vorticity equation

⇨ The <u>**barotropic vorticity equation**</u> can be written:

$$\frac{D}{Dt}(f + \xi) = -(f + \xi)\nabla.\mathbf{v}$$

↳ The substantial derivative of the **absolute vorticity**, the sum of the relative vorticity, and the planetary vorticity ($\xi_a = f + \xi$) equals the absolute vorticity times the divergence.

⇢ **The divergence** can be thought of as a **source** of absolute vorticity.

↪ For <u>**non-divergent barotropic flow**</u>, the absolute vorticity is conserved with motion:

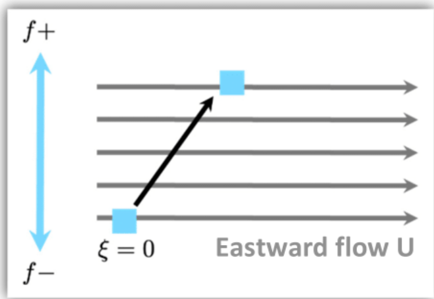
$$\frac{D}{Dt}(f + \xi) = 0$$

3.4.b) Parcel displacement in a planetary vorticity gradient

⇨ Let's consider a parcel of fluid in an eastward flow U:

▪ In a **non-divergent barotropic framework**, the ***absolute vorticity is conserved*** (see **#WAVES3.4a**) following the parcel: $\xi_a = f + \xi = cst$.

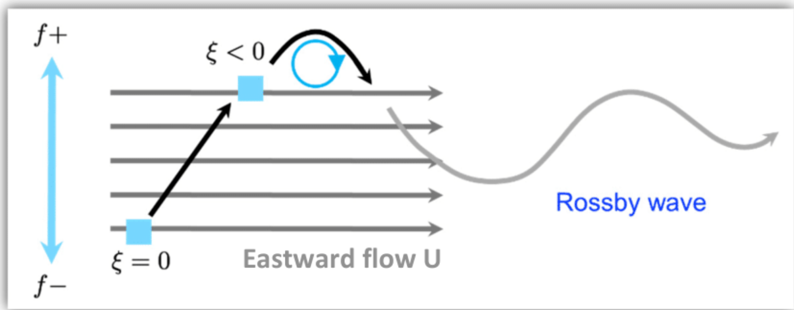
▪ On a planet with some **curvature**, i.e. the planetary vorticity (Coriolis parameter) f varies with latitude (larger f in the north - smaller f in the south).



⇨ Let's assume that at the origin, this parcel of fluid has no relative vorticity ($\xi = 0$). Imagine that for some reason, there is a perturbation that **displaces the parcel** (a little bit) to the north, where the planetary vorticity f is greater ($f \nearrow$).

⇨ In agreement with the conservation of absolute vorticity ($\xi_a = f + \xi = cst$), the relative vorticity of the flow must compensate for this increase in f ($\xi \searrow$) and ξ must become negative ($\xi < 0$). Negative relative vorticity is associated with a **clockwise flow curvature**. It introduces a clockwise secondary circulation into the fluid.

↳ So, the flow must curve back down towards the south and the parcel will return to its latitude of origin. This is a *stable situation*, i.e. the solution oscillates in such a way that the **force that restores** it to its position of origin is somehow proportional to the distance from the origin position.



⇨ You can imagine it **overshooting** and going back down south, in which case it will come back north with positive relative vorticity.

⇨ There is a kind of restoring force such that if the parcel goes north of a certain reference latitude, it will be pulled back towards the south, and if the parcel goes south of its equilibrium latitude, it will be pulled back north. This conservation of absolute vorticity triggers an oscillation of the flow in latitude. It will create a **wave**, a **Rossby wave**. A wave for which the **restoring force** is not just the Coriolis force, but the variation of the Coriolis force with the latitude.

✋ We need **variable *f*** for this to happen, so Rossby waves **cannot work on an *f*-plane**.

3.4.c) Barotropic Rossby wave dispersion relation

⇨ We will describe this **non-divergent barotropic wave motion** in mid-latitudes where we can make the β approximation for the variation of the Coriolis parameter with latitude.

↳ Around a local reference latitude, where $f = f_0$, the variation in f can be approximated by a linear term: $f = f_0 + \beta y$, with β being a linearization of df/dy ($\beta = 2\Omega cos(\phi_0)/R_{Earth}$). As f increases from south to north, β is always positive ($\beta > 0$) and it is maximum at the equator.

⇨ We express the **conservation of absolute vorticity**: $\boxed{\frac{D}{Dt}(f + \xi) = 0 \Leftrightarrow \frac{D}{Dt}(\xi + \beta y) = 0}$

▪ The constant background westerly flow is U
▪ The wave perturbations are u, v

⇨ We develop the substantial derivative:

$$\frac{D}{Dt}(\xi + \beta y) = \frac{\partial}{\partial t}(\xi + \cancel{\beta y}) + (U + u)\frac{\partial}{\partial x}(\xi + \cancel{\beta y}) + v\frac{\partial}{\partial y}(\xi + \beta y) = 0$$

↪ Since y does not vary locally in time or x, the equation simplifies.

⇨ We **linearize the equation** and consider that the wave perturbations are small compared to the mean flow. We cross-out all the quadratic terms:

$$\frac{\partial \xi}{\partial t} + (U + \cancel{u})\frac{\partial \xi}{\partial x} + v\cancel{\frac{\partial \xi}{\partial y}} + \beta v = 0$$

⇨ Using the definition of the stream function ψ, $u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}, \xi = \nabla^2 \psi$, Non-divergent flow

We can write the linear perturbation equation in ψ:

$$\boxed{\frac{\partial}{\partial t}\nabla^2 \psi + U\frac{\partial}{\partial x}\nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} = 0}$$

⇨ We are going to look for **wave-like solutions (plane-wave solutions)**:

$$\boxed{\psi = Re\, \tilde{\psi}\, e^{i(lx+my-\omega t)}}$$

▪ They have the form of an amplitude coefficient times an imaginary exponential:

- l is the **zonal wavenumber** (2π divided by the x-wavelength),
- m is the **meridional wavenumber** (2π divided by the y-wavelength)
- ω is the **angular frequency** (2π divided by the period).

▪ Taking a derivative of this trigonometric function yields the same function multiplied by some algebraic coefficients:

$$\frac{\partial}{\partial t} \rightarrow -i\omega \times \qquad \frac{\partial}{\partial x} \rightarrow il \times \qquad \nabla^2 \rightarrow -k^2 \times = -(l^2 + m^2) \times$$

↳ Substituting the solution and its derivatives into the linear vorticity equation gives:

$$-i\omega\left(-(l^2 + m^2)\right) + il\left(-(l^2 + m^2)\right)U + \beta i l = 0$$

⇨ We obtain a relationship between ω, l and m (with two other geophysical parameters U and β). This is the **dispersion relation for barotropic Rossby waves**:

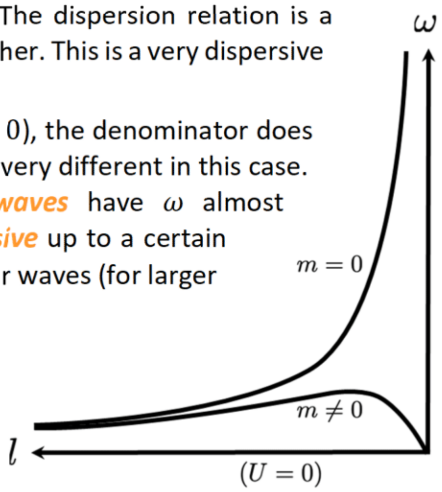
$$\boxed{\omega = Ul - \frac{\beta l}{l^2 + m^2}}$$

• The phase speed $c = \frac{\omega}{l}$ is equal to $U - \beta/k^2$. $k^2 = l^2 + m^2$ and β are always positive. Thus, relative to the background flow U, the **Rossby waves always propagate westwards**.

• With $m = 0$ (i.e. the waves have an infinite meridional extent, i.e. meridionally they cover the entire planet), ω is proportional to $-\beta/l$. Relative to the base state flow, this term is negative, so we plot it in the negative quadrant of the dispersion diagram. The dispersion relation is a hyperbola, so as the zonal scales get larger, the frequencies get higher. This is a very dispersive large-scale wave. It is called a **Rossby-Haurwitz wave**.

• Once you set a meridional scale to your structure ($m \neq 0$), the denominator does not disappear. When $l = 0$ then $\omega = 0$. The dispersion relation is very different in this case. For the meridionally-confined structures, the ***long Rossby waves*** have ω almost proportional to l, which means that they are almost ***non-dispersive*** up to a certain point. The maximum ω is found for $l = m$, and then for the shorter waves (for larger l), they become very **dispersive**.

Non-dispersive waves: all the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape



3.4.d) Rossby wave properties

• The **phase speed** is: $\boxed{c = \frac{\omega}{l} = U - \frac{\beta}{l^2 + m^2}}$

↪ With $k^2 = l^2 + m^2$ and β always positive, the **Rossby waves always propagate westwards**, relative to the background flow U.

↪ The longer the wavelength (smaller k^2) the faster the wave.

• The **group speed** on the other hand has a direction that depends on the sign of its numerator:

$$\boxed{c_g = \frac{\partial \omega}{\partial l} = U + \frac{\beta(l^2 - m^2)}{(l^2 + m^2)^2}}$$

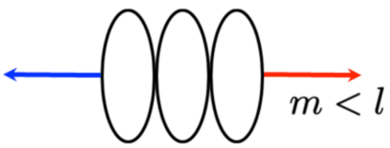
$$\frac{\partial \omega}{\partial l} = U - \beta \boxed{\frac{\partial}{\partial l}(l(l^2+m^2)^{-1})} = \frac{1}{(l^2+m^2)} + l(-(l^2+m^2)^{-1} \times 2l)$$

$$= \frac{(l^2+m^2) - 2l^2}{(l^2+m^2)^2} = -\frac{l^2 - m^2}{(l^2+m^2)^2}$$

↪ Relative to the background current, the direction of propagation of the wave's energy **depends on the shape of the wave**:

➢ If $l = m$, the group speed is zero.

➢ If $l > m$, i.e. waves with a larger meridional scale than their zonal scale, then the ratio term in the group speed formula is always positive. The phase speed of these waves will be to the west, while their group speed will be to the east.



➢ If $l < m$, the waves are elongated in the zonal direction and the ratio term is negative. The phase speed and the group speed are both to the west. These waves are more non-dispersive and are easier to observe because they will not lose their shape as they propagate westwards.

⇨ From the dispersion relation, it follows that:

⇢ **Rossby waves are dispersive, the longer the wavelength (smaller k^2) the faster the wave** (similar to deep-water waves, see **#WAVES2.3e**).

⇢ **Rossby waves closer to the equator are also faster** (β is maximum at the equator and zero at the poles).

⇢ **The group velocity depends on the ratio of zonal to meridional scales** (for larger meridional scales/smaller zonal scales, the group velocity is eastwards).

3.4.e) Rossby wave propagation mechanism

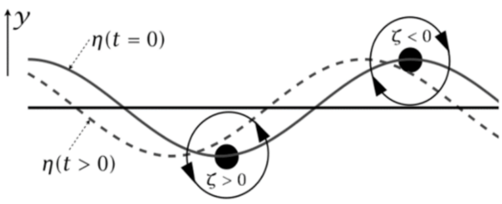
⇨ *Why do Rossby waves propagate to the west?*

• **Remember the parcel** which had been displaced from its position of origin. To the north, it has acquired negative relative vorticity resulting in a clockwise circulation. To the south, positive relative vorticity has been induced, i.e. an anti-clockwise circulation.

• Now imagine a **streamline of absolute vorticity** following the parcel. It has been moved to the north or to the south, portraying a wave.

⇨ **How would the steamline be displaced by this secondary circulation?**

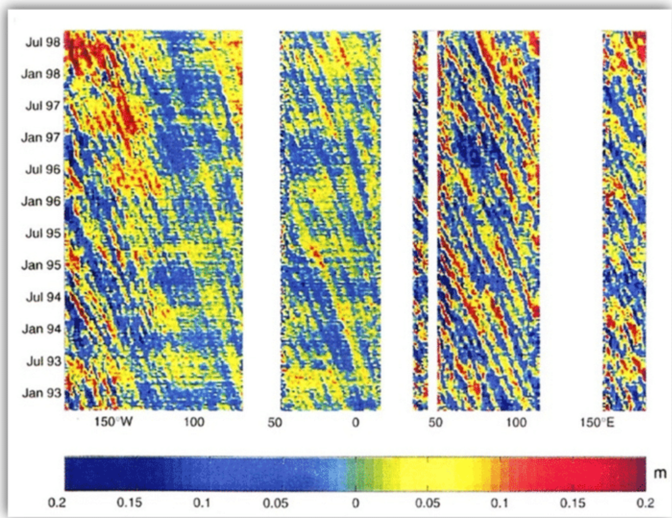
It will be pushed away from the origin in the west, and towards it in the east. Thus, at a later stage, the streamline will follow the dashed curve, effectively shifting it to the left on the diagram. The Rossby wave thus propagates westwards.



↳ ***The secondary circulation induced by the constraint to conserve the vorticity produces westward propagation.***

3.4.f) Observations

⇨ *Can we see Rossby waves in the observations?*



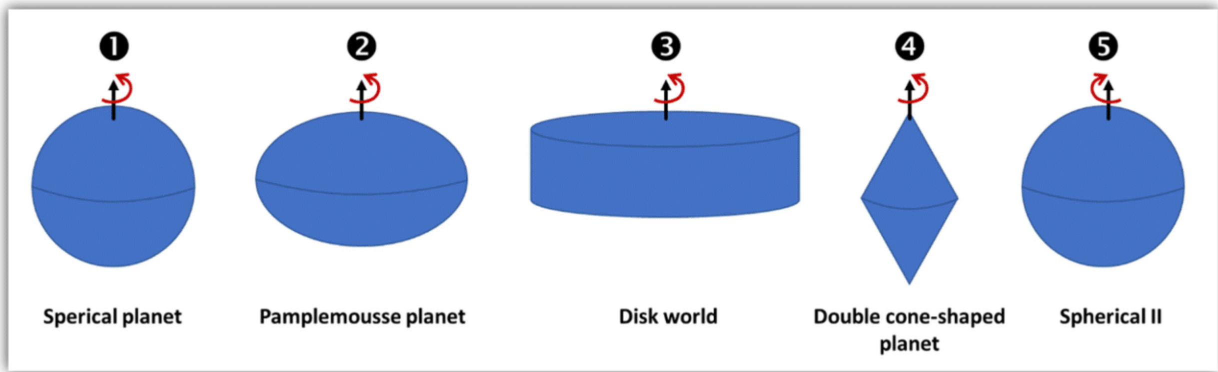
Here is a (rather old) global longitude-time plot (hovmöller diagram) of Sea Level anomalies (perturbations) at 25°S using Topex/Poseidon altimetry data from 1993 to 1998. The longitude in degrees covers the 3 tropical oceans: Pacific, Atlantic, and Indian Oceans. The white bands are the land with no observations (America, Africa, Madagascar, and Australia).

The **diagonal stripes** are the signature of westward propagation. It takes about **five years to cross the Pacific basin**.

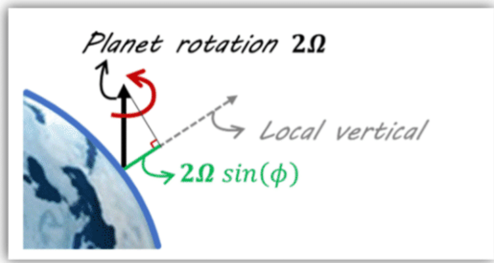
↳ So, there is some evidence for Rossby wave-like behaviour in satellite altimetry. Note that it is more difficult to detect Rossby waves in sea surface temperature anomaly data.

3.4.g) The Rossby wave game

⇨ *Can Rossby wave exist on the following planets?*



↳ Remember that f is the **local vertical projection** of the planet's rotation vector, as illustrated below:



☑ *Planet ❶ is our planet, we live on this planet. f varies with latitude. Of course, there are Rossby waves!*
☑ *Planet ❷ is an exaggerated version of our planet. f varies with latitude, Rossby waves exist.*
☒ *Planet ❸, it is like living on a record player. There is no variation in f, it is like an f-plane. Rossby waves cannot develop on this planet. This is the same as a tube planet ($f = 0$ on the side and $f = 2\Omega$ on the north/south faces).* ☒ *On planet ❹, the projection of the rotation onto the local vertical remains constant on each hemisphere. It is like an f-plane. Rossby waves cannot exist.* ☑ *Planet ❺ is our planet seen upside-down, like from Australia, South America, or Antarctica. f varies with latitude, Rossby waves exist. The planet is just rotating in the opposite direction.*