

General Properties of Waves



CHAPTER 1

General Properties of Waves

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This chapter describes the general properties of the **ocean surface waves**. We will discuss how they are **forced** by the wind (see #WAVE1.1a) and see the **prevailing timescales** or frequencies at which ocean waves are observed (see #WAVE1.1b). We will then describe the fluctuations of the sea surface and they can be represented **mathematically** (see #WAVE1.2). We will detail various properties of waves, such as the wavenumber, the frequency, and the speed. We will then describe the **interferences** and **modulation** that occur when two waves with similar frequencies are summed-up (see #WAVE1.3). This will lead us to define the **group speed**. Finally, we will discuss the phenomenon of **dispersion** and introduce the **dispersion relation** (see #WAVE1.4).

WAVES1.1: General Properties of Waves

1.1.a) Wind-forced waves

⇒ Let's start with a question: *Where do surface ocean waves come from?*

👉 Waves are most commonly caused by **wind**. Wind-driven waves, or surface waves, are created by friction between the wind and surface water. The two fluids in contact at the ocean surface have different speeds, resulting in **frictional stress** at the interface (proportional to the square of the speed difference). At the sea surface, most of the energy transferred results in waves, and a small amount generates wind-driven currents.

⇒ Here is a **hypothesis** to explain how ocean waves might grow:

❶ Let's just imagine the surface of the ocean. It is almost flat with a few undulations to it and a wind is blowing across it.

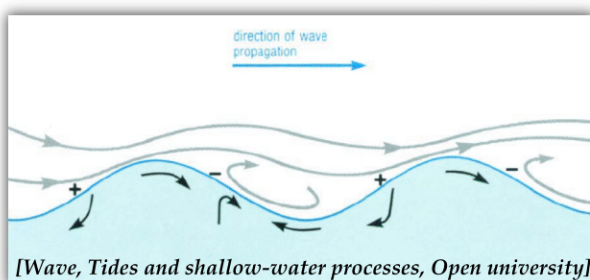
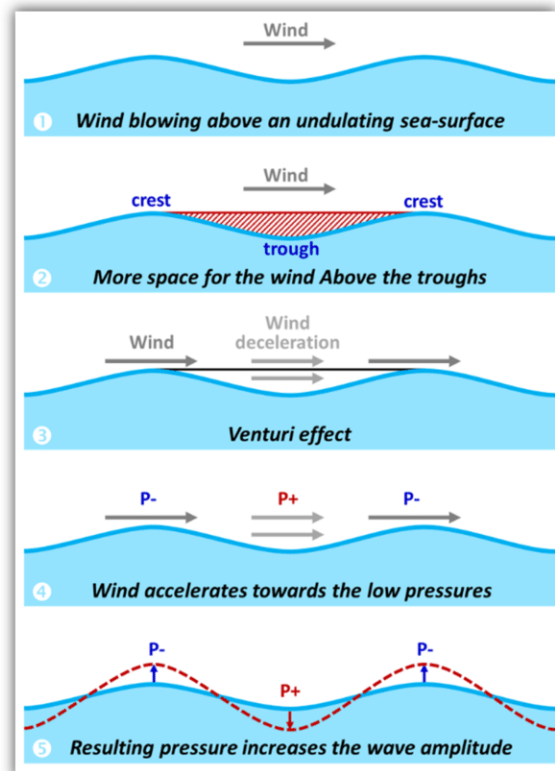
❷ Near the sea-surface, there is a little bit more space above the trough than above the crest (red striped area).

❸ So, the wind will take up more space above the trough and due to conservation of mass, it will slow down. This is the **Venturi effect**.

❹ **Bernoulli's theorem** (see #AppendixA) states that changes in the wind speed are associated with a **pressure gradient force**. The wind accelerates from high (**P+**) to low (**P-**) pressures and it decelerates toward higher pressures (**P+**).

❺ As a result, relative to the average pressure, there is slightly less pressure (**P-**) above the crests and slightly more (**P+**) above the troughs. The ocean surface is thus **pushed up** at the crests and **pushed down** in the troughs, increasing the wave amplitude.

👉 The wind blowing over a slight undulation makes the undulation get slightly bigger.



⇒ This is an oversimplification because the wave not only grows in place, but actually **moves** (propagates) **in the same direction as the wind** as it grows.

👉 The diagram on the side is a little bit more realistic. There is a slight phase difference between the vertical acceleration and the position of the crests and troughs, which is consistent with the propagation of the wave.

⇒ This is the **sheltering model of wave generation** (by H. Jeffreys, 1925). The back of the wave, which is exposed to the wind, experiences a higher pressure than the front face, which is sheltered from the force of the wind. The presence of waves thus modifies the airflow (with eddies forming in front of each wave), creating a pressure difference that pushes the wave in the direction of the wind, along with its amplification.

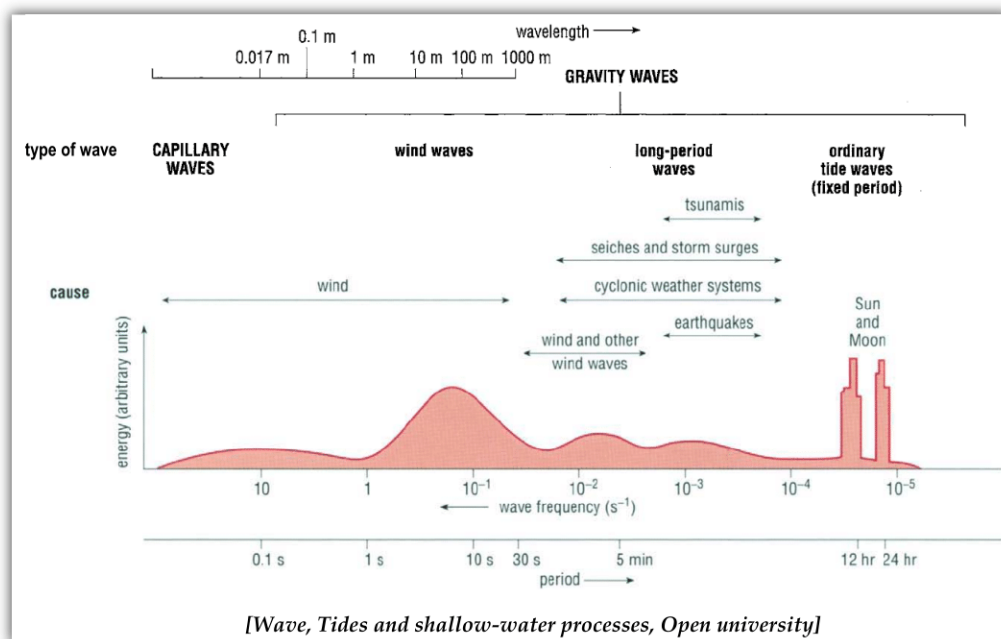
⇒ Although this model does not explain the initial undulation, it successfully applies to situations where the wind speed exceeds 1 m/s and is faster than the wave speed. The initial wave must be steep enough for the effect to work.

📖 More potentially hazardous waves can be caused by severe weather, such as a hurricane. The strong winds and pressure from this type of severe storm can cause a **storm surge**, a series of long waves that originate far from shore in deeper water and intensify as they move closer to land. Other dangerous waves can be caused by **underwater disturbances** that displace large amounts of water quickly such as earthquakes, landslides, or volcanic eruptions. These very long waves are known as **tsunamis**. Storm surges and tsunamis are not the kind of waves you think as of crashing down on the shore. These waves roll upon the shore like a massive sea-level rise and can reach far inland.

1.1.b) Observed spectrum

⇒ The diagram below is the **observed spectrum** of the ocean surface waves. It represents different types of waves, showing the relationship between wavelength (see #WAVE1.2a), wave frequency and period (see #WAVE1.2b), the nature of the forces that caused them, and the relative amount of energy in each type of wave.

⇒ It gives an idea of **how much energy is associated with different wave periods** (horizontal axis). Short waves with periods of less than one second are on the left side of the distribution, and longer waves with periods of 12 and 24 hours are represented on the right.



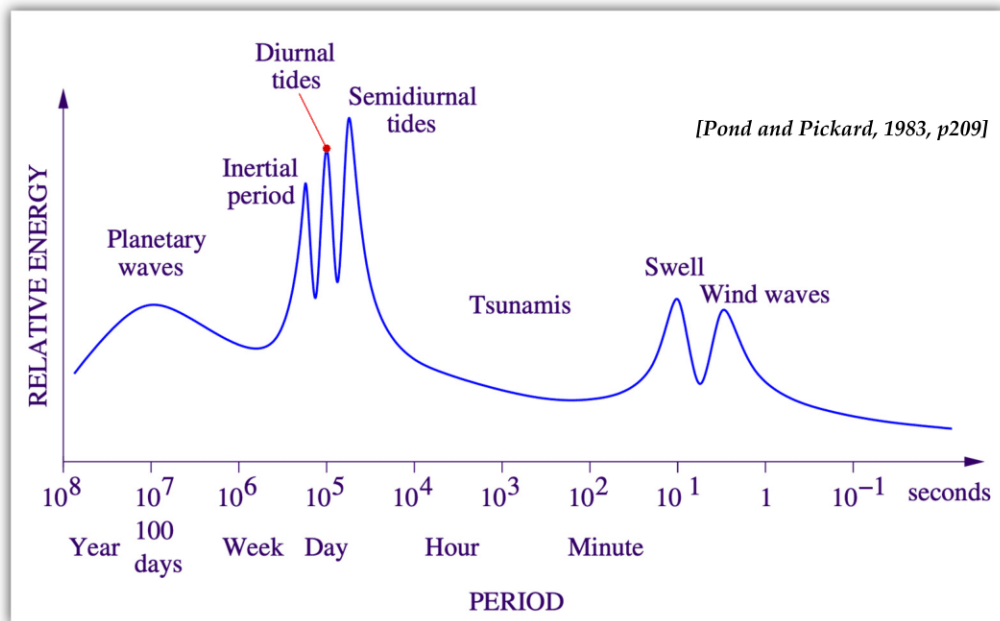
⇒ The observed spectrum of ocean surface waves is not evenly distributed (uniform). There are some periods that are more energetic than others.

- The largest peak is for **wind-driven waves** with periods between 1 and 30 seconds.
- There is a relatively small amount of energy for the slower periods, i.e. **swells, weather systems**.

- Notably, not much energy is associated with **tsunamis**. Not because they are not energetic. It is because there are not that many of them (fortunately), so they do not contribute much to the distribution.

- There are also two sharp peaks for the long periods. These are the **tidal frequencies** associated with the sun and the moon. These waves and their forcing will be discussed in #Chapter5.

⇒ Below is another way of looking at the energy of the ocean surface waves. The diagram below is a global wave spectrum, which shows the period in which the wave energy resides as a function of the wave frequency (f). But it is actually the periods of the waves (T) that are written on the horizontal axis. Because $T = \frac{1}{f}$, the horizontal axis has a non-linear scale, with powers of 10 increasing to the left.

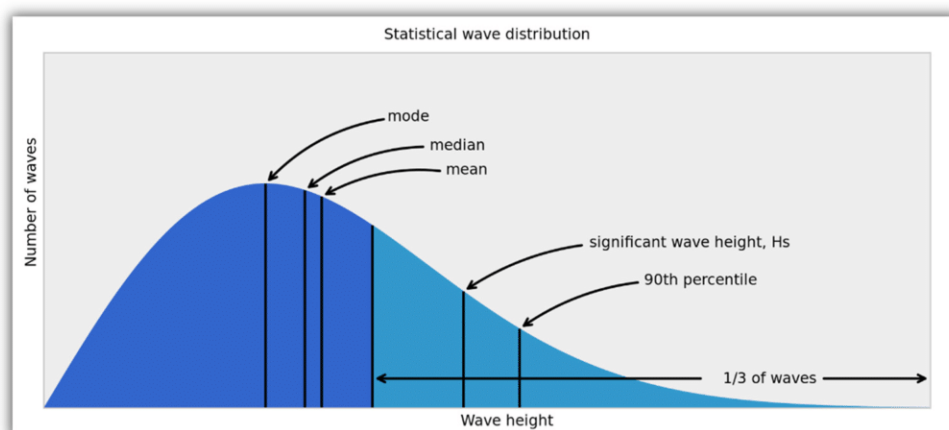


This diagram is also quite schematic. It illustrates that (from right to left):

- **Wind waves** and **swells** account for most of the energy for periods of a few seconds to a minute.
- For longer periods, we find **tsunamis**. There is a slight dip in this spectrum because although tsunamis are very energetic, there are not very many of them.
- Then, there are 3 peaks: **semidiurnal tides**, **diurnal tides**, and the **inertial period**.
 - There are actually two frequencies for semi-diurnal tides due to the sun and the moon.
 - The sun and the moon are also responsible for diurnal tides (two more frequencies).
 - The inertial period is related to the rotation of the earth. If you let a parcel of water with an given initial velocity just go freely on Earth, the effect of the rotation will act on the parcel as if there were a force acting on it – the (fictitious) Coriolis force, and the parcel will just go round in an inertial circle (see **#WAVES3.2d**). The period associated with the inertial motion is a function of latitude, and this period is captured in this spectrum.

1.1.c) Wave height distribution

⇒ Let's now discuss the **amplitude** of the waves. The schematic below shows a **statistical distribution of the wave height** at one location.



⇒ In this diagram, the horizontal axis is the amplitude of the wave, i.e. the height between the trough and the peak of the wave, from small waves (on the left) to larger waves (on the right). The vertical axis counts how many waves are of a given height, i.e. the distribution of wave heights. It is a continuous function, similar to a probability density function. The area under this curve is equal to 1, i.e. all the waves.

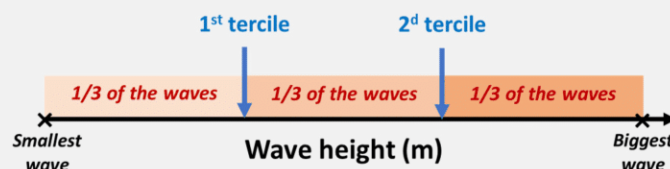
✎ The wave height distribution function is **not a symmetric distribution**:

- There are many waves that are not very big and then there is a **very long tail** on the right side of the distribution with a very small number of very large waves.
- This is a unimodal distribution that is **skewed to the right**. Most of the waves are smaller than the mean, and the mode (the peak of the distribution) is to the left of the mean.

⇒ If you are interested in the **possible damage that might be caused by waves** (for example, if you are building a structure such as an oil rig or a harbour), you will not be interested in the average wave height. This mean quantity is biased by the many small waves that are harmless to any marine structure. You need to quantify the **average height of the really big waves** that make up the right tail of the distribution.

✎ To determine the typical size of a big wave, we estimate the **average of the third tercile group of the wave height distribution**.

The **terciles** are the two cutoff point values that divide the population into three groups of equal size (each containing 1/3 of the number of the waves). The **third tercile group**, with a wave height greater than the second tercile, represents the 1/3 of the waves with the largest amplitudes.



✎ In the wave height distribution (on the previous page), the third tercile group is highlighted in a lighter colour, compared to the rest of the distribution, which is shown in dark blue. The **significant wave height**, labelled H_S (or $H_{1/3}$), is defined as the **average of the third tercile group**. This value informs you of the **typical size of an extreme wave event**, which is more useful than the average wave height when designing a marine structure.

WAVES1.2: Wave Kinematics

⇒ In this section, we will describe these waves. To simplify the theory of surface waves, we will assume that **the waveform is sinusoidal**, and thus can be represented by **trigonometric functions** (sinusoidal functions). We begin by examining the dimensions of an idealized wave in space (see #WAVE1.2a) and time (see #WAVE1.2b) and reviewing the terminology used to describe it.

1.2.a) Basic properties in space

⇒ On the next page is a **sinusoidal function** that schematizes the **propagation of an ocean wave in space**.

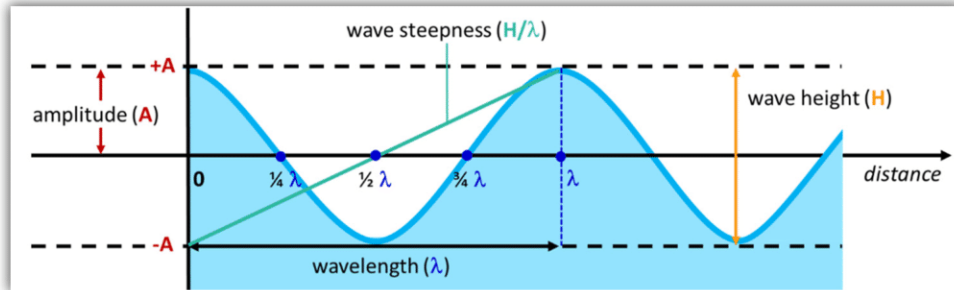
✎ It shows how the displacement of the water level varies over distance at a fixed point in time (~ a snapshot of the wave).



⇒ Below, this wave propagates to the right (1D in space) as x increases.

Here is some **vocabulary** to describe the characteristics of the wave in space:

- The **amplitude** of the wave (A) is the difference in height between the midpoint of the wave and its crest (or peak). The unit is meters.
- The **wave height** (H) is the total overall vertical change in height between the trough and crest of the wave. **The wave height is equal to twice the wave amplitude** ($H = 2 \times A$).

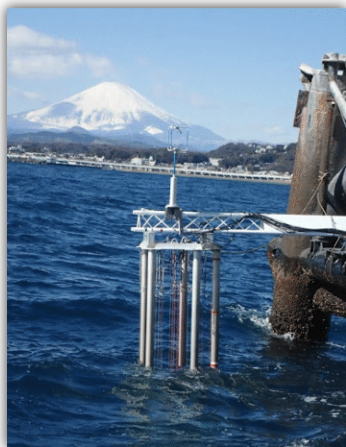


- The **wavelength** (λ) is the **horizontal distance between one crest and the next crest**, or between two successive troughs. More generally, λ is the distance between any phase of the wave and the same phase again.

⇒ In the above schematic, the water displacement (η) is zero at $x = \frac{1}{4} \lambda$ and at $x = \frac{3}{4} \lambda$.

- For sailors, the absolute height of a wave is less important than its **steepness**, defined as the wave height divided by the wavelength (H/λ). In the open ocean, most wind-generated waves have a steepness of about 0.03 to 0.06. Waves with greater steepness can be a problem for shipping.

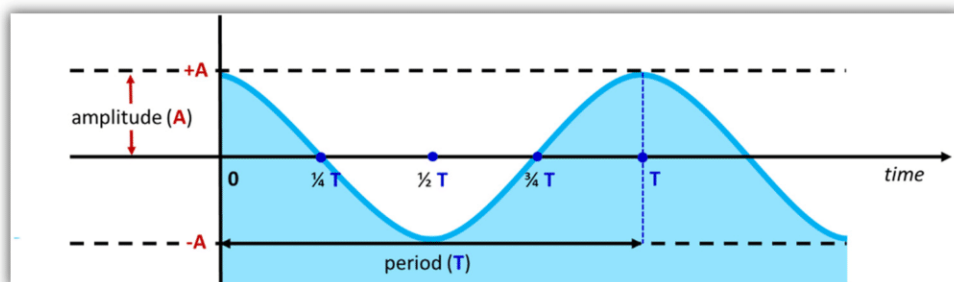
1.2.b) Basic properties in time



⇒ Let's now examine the **wave properties in time**. Below is another sinusoid that illustrates the displacement of the water level with time at a fixed point (as measured by a wave gauge for example). Time increases on the right.

- The **wave amplitude** (A) and the **wave height** (H) are defined as in #WAVE1.2a. The displacement of the water level (η) at a fixed point varies between $-A$ (at the trough) and $+A$ (at the peak).
- The time interval between two successive peaks (or two successive troughs) passing at a fixed point is called the **period** (T) and is usually measured in seconds.

⇒ In the diagram below, the water displacement is zero at $t = \frac{1}{4} T$ and $t = \frac{3}{4} T$.



📖 Note that the steeper the wave, the further the **sea-level fluctuations depart from a simple sine curve**. Note also that the surface displacement has a trochoidal shape (see #WAVES2.1e). The sinusoidal form of wave motion is sufficient for our purposes because it provides solutions to linear equations. It will be used throughout the course.

1.2.c) Temporal frequency and wavenumber

- Let's recall the relationship between **period (T)** and **frequency**.

↪ The number of peaks (or the number of troughs) that pass through a fixed point in a time interval of one second is called the frequency of a wave. It is denoted by f , and $f = 1/T$. Its unit is the inverse second or per second and is denoted by s^{-1} . It corresponds to the unit **Hertz**.

↪ If we use a sinusoidal model to represent the oscillation of the wave in time, we can make an analogy between the period of a wave and a cycle that has an angle (in radians) of 2π . We can thus define the **angular frequency** (pulsatance) ω as the rate of change of the phase of a sinusoidal waveform. One revolution is equal to 2π , hence $\omega = 2\pi/T = 2\pi f$, and the water displacements represented in the \sin curve in #WAVE1.2b could be written: $\eta(t) = A \cos(\omega t)$. ω is in rad/s .

- The **wavelength (λ)** is the distance between two successive crests in the direction of the propagation.

↪ The **wavenumber** is the **spatial frequency** of a wave (⚠ not to be confused with the number of waves). It is measured in radians per unit of distance. Just as angular frequency is related to the number of waves per unit time, **wavenumber** depends on the number of waves per unit distance. It is defined as $k = 2\pi/\lambda$, and the water displacements represented in the sinusoidal curve in #WAVE1.2a could be written as: $\eta(x) = A \cos(kx)$. k is in rad/m .

1.2.d) Wave kinematics (1D)

↪ The sinusoidal model describes the water level oscillating up and down in space (see #WAVE1.2a) and at the same time, it is also oscillating up and down in time (see #WAVE1.2b). In this simple example, it propagates to the right (1D in space). So, we need to combine these two descriptions (the sinusoid in space (see #WAVE1.2a) and the sinusoid in time (see #WAVE1.2b)) into one description of a propagating wave.

But these two descriptions are not independent...

If **you walk along the blackboard with a piece of chalk** that you move regularly up and down (at a given period T_1), you will draw a sinusoid with a certain **wavelength**.

↪ If you do this again but this time you move the piece of chalk faster, so that the period T_2 is smaller than the period T_1 . You walk along the blackboard at the same speed. You will draw a second sinusoid with a shorter **wavelength**.

↪ If you do it again a third time, moving the piece of chalk at the period T_1 , but this time you walk faster along the blackboard. You will again draw a sinusoid with a longer **wavelength** than the first sinusoid.

↪ **There is a relationship between speed, period/frequency, and wavelength/wavenumber.**

↪ Let's have a look at the **mathematical way** to represent a wave. Here is a first basic formula that represents a wave function. η represents the water displacement at any given time (t) and space (x). η is modelled by a trigonometric function:

$$\eta(x, t) = A \cos(kx - \omega t + \phi)$$

- A is the **amplitude** of the wave.
- It is multiplied by a **cosine function** that represents the oscillation of the water displacements. Inside the cosine function:

↪ We find the **spatial coordinate** x and the **time** t .

↪ There are **some coefficients** in front of the spatial and temporal coordinates. They make the arguments inside the cosine function non-dimensional and expressed in radians. k will be in rad/m , while ω will have a dimension of a frequency in rad/s .

$\left\{ \begin{array}{l} k \text{ is the } \mathbf{horizontal\ wavenumber} \text{ (} 2\pi \text{ divided by the wavelength } \lambda, k = 2\pi/\lambda), \\ \omega \text{ is the } \mathbf{angular\ frequency} \text{ (} 2\pi \text{ divided by the period } T, \omega = 2\pi/T). \end{array} \right.$

↪ There is also a **phase** ϕ that is added. It accounts for the fact that the water displacement does not necessarily start from zero. It can be anything from 0 to 2π .

👉 How many parameters do we need to describe this one-dimensional propagating wave?

- 1) the amplitude A ,
- 2) the horizontal wavenumber in the x -direction ($k = 2\pi/\lambda$),
- 3) the angular frequency ($\omega = 2\pi/T$),
- 4) the phase ϕ .

👉 We need **4 parameters** to describe this one-dimensional propagating wave.

⇒ The **phase speed** of the wave is $c = \lambda/T = \omega/k$

⇒ There are **many ways to write a waveform**, some of them are more useful than others. The cosine function presented earlier may be easier to understand.

WRITING n°2: We can write any trigonometric expression in terms of a complex exponential, such that:

$$\eta = \text{Re } \tilde{A} e^{i(kx - \omega t)} = \text{Re } \tilde{A} e^{ik(x - ct)}$$

- The amplitude becomes a complex number, and is written as \tilde{A} .
- The arguments for the cosine are now the arguments of a complex exponential.
- kx and ωt are the same as before.

👉 In this notation, we have lost the phase ϕ , because the amplitude is complex and contains the phase. So we still have **4 parameters** to describe this one-dimensional wave, because we have to define the real and the imaginary parts of \tilde{A} .

WRITING n°3: The water level fluctuations can also be written as:

$$\eta = \text{Re } \tilde{A} e^{i\theta}$$

👉 It is written as the real part of a complex amplitude (\tilde{A}) times the complex exponential $e^{i\theta}$, where θ is the phase of the wave ($\theta = kx - \omega t = k(x - ct)$).

WRITING n°4: If you do not like the complex amplitude, you can use a real amplitude (A), but in this case, you have to put back the phase back, so that:

$$\eta = \text{Re } A e^{i\theta} e^{i\phi}$$

WRITING n°5: If you do not like the complex exponential and you do not like to talk about the phase, you can write the waveform like this:

$$\eta = A_1 \cos(kx - \omega t) + A_2 \sin(kx - \omega t)$$

👉 The amplitude is real, but it has two components A_1 and A_2 that are associated with cosine and sine functions, respectively. This is equivalent because the complex amplitude can actually be written as: $\tilde{A} = A_1 - iA_2 = A e^{i\phi}$.

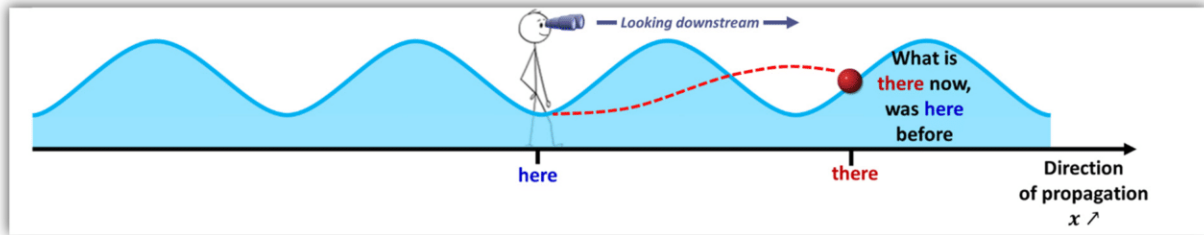
⇒ The **derivatives of the waveform** become coefficients:

$$\frac{\partial \eta(x, t)}{\partial t} \rightarrow -i\omega \eta(x, t) \qquad \frac{\partial \eta(x, t)}{\partial x} \rightarrow ik \eta(x, t)$$

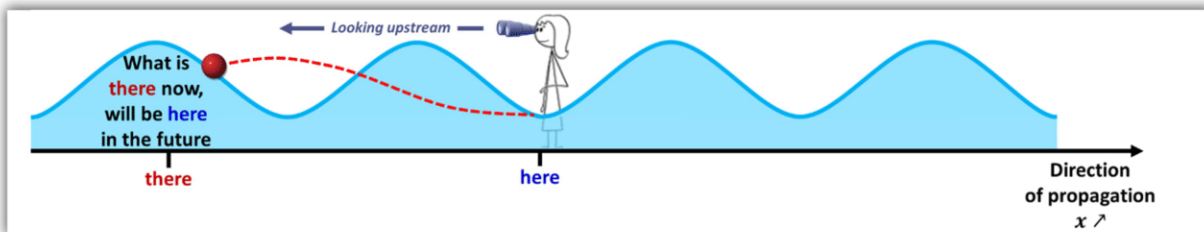
👉 Why is there a minus sign in front of ω ?

⇒ In our example, we have chosen a wave that propagates to the right as x increases. The cosine function develops in this direction.

▪ As illustrated below, if you look **downstream** ($x > 0$), you are looking into the **past**: What is **there** ($x > 0$) now, was here before ($t < 0$).



▪ As illustrated below, if you look in the negative x -direction (**upstream**), you are looking at oscillations that will arrive at your position in the **future**: What is over **there** now ($x < 0$), will be here in the future ($t > 0$).



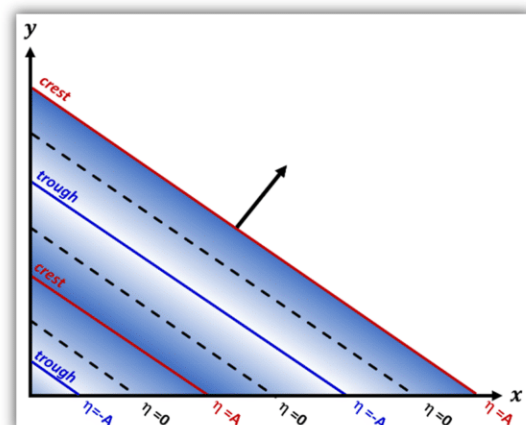
⇒ To describe the propagation, we need to express the structure of the wave in terms of space and time as angles in the arguments of the trigonometric function. Thus, since **we have chosen that the wave propagates in the direction of x increasing**, we must reconcile this sign difference in the space/time coordinates associated with the direction of the wave propagation: the fluctuations downstream ($x > 0$) were at $x = 0$ before ($t < 0$) or the fluctuations upstream ($x < 0$) will arrive at $x = 0$ in the future ($t > 0$).

👉 To do this, we need to introduce **a minus sign** so that the space and time terms have opposite signs. We can put the minus sign either in front of the terms related to space or time. **By convention**, we put the minus sign in front of ω .

1.2.e) Plane-wave propagation (2D)

⇒ Let's have a look at the **mathematical formulation** of a wave propagating in a given direction in a **plane**. η represents the water displacements at any given time (t) and spatial position in **two dimensions** (x and y). As in **#WAVES1.2d**, $\eta(x, y, t)$ is modelled by a trigonometric function:

$$\eta(x, y, t) = A \cos(lx + my - \omega t + \phi)$$



👉 We need **five parameters** to describe this two-dimensional propagating wave.

- 1) The amplitude **A** ,
- The 2 horizontal wavenumbers:
 - 2) the wavenumber in the x -direction ($l = 2\pi/\lambda_x$)
 - 3) the wavenumber in the y -direction ($m = 2\pi/\lambda_y$),
- 4) The angular frequency ω ,
- 5) The phase ϕ .

⇒ As in #WAVES1.2d, there are other ways to write a plane-wave:

WRITING n°2: We can use a complex exponential: $\eta = \text{Re } \tilde{A} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$

→ The amplitude is a complex number that represents for the phase of the wave.

→ The spatial terms in the arguments of the complex are in bold because they are vectors:

$\mathbf{k} = \begin{pmatrix} l \\ m \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and the vector product resolves to $\mathbf{k} \cdot \mathbf{x} = lx + my$.

→ ωt is the same as before.

WRITING n°3/4: With $(\theta = lx + my - \omega t)$, the two-dimensional propagating sea-level fluctuations can also be written as: $\eta = \text{Re } \tilde{A} e^{i\theta}$ or $\eta = \text{Re } A e^{i\theta} e^{i\phi}$

WRITING n°5: Without the complex exponential and phase, the waveform can be written with two components for the amplitude: $\eta = A_1 \cos(lx + my - \omega t) + A_2 \sin(lx + my - \omega t)$

⇒ For a two-dimensional (plane) wave, the wavenumber \mathbf{k} is a vector and its components (l, m) are the projections of \mathbf{k} onto the spatial coordinate system (x, y) , such that $k^2 = l^2 + m^2$.

▪ The wave phase being $\theta = lx + my - \omega t$, it is easy to derive that:

→ $\frac{\partial \theta}{\partial x} = l$ and $\frac{\partial \theta}{\partial y} = m$, so that the spatial gradient of θ is equal to the wavenumber:

$$\nabla \theta = (l, m) = \mathbf{k}$$

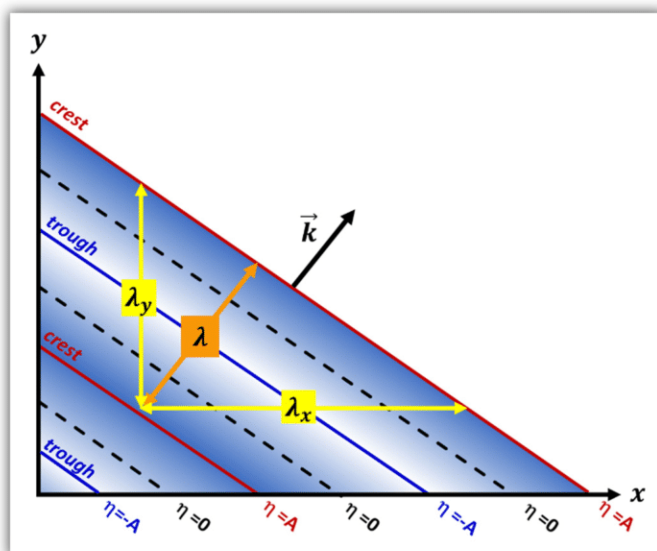
→ The temporal gradient of θ is equal in magnitude to the angular frequency: $\frac{\partial \theta}{\partial t} = -\omega$

▪ The **derivatives of the waveform** become coefficients:

$$\frac{\partial}{\partial t} \rightarrow -i\omega, \quad \frac{\partial}{\partial x} \rightarrow il, \quad \frac{\partial}{\partial y} \rightarrow im, \quad \text{and } \nabla^2 \rightarrow -(l^2 + m^2) = -k^2$$

👉 The direction of propagation of this plane wave is given by the vector wavenumber \mathbf{k} .

⇒ If we consider the wavelength λ of the plane-wave, it is the distance between two successive crests (or troughs) **in the direction of the propagation**. It is equal to $\lambda = 2\pi/k$.



→ We can measure the wavelength in the x (λ_x) and y (λ_y) directions (see the figure on the side). Curiously, you will see that the wavelength in the x (or y) direction is longer than the actual wavelength.

→ It is tempting to think of the wavelengths as components of a vector, but it does not work that way!

The zonal and meridional wavelengths are not components of a vector, unlike the zonal (l) and meridional (m) wavenumbers.

→ λ_x and λ_y along x - and y -directions are projected onto the direction of propagation, and not the other way around!

→ Given that $k^2 = l^2 + m^2$, with $k = 2\pi/\lambda$, $l = 2\pi/\lambda_x$, and $m = 2\pi/\lambda_y$, we can show that:

$$1/\lambda^2 = 1/\lambda_x^2 + 1/\lambda_y^2$$

→ This leads to: $\lambda = \sqrt{\frac{\lambda_x^2 \lambda_y^2}{\lambda_x^2 + \lambda_y^2}}$

1.2.f) Period, wavelength, and speed

⇒ Let's review some basic **terminology** used to describe the propagation of a plane-wave:

- The **wavelength in the x -direction** λ_x is equal to 2π divided by the x -component of the wavenumber (l): $\lambda_x = 2\pi/l$.

- The **wavelength in the y -direction** λ_y is equal to 2π divided by the y -component of the wavenumber (m): $\lambda_y = 2\pi/m$.

✚ **Wavelength is NOT A VECTOR!**

- The **real wavelength in the propagation direction** λ is 2π divided by the total wavenumber (k). The **wavenumber is a vector** that is resolved by its two components (l, m), so that:

$$\lambda = 2\pi/k \quad \text{with} \quad k^2 = l^2 + m^2$$

- The **period** of the wave T is 2π divided by the angular frequency: $T = 2\pi/\omega$

- The **phase speed** of the wave is $c = \omega/k$. The phase-speed in the x and y directions are:

$$c_x = \omega/l \quad \text{and} \quad c_y = \omega/m$$

✚ Similar to the wavelength, the **phase speed is NOT A VECTOR!**

✚ If someone gives you the phase speed in the x -direction (c_x) and the phase speed in the y -direction (c_y), you cannot resolve them to get the phase speed of the wave (c , in the direction of propagation). You would need to know the component of the wavenumber vector (l and m), to find the real wavelength and calculate the phase-speed.

✚ Similar to the wavelength, the phase speed of the wave in the x or y directions (c_x or c_y) are greater than the actual phase-speed of the wave, unlike the components of a vector, which are always less than the resultant.

📖 Imagine a lighthouse firing a beam at a wall. The beam is spinning, so the speed at which the projected light travels along the wall can be very large compared to the speed at which the light is moving towards the wall. But nothing is actually traveling along the wall.

WAVES1.3: Group Speed

⇒ So far, we have restricted our analysis to a wave with a single frequency ω – a monochromatic wave. But as we saw previously (see #WAVES1.1b), there is a whole range of frequencies in ocean waves. In this section, we will look at the relationship between different frequencies in a wave phenomenon. It triggers some **interference**, a **modulation** of the signal, and some **dispersion**.

1.3.a) Interference and modulation

⇒ Let's start with two frequencies.

✚ Imagine two waves propagating in the same direction, with the same amplitude, but at slightly different frequencies. In the illustration, the blue wave has a slightly higher frequency (shorter period) than the red wave.

- There is a position in time when the two crests of the two wave trains coincide. At this point, both signals are **in phase** and they **reinforce** one another.

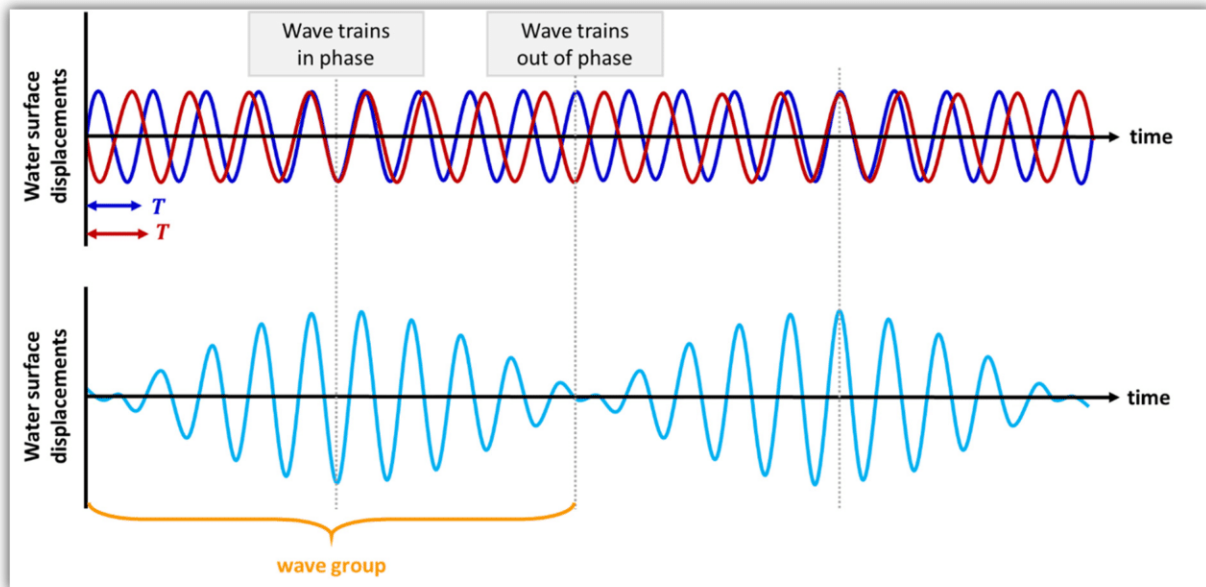
- Moving to the right, since the red wave is oscillating a little faster than the blue wave, the waves gradually shift phases, to a point in time where the crests of one wave coincide with the troughs of the other. Here, the waves are **out of phase** and they **cancel** each other out perfectly.

- Then, they come back into phase, and later on they reinforce one another... etc.

⇒ Look at the **resulting amplitude**:

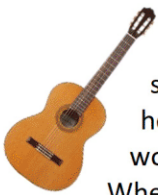
- When the waves are in phase, their amplitudes are added and the resulting wave has about twice the amplitude of the two original waves.

- When the two waves are out of phase, their amplitudes cancel each other out, and the surface water has minimal displacement.



⇒ The resulting signal oscillates at nearly the same frequency as the original signals, but there is a **modulation of the amplitude**: a sequence of packets of large amplitude, small amplitude, large amplitude, ... etc, forming **wave groups**.

✚ The two component wave trains thus interact, each losing its individual identity, and combine to form a series of **wave groups**, separated by regions almost free from waves. This is how **AM radio** works, where AM stands for Amplitude Modulation.



Tuning a guitar: A guitar has 6 strings. To tune it, you work them by pairs. If you pluck one string with your finger at the fifth fret, it should have the same frequency as the next string. So you can pluck them together and they should sound the same. If you hear a horrible woo-woo-woo sound, you need to turn the tuning pegs, until you hear a perfect woo sound without any modulation.

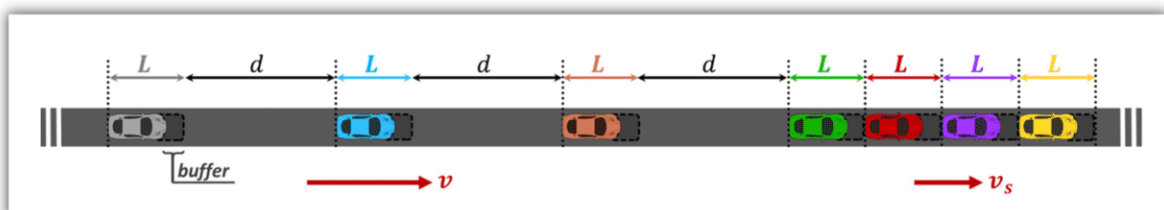
When two strings are not perfectly tuned, you hear the **beat frequency**. This is the **frequency of the modulation packet**. When both strings vibrate at the same frequency, the modulation disappears. The beat frequency is the difference between the two frequencies.

1.3.b) Propagation of a wave group

⇒ In the illustration above, we had two waves that are very similar to one another (very similar wavelengths, very similar frequencies). They both propagate in the x -direction. *What about the wave group packet? How does it propagate? The same way? Does it propagate at the same speed?*

✚ **Not necessarily.** The wave group packet could propagate at a different speed. It could even be traveling in the opposite direction.

⇒ Sometimes, we observe this in traffic. An abrupt slowdown in concentrated traffic (compression wave) can travel as a pulse (a shock wave) along the line of cars. It can travel either downstream (in the direction of traffic) or upstream, or it can be stationary. It depends on the speed of the cars and the distance between the cars.



⇒ Below is an example of a wave packet propagating downstream, i.e. in the same direction as the individual crests and troughs. We can mark an individual crest with a cross and follow it as it propagates. At the same time, the wave packet is propagating and we can mark the center of the packet (gray rectangle) with a circle.

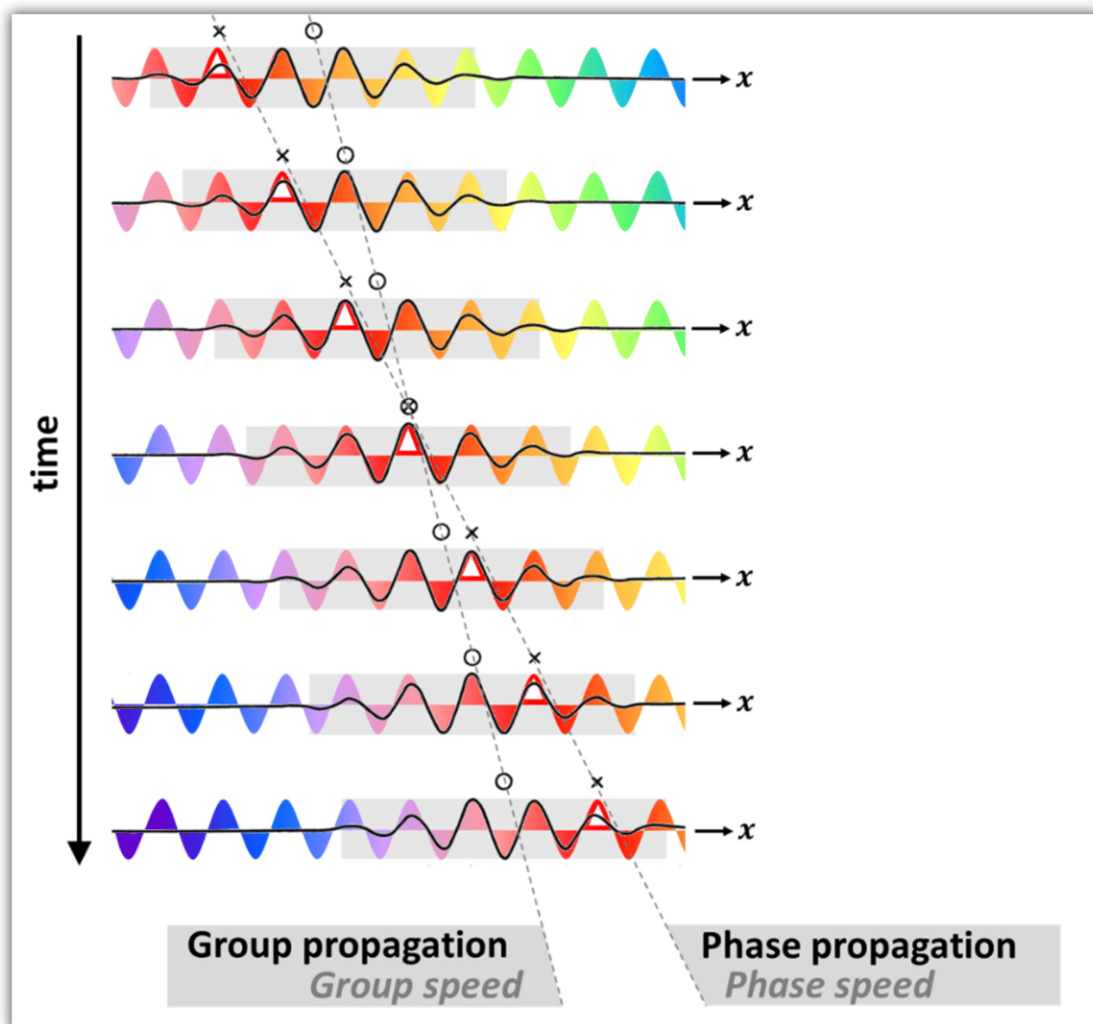
⇒ The **wave and the wave group do not propagate at the same speed**. The individual crests are traveling faster than the packet.

- The propagation of the individual crests is the **phase propagation** and the speed of the individual crests is the **phase speed c** .
- The propagation of the packets is **the group propagation** and the speed of the group wave is the group speed c_g

⇒ This is how surface gravity waves behave in **deep-water** (see #WAVES2.3e): we will show that for deep-water waves, the group speed is half the phase speed.

⇒ Imagine a packet moving along and individual crests are propagating through the packet amplifying and then going out the front and fading away.

What's interesting is that the **energy** associated with that perturbation **propagates at the group speed** and the information propagates at the group speed.



1.3.c) Group speed formulation

⇒ Let's formalize the expression of the group speed.

✎ Here is the mathematical description of two waves that are very similar:

- They have the same amplitude and the same phase (\tilde{A}).
 - They have very similar wavenumbers $k_1 = k + \Delta k$ and $k_2 = k - \Delta k$.
 - They have very similar frequencies $\omega_1 = \omega + \Delta\omega$ and $\omega_2 = \omega - \Delta\omega$.
- ✎ where Δk and $\Delta\omega$ are very small.

⇒ We write down the wave function for the **sum of these two waves** (as in #WAVES1.2d):

$$\eta = \text{Re } \tilde{A} [e^{i(k_1x - \omega_1t)} + e^{i(k_2x - \omega_2t)}]$$

✎ We substitute the definition of k_1 , k_2 , ω_1 , and ω_2 using k , Δk , ω , and $\Delta\omega$. It follows:

$$\eta = \text{Re } \tilde{A} e^{i(kx - \omega t)} [e^{i(\Delta kx - \Delta\omega t)} + e^{-i(\Delta kx - \Delta\omega t)}]$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

• It can be written as the original wave ($\text{Re } \tilde{A} e^{i(kx - \omega t)}$) times the sum of two complex exponentials with Δk and $\Delta\omega$. This term is actually two times the cosine of $(\Delta kx - \Delta\omega t)$. It gives:

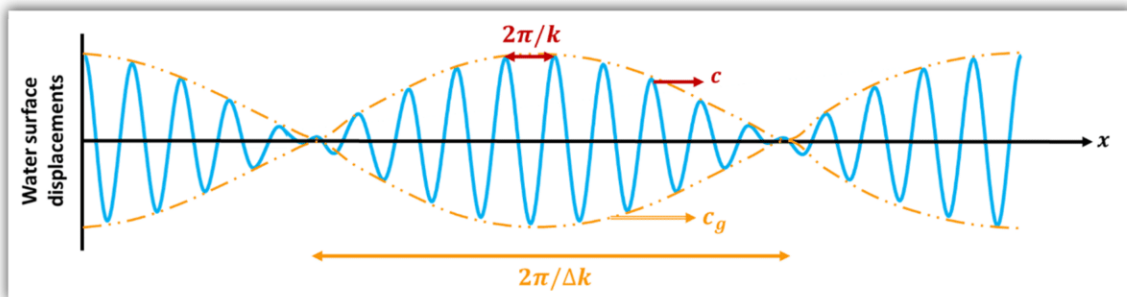
$$\eta = \text{Re } \tilde{A} e^{i(kx - \omega t)} \times 2 \cos(\Delta kx - \Delta\omega t)$$

Rapidly varying signal *Slowly varying envelope*
Original signal *Modulation*

• This is the original wave multiplied by 2 times the cosine of another wave.

✎ The parameters of this envelope wave are, instead of k and ω , Δk and $\Delta\omega$:

- Δk is small: the effective **wavenumber** of this modulating function is **small**, which means the effective **wavelength** of this modulating function is **long**.
- $\Delta\omega$ is small: the effective **frequency** of this modulating function is **small**, which means it has a **long period**.



⇒ The reconstructed signal we obtained is the original wave modulated by this envelope term, which has a very long wavelength and a very long period. This is the modulating packet. That is the modulating packet is also propagating because this term ($2 \cos(\Delta kx - \Delta\omega t)$) is also a waveform describing a longer wave with a lower frequency.

⇒ We can work out the effective speed of the modulating envelope. The speed at which a packet will move. This is the ratio of the angular frequency to the wavelength (see #WAVES1.2d or #WAVES1.2f) of the modulating packet:

$$c_g = \Delta\omega / \Delta k$$

✎ If we take the limit where Δ becomes very small, we can actually express this as a function:

$$c_g = \frac{\partial\omega}{\partial k}$$

This is the **group speed**, and this is a **general property of waves**.

WAVES1.4: Dispersion relation

⇒ The **phase speed**, the speed at which an individual wave will propagate ($c = \omega/k$), and the **group speed**, the speed at which information/energy propagates ($c_g = \partial\omega/\partial k$), are both a **function of the wavenumber (k) and the frequency (ω)**.

⇒ The relationship between frequency and wavenumber ($\omega = f(k)$) is called the **dispersion relationship**.

1.4.a) Dispersion

⇒ ω/k and $\partial\omega/\partial k$ are not necessarily equal for all values of k .

⇒ If the phase speed, ω/k , or the group speed, $\partial\omega/\partial k$, varies with the spatial scale of the wave (the wavelength λ or the wavenumber $k = 2\pi/\lambda$), then there will be some complex interesting behaviours.

If the phase speed depends on the spatial scale, it means that some wavelengths might go faster than others. This is **dispersion**.

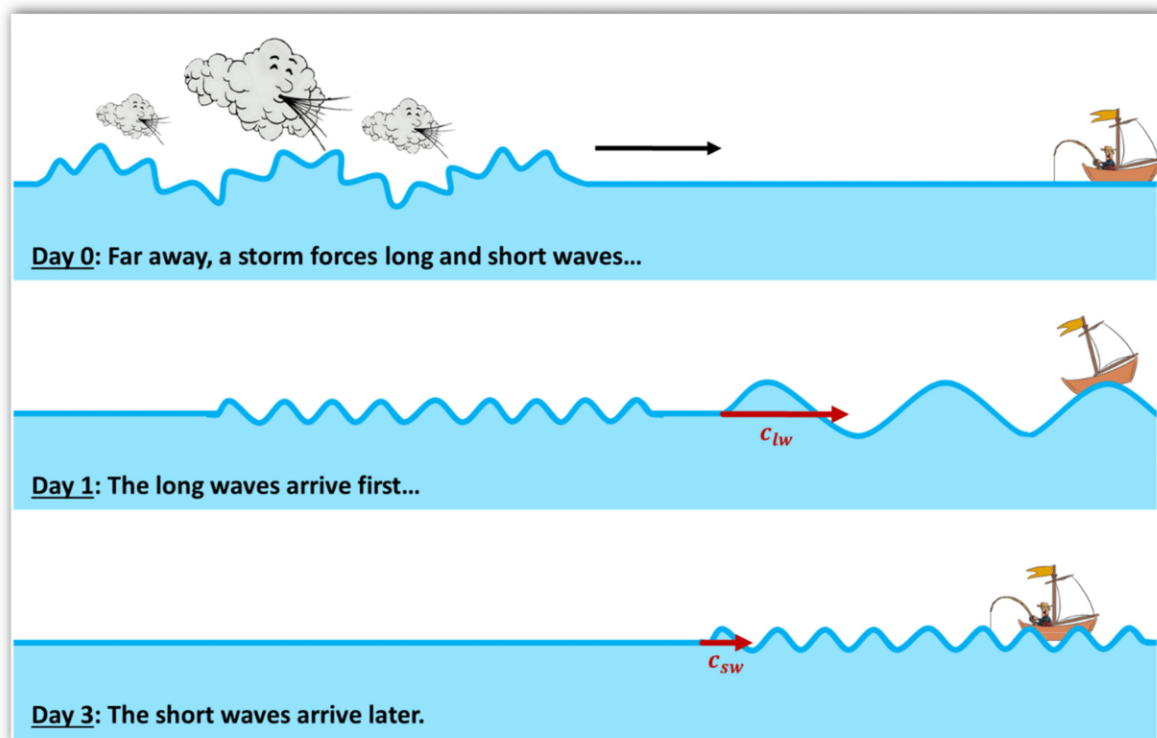
CASE1: We might have a dispersive system in which **short waves go slow, and long waves go fast**.

⇒ In this case, *what would happen to a wave that was a mixture of short and long waves?* The long waves would all go pretty fast and the short waves would go slower.

⇒ This is illustrated below. Imagine, you are on a boat, out at sea. At some distance, there is a **storm that generates waves**. These waves are a mixture of long and short waves propagating in your direction:

- The first thing that arrives is a **long swell** associated with the long waves that arrive first. It is not a very comfortable feeling and you may feel sick.
- Over the next few days, you see the sea-surface all choppy and disturbed because the **short waves have finally arrived**, the long waves are long gone.

⇒ The wavelengths get **dispersed**, they get **separated**. It is **dispersion**.



⇒ This is what happens for **deep-water waves** (see #WAVES2.3e).

CASE2: In contrast, we might find a **dispersive** system where **short waves go faster than longer waves**.

CASE3: Finally, we might have a system in which the short waves and the long waves propagate at the same speed. This means that the phase-speed (c) is a constant, independent of the wavelength or the frequency. All wavelengths and all frequencies have the same speed.

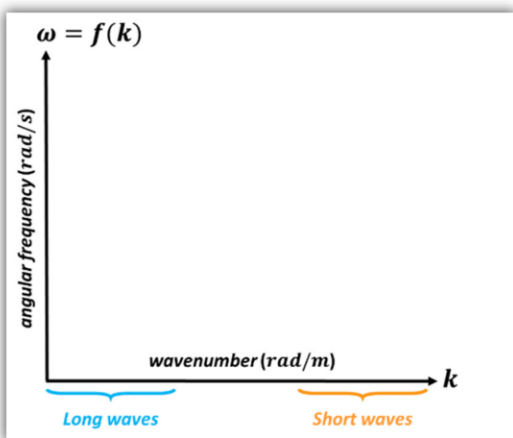
→ In this particular case, the relationship between ω and k is linear. We have:

$$\frac{\partial \omega}{\partial k} = \frac{\omega}{k}, \quad c_g = c$$

→ This system is called **non-dispersive**. The dispersion relation is linear ($\omega = ck$) and passes through the origin ($\omega = 0$ when $k = 0$).

1.4.b) Dispersion diagram

→ We represent the relationship between frequency and wavenumber ($\omega = f(k)$) on a diagram. This diagram is called a **dispersion diagram**.



- The horizontal axis is the wavenumber ($k = 2\pi/\lambda$), ranging from small wavenumbers (long waves) to large wavenumbers (short waves).
- The vertical axis is the frequency ($\omega = 2\pi/T$)

→ If we know the physical system, we know the relationship between ω and k . We can plot it as a curve on the dispersion diagram by assigning a value of ω to each value of k . We can then calculate the phase speed and the group speed for any wavelength.

→ On this graph:

→ **the phase speed** (c) is the arrow that points from the origin toward the curve (the ratio $\frac{\omega}{k}$)

→ **the group speed** (c_g) is the tangent to the curve ($\frac{\partial \omega}{\partial k}$)

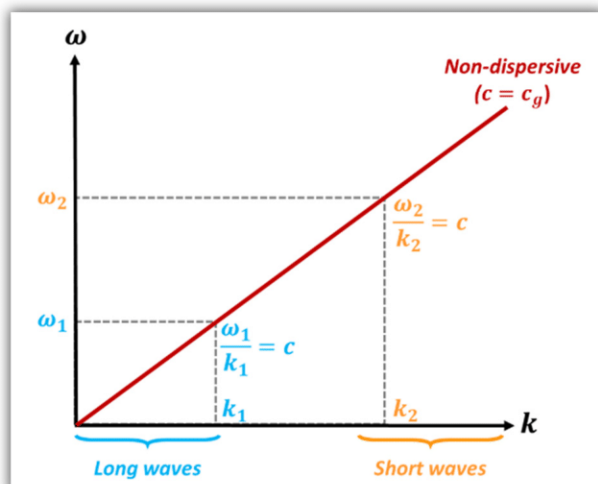
→ Below are dispersion diagrams with the 3 dispersion curves illustrating the 3 cases mentioned previously (in #WAVES1.4a).

• **Non-dispersive waves:** This is the case of boundary Kelvin waves (see #WAVES3.3).

→ In this case, the relationship between ω and k is linear and its representation on the dispersion diagram is a **straight line**.

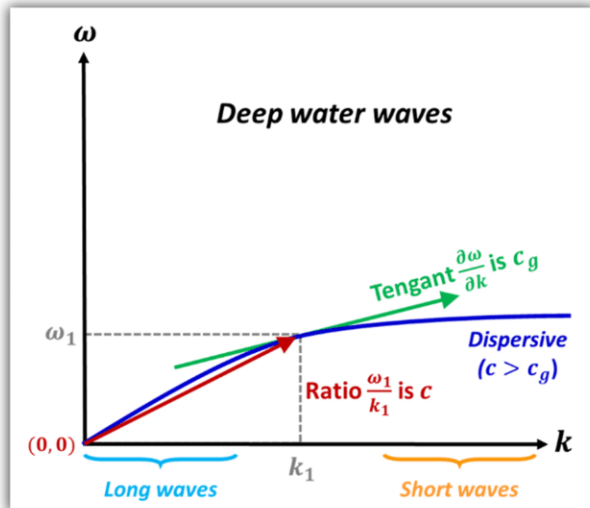
→ For any value of k , the slope of the curve, i.e. the **ratio ω/k remains constant**. It is equal to the phase speed of the waves, which is constant (for all k). For **long waves** (small k), ω is small, and for **shorter waves** (larger value of k), the frequency is proportionally larger.

→ The slope of the curve is also equal to the group speed $\partial \omega / \partial k$ ($c_g = c$).



For non-dispersive waves, all the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change its shape during its propagation

• **Dispersive waves, for which long waves go faster than short ones:** This is the case for **deep-water waves** (see #WAVES2.3e).



→ For any value of k , we look at the ratio ω/k to estimate the **phase speed**. It is the slope from the origin of the diagram (0,0) to any point on the curve (in dark red in the dispersion diagram on the side). And, the phase speed changes as the waves get shorter.

→ For deep-water waves, the larger the wavenumber, the smaller the ratio ω/k : long waves propagate faster than short waves.

→ To estimate the **group speed** ($c_g = \frac{\partial \omega}{\partial k}$), we draw the tangent to the dispersion curve (in green in the dispersion diagram on the side).

→ For any value of k (long waves and short waves), the group speed of deep-water waves is less than the phase speed. As a result, the modulation packet propagates slower than the individual wave crests, and the individual crests propagate through the packet, amplifying and then going out in the front and fading away (see #WAVES1.3b).

→ The group speed is also a function of the wavelength. For deep-water waves, the group speed associated with long waves is larger than for short waves.

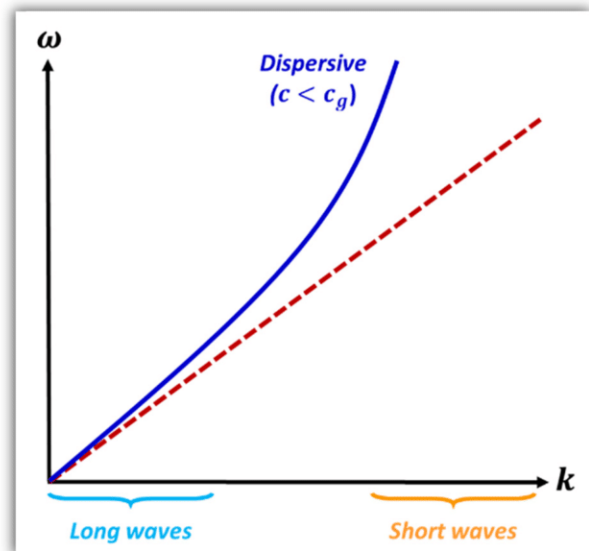
⇒ This is **normal dispersion**.

→ In #WAVES2.3e, we will see that for deep-water waves, the group speed is half of the phase speed.

• **Dispersive waves, for which short waves travel faster than long ones:** In such a physical system, the group speed is larger than the phase speed. So, the group packets propagate faster than the individual waves.

→ It is a rather unusual behaviour. This is the case of capillary waves, the tiny ripples on the surface of the water, for which the restoring forces is the surface tension.

These dynamics are beyond our scope of interest for this course.



➤➤ Throughout the course, we will derive **dispersion relations** for different types of ocean waves (for deep and shallow-water surface gravity waves in #Chapter2, for some geophysical waves in #Chapter3, and for internal waves in #Chapter4).