Geophysical Fluid Dynamics

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M2 SOAC - DC

Course outline

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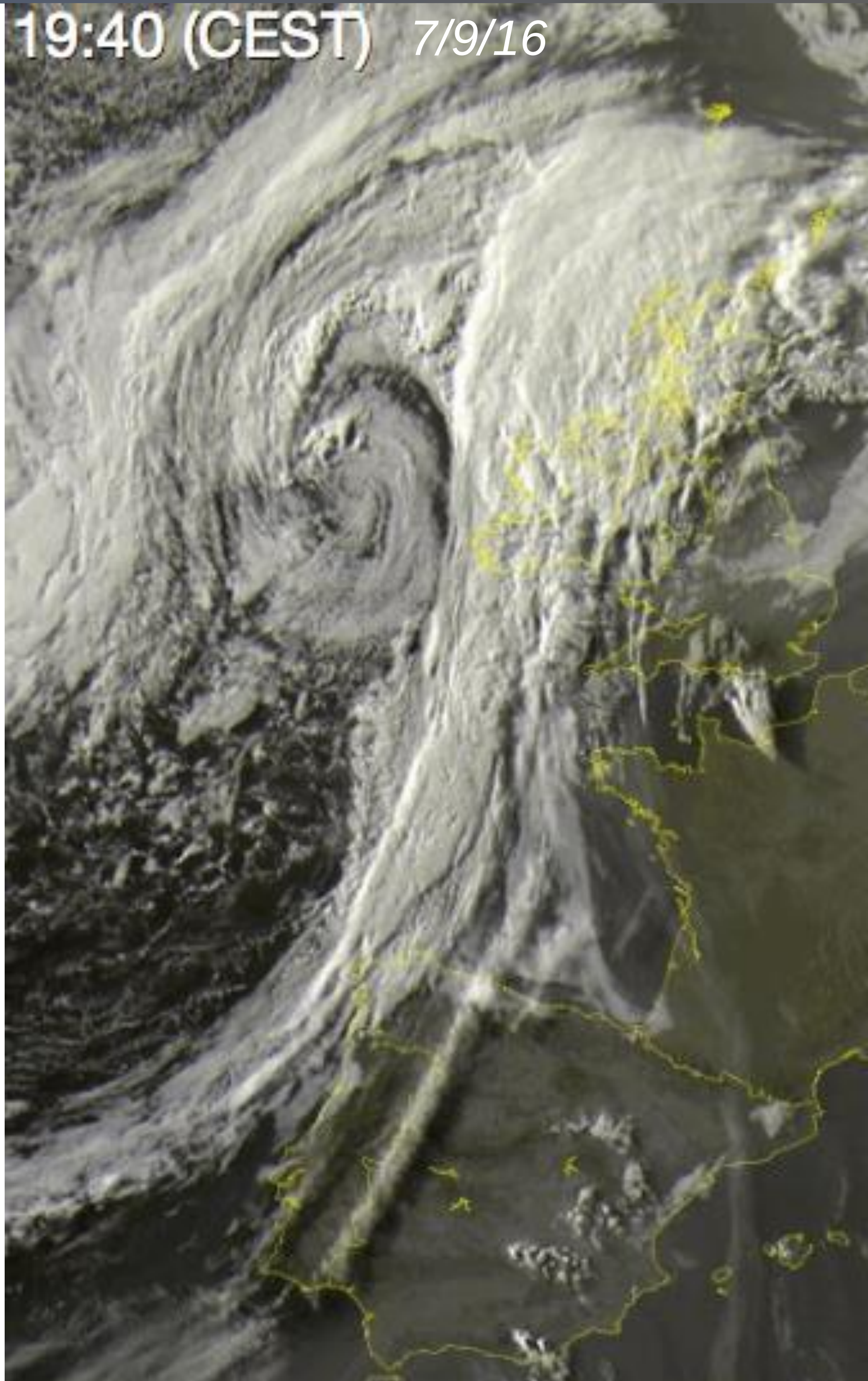
1) Shallow water and vorticity
2) Quasi-geostrophic theory
3) Rossby waves and instability
4) Gravity waves and tropical dynamics
5) Scale interactions in the atmosphere and ocean

Some things I hope you already know about:
Partial differential equations, vector calculus, the Coriolis force and geostrophy, the basic equations of motion, vorticity and divergence.

Books:
Introduction to GFD - Cushman-Roisin
Introduction to Dynamical Meteorology - Holton
Geophysical Fluid Dynamics - Pedlosky
Atmospheric and Oceanic Fluid Dynamics - Vallis
El Niño - Philander

Questions ?
- before the exam you are always priority number one.
- after the exam you might find it hard to get my attention.



Chapter 1: Shallow water and vorticity

Some concepts to discuss

rotation, stratification,
Development, balance, nonlinearity,
homogeneous-boussinesq-anelastic,
barotropic-baroclinic,
stationary-transient

The variables we use

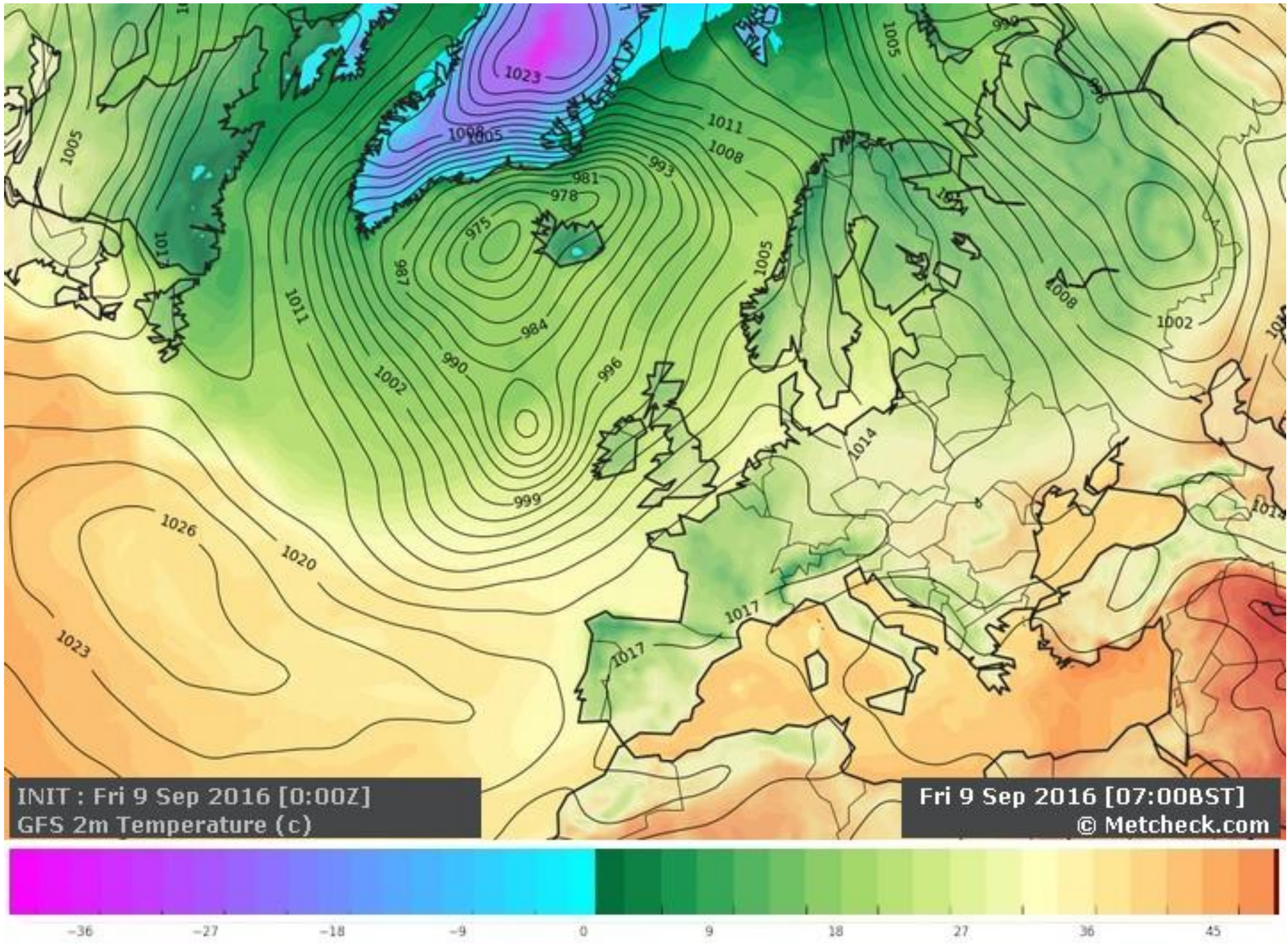
wind/current, pressure, density...
layer thickness, vorticity, divergence,
streamfunction, velocity potential

The shallow water equations

coordinate transformation
reduced gravity,
external and internal modes

Circulation and vorticity

the circulation theorem
the vorticity equation
potential vorticity



The momentum equations

The familiar (!) equations for x and y momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

development   nonlinear   rotation   balance

$$\frac{\partial p}{\partial z} = -\rho g$$ *hydrostatic*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ *continuity*

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = k\frac{\partial^2 \rho}{\partial z^2}$$ *density (thermodynamic)*

notes:

Strictly speaking, if the flow is balanced then it will not develop. Not much use for prediction.

The nonlinear terms are the advection terms. A linear system can still develop through wave solutions, and can still transport perturbation properties with the flow.

We have five variables: three wind components, density and pressure.

There are five equations if we add hydrostatic balance, continuity and a thermodynamic/density equation. (if we add temperature as a variable then it is linked to density and pressure by the equation of state).

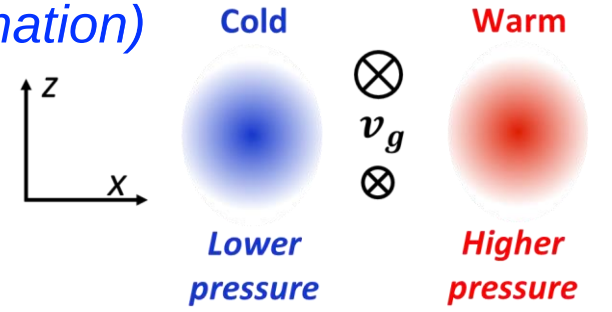
For an incompressible hydrostatic fluid, only the momentum and thermodynamic/density equations are prognostic. Continuity and hydrostatic balance are diagnostic.

Geostrophic hydrostatic flow

Geostrophic flow is possible with no density variations (homogeneous). But when density varies, things get interesting:

Geostrophic - hydrostatic flow -> Thermal Wind Balance (with the Boussinesq approximation)

$$\frac{\partial p}{\partial x} = \rho_0 f v_g, \qquad \frac{\partial p}{\partial z} = -\rho g, \qquad \frac{\partial^2 p}{\partial x \partial z} = \rho_0 f \frac{\partial v_g}{\partial z} = -g \frac{\partial \rho}{\partial x}$$



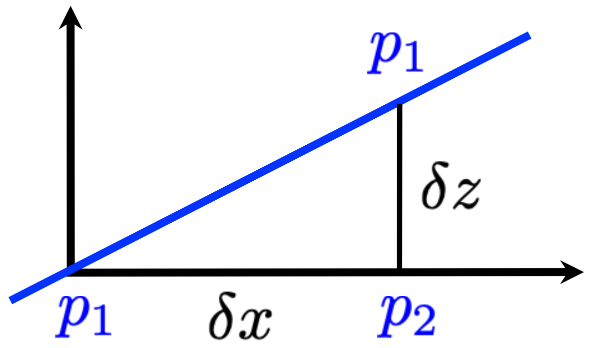
So horizontal gradients of density (or temperature) are related to vertical gradients of geostrophic flow.

And now in pressure coordinates without the approximation on density:

$$\left.\frac{\partial p}{\partial x}\right|_z = -\frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_p = \rho f v_g, \qquad \frac{\partial p}{\partial z} = -\rho g$$

$$\left.\frac{\partial z}{\partial x}\right|_p = \frac{f v_g}{g}, \qquad \frac{\partial z}{\partial p} = -\frac{1}{\rho g}$$

$$\frac{\partial^2 z}{\partial x \partial p} = \frac{f}{g}\frac{\partial v_g}{\partial p} = -\frac{1}{g}\left.\frac{\partial}{\partial x}\right|_p \left(\frac{1}{\rho}\right)$$



So without approximation we can say that vertical (pressure) gradients of the geostrophic wind depend on horizontal gradients of density.
This analysis also reveals that if density is a function of pressure, then the geostrophic wind must be vertically uniform -> “Barotropic flow”.

Density and its variations

The way in which density varies can have important consequences for the flow. Here are the definitions for various levels of approximation:

Homogeneous: $\rho = \rho_0 \quad (\rho' = 0), \quad p = p_0(z) + p'(x, y, t) \quad \Rightarrow \frac{\partial \mathbf{v}}{\partial z} = 0$

Boussinesq: $\rho = \rho_0 + \rho'(x, y, z, t), \quad p = p_0(z) + p'(x, y, z, t)$

Anelastic: $\rho = \rho_0(z) + \rho'(x, y, z, t)$

Barotropic: $\rho = \rho(p) \quad \Rightarrow \quad \frac{\partial \mathbf{v}_g}{\partial z} = 0$

Baroclinic: $\rho \neq \rho(p)$

Later we will use the *shallow water* model.

This represents a Boussinesq fluid with a set of homogeneous layers. Density is piecewise constant. Pressure varies continuously in the vertical and in the horizontal (but horizonal gradients of pressure will be piecewise constant).

Barotropic and baroclinic flow

We have seen that in some circumstances the flow is vertically coherent. Depth independent flow is associated with the “barotropic” component, also referred to as the “external” mode (more on this later). Bartotropic flow can exhibit many phenomena: vortices, Rossby waves, jets and instability. It is a good starting point for theories of the large scale ocean circulation.

When density surfaces cross pressure surfaces the flow is “baroclinic”. The baroclinic component is associated with horizontal temperature gradients: fronts and developing cyclones; ocean eddies on the thermocline. Baroclinic processes are necessary to liberate potential energy and generate *circulation*. Baroclinic instability occurs on a preferred scale (the Rossby radius) and is important for generating geostrophic turbulence.

Stationary and transient flow

Stationary waves $\phi = [\phi] + \phi^*, \quad [vT] = [v][T] + [v^*T^*]$

This is the departure from the zonal mean. The flux produced by any flow structure, the time mean for example, can be decomposed into components effected by the zonal mean (Hadley, Ferrel cells) and by the stationary waves.

Transient eddies $\phi = \overline{\phi} + \phi', \quad \overline{vT} = \overline{v}\,\overline{T} + \overline{v'T'}$

This is the departure from the time mean. The flux due to time variations is an important part of the mean flux.

Transient eddy "forcing"

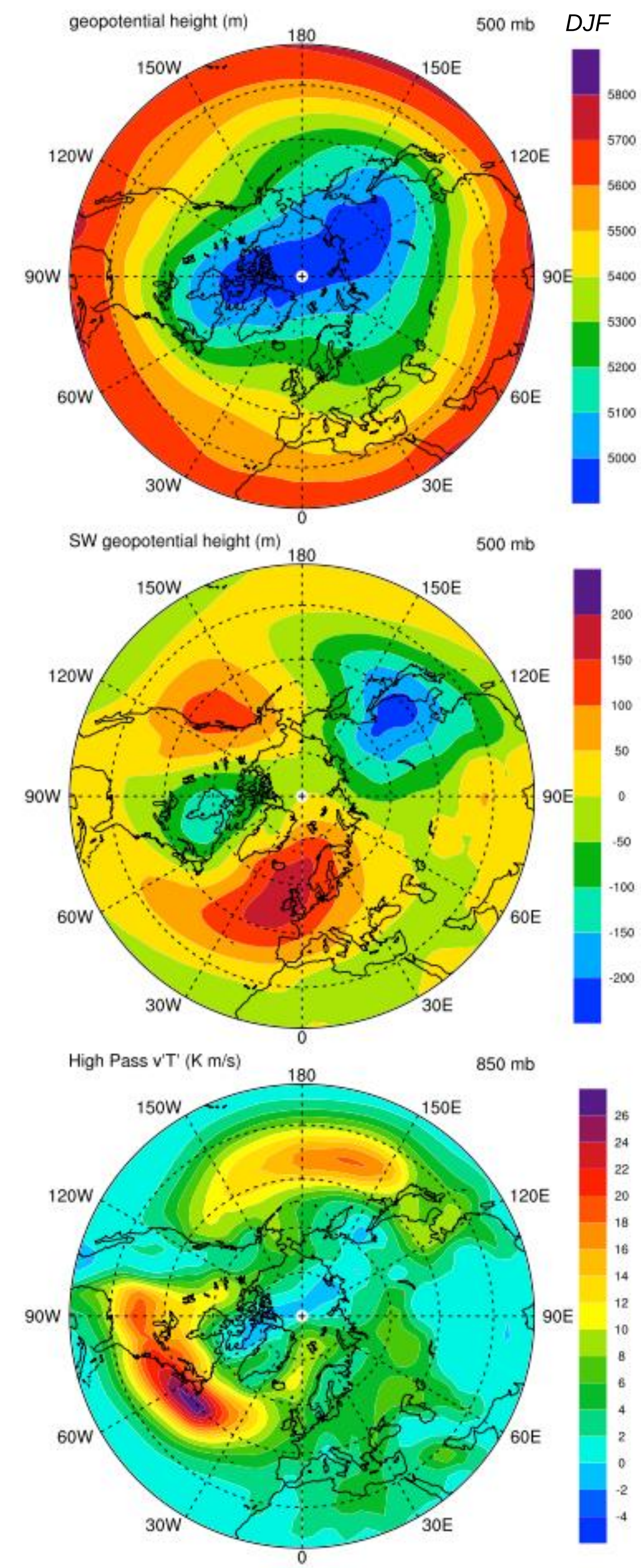
Consider the maintenance of the time mean flow:

$$\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T = \mathcal{F} - \mathcal{D}, \quad \overline{\mathbf{v}}.\nabla\overline{T} = -\overline{\mathbf{v}'.\nabla T'} + \overline{\mathcal{F}} - \overline{\mathcal{D}}$$

The mean effect of transient eddies is sometimes viewed as a forcing term the contributes to to the maintenance of the time-mean state.

These mechanisms depend on nonlinear terms, and may lead to nonlinear behaviour on longer timescales (but not necessarily).

Nonlinearity often manifests as asymmetry



Some scaling parameters

The importance of rotation: the Rossby number

Compare the advection term with the Coriolis force $u\frac{\partial u}{\partial x} \Big/ fu \rightarrow$ $R_o = \frac{U}{fL}$

When the Rossby number is small the flow is close to geostrophic

The importance of stratification: the Froude number

For nonrotating steady flow $u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ *compare vertical divergence with horizontal divergence*

$$\frac{\partial w}{\partial z} \Big/ \frac{\partial u}{\partial x} \rightarrow F_r = \frac{U}{NH} = \frac{U}{\sqrt{g'H}}$$

The Froude number is the ratio of flow speed to internal gravity wave speed.

When the Froude number is small stratification is important, vertical excursions of the flow are limited.

Rotation vs Stratification: the Burger number

$$(B_u)^{\frac{1}{2}} = \frac{R_o}{F_r} = \frac{NH}{fL} = \frac{\sqrt{g'H}}{fL}$$

When the Burger number is ~1, vorticity advection balances vortex stretching.
This occurs at a special spatial scale...

The Rossby radius: $L_R = \frac{NH}{f} = \frac{\sqrt{g'H}}{f}$

Recall

$$N^2 = \frac{g}{\rho}\frac{\partial \rho}{\partial z} = \frac{g'}{H}$$

$$(g' = g\Delta\rho/\rho)$$

Equation sets and variables

The primitive equations u, v, w, p, ρ

essentially five variables, three prognostic equations and two diagnostic equations

The shallow water equations u, v, h

three variables, three prognostic equations

The quasi-geostrophic equations ψ, q

one variable, one prognostic equation, one definition

Streamfunction and velocity potential (revision)

The vector horizontal velocity can be written as two scalars $\mathbf{v} = (u, v) = -\nabla\phi + \hat{\mathbf{k}}_\wedge\nabla\psi$

$$\Rightarrow u = -\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}$$

Φ is the velocity potential. Divergent flow emanates from maxima of Φ.
ψ is the streamfunction. Nondivergent flow circulates clockwise round maxima of ψ.

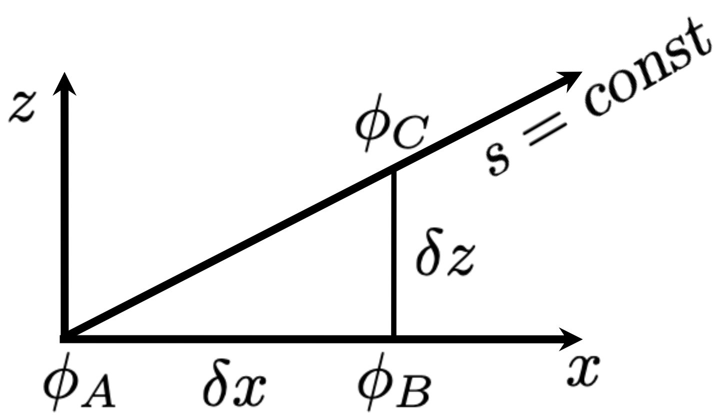
If the flow is either nondivergent or irrotational we can economise one variable.
Quasi-geostrophic flow is nondivergent so we only need ψ.

Furthermore, divergence, $D = \nabla.\mathbf{v} = -\nabla^2\phi$ *and relative vorticity,* $\xi = \hat{\mathbf{k}}.\nabla_\wedge\mathbf{v} = \nabla^2\psi$

Alternative vertical coordinates

We can simplify the equations if we use a conserved quantity as the vertical coordinate. In this frame of reference there is no "vertical velocity", rendering the system two-dimensional. So we can reduce our equation set by using coordinate systems based on density in the ocean or potential temperature in the atmosphere. But the price we pay for this simplification is to complicate the boundary conditions: coordinate surfaces outcrop, they move in time, and our coordinates are no longer orthogonal.

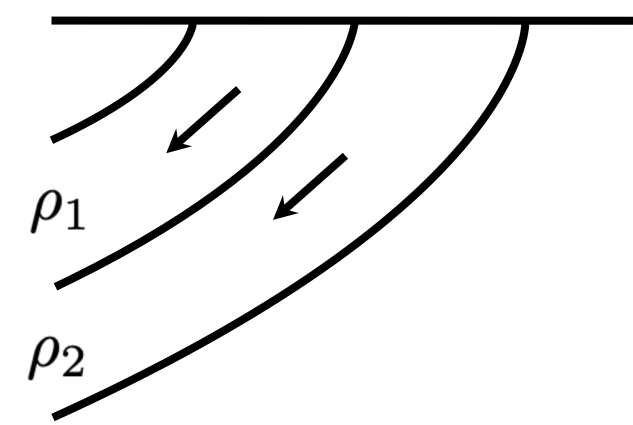
General coordinate transformation:



$$\frac{\phi_C-\phi_A}{\delta x} = \frac{\phi_C-\phi_B}{\delta z}\left(\frac{\delta z}{\delta x}\right) + \frac{\phi_B-\phi_A}{\delta x}$$

$$\delta x,\ \delta z \to 0$$

$$\left.\frac{\partial\phi}{\partial x}\right|_s = \frac{\partial\phi}{\partial z}\left(\left.\frac{\partial z}{\partial x}\right|_s\right) + \left.\frac{\partial\phi}{\partial x}\right|_z$$

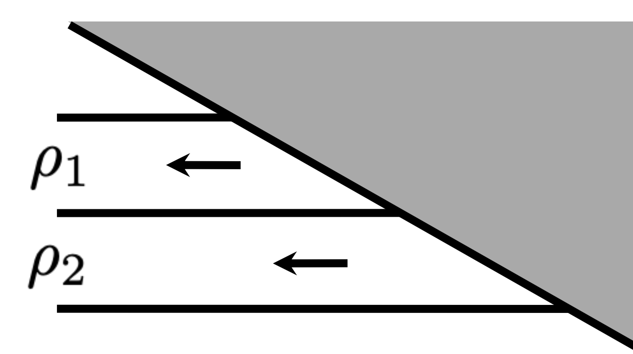


$$\Rightarrow \left.\frac{\partial\phi}{\partial x}\right|_z = \left.\frac{\partial\phi}{\partial x}\right|_s - \frac{\partial\phi}{\partial z}\left.\frac{\partial z}{\partial x}\right|_s \qquad [1]$$

rule 1

$$\frac{\partial\phi}{\partial z} = \frac{\partial s}{\partial z}\frac{\partial\phi}{\partial s} \qquad [2]$$

rule 2



Density coordinates

Let's transform the primitive equations to density coordinates for isopycnal flow in a Boussinesq fluid:

Hydrostatic equation: $\frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad \frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$ *(rule 2)*

Define "Montgomery potential" as $P = p + \rho g z, \quad \Rightarrow$ $\frac{\partial P}{\partial \rho} = g z$

Momentum equations: $\frac{Du}{Dt} - fv = -\frac{1}{\rho_0} \left. \frac{\partial p}{\partial x} \right|_z = -\frac{1}{\rho_0} \frac{\partial P}{\partial x}$ *(rule 1)*

and if ρ is conserved, no equivalent of vertical velocity so

$$\frac{\partial u}{\partial t} + u \left. \frac{\partial u}{\partial x} \right|_\rho + v \left. \frac{\partial u}{\partial y} \right|_\rho - fv = -\frac{1}{\rho_0} \frac{\partial P}{\partial x}$$
$$\frac{\partial v}{\partial t} + u \left. \frac{\partial v}{\partial x} \right|_\rho + v \left. \frac{\partial v}{\partial y} \right|_\rho + fu = -\frac{1}{\rho_0} \frac{\partial P}{\partial y}$$

Continuity: $\left. \frac{\partial u}{\partial x} \right|_z + \left. \frac{\partial v}{\partial y} \right|_z + \frac{\partial w}{\partial z} = 0$

apply rules 1 and 2 and after some manipulation:

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \left. \frac{\partial}{\partial x} \right|_\rho \left(u \frac{\partial z}{\partial \rho}\right) + \left. \frac{\partial}{\partial y} \right|_\rho \left(v \frac{\partial z}{\partial \rho}\right) = 0$$

This is mass conservation expressed in terms of a flux of layer thickness. The final step to a layer model is to discretize: $\frac{\partial z}{\partial \rho} = \frac{h}{\Delta \rho}$

details

1) Hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g$$

$$[2] \Rightarrow \frac{\partial p}{\partial z} = \frac{\partial \rho}{\partial z}\frac{\partial p}{\partial \rho} = -\rho g$$

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$$

Define

$$P = p + \rho g z$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + gz + \rho g \frac{\partial z}{\partial \rho}$$

$$\frac{\partial P}{\partial \rho} = gz$$

Hydrostatic equation in terms of "Montgomery potential" *P*.

2) Thermodynamic equation (density equation)

in *any* coordinate system (Boussinesq fluid - incompressible)

$$\frac{D\rho}{Dt} = 0$$

in *z* coordinates

$$\frac{\partial \rho}{\partial t} + u\left.\frac{\partial \rho}{\partial x}\right|_z + v\left.\frac{\partial \rho}{\partial y}\right|_z + w\frac{\partial \rho}{\partial z} = 0$$

i.e.

$$\frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} + \frac{dy}{dt}\frac{\partial \rho}{\partial y} + \frac{dz}{dt}\frac{\partial \rho}{\partial z} = 0$$

On a surface of constant *ρ*, *z* varies. To make *z* the variable and *ρ* the coordinate, we rewrite this equation swapping the variables:

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + \frac{d\rho}{dt}\frac{\partial z}{\partial \rho} = w$$

and the last term is zero because *ρ* is conserved. This gives us an equation for *w*.

3) *x* momentum equation

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\left.\frac{\partial p}{\partial x}\right|_z$$

$$[1] \Rightarrow = -\frac{1}{\rho_0}\left[\left.\frac{\partial p}{\partial x}\right|_\rho - \frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\rho\right]$$

$$= -\frac{1}{\rho_0}\left.\frac{\partial}{\partial x}\right|_\rho (p + \rho g z) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

since

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left.\frac{\partial}{\partial x}\right|_\rho + v\left.\frac{\partial}{\partial y}\right|_\rho$$

(no vertical term because *ρ* is conserved)

$$\rightarrow \frac{\partial u}{\partial t} + u\left.\frac{\partial u}{\partial x}\right|_\rho + v\left.\frac{\partial u}{\partial y}\right|_\rho - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

likewise

$$\frac{\partial v}{\partial t} + u\left.\frac{\partial v}{\partial x}\right|_\rho + v\left.\frac{\partial v}{\partial y}\right|_\rho + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

4) continuity equation

z coordinates:

$$\left.\frac{\partial u}{\partial x}\right|_z + \left.\frac{\partial v}{\partial y}\right|_z + \frac{\partial w}{\partial z} = 0$$

(ignore *dv/dy* term for the moment)

$$[1] \text{ and } [2] \Rightarrow \left.\frac{\partial u}{\partial x}\right|_\rho - \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\rho + \frac{\partial w}{\partial \rho}\frac{\partial \rho}{\partial z} = 0$$

$$\text{multiply by } \frac{\partial z}{\partial \rho}$$

$$\frac{\partial z}{\partial \rho}\left.\frac{\partial u}{\partial x}\right|_\rho - \frac{\partial u}{\partial \rho}\left.\frac{\partial z}{\partial x}\right|_\rho + \frac{\partial w}{\partial \rho} = 0$$

details

the last term can be expanded

$$\frac{\partial w}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{Dz}{Dt}\right) = \frac{\partial}{\partial \rho}\left[\frac{\partial z}{\partial t} + u\left.\frac{\partial z}{\partial x}\right|_{\rho}\right]$$

$$= \frac{\partial^2 z}{\partial \rho \partial t} + \frac{\partial u}{\partial \rho}\left.\frac{\partial z}{\partial x}\right|_{\rho} + u\frac{\partial^2 z}{\partial \rho \partial x}$$

which leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \frac{\partial z}{\partial \rho}\left.\frac{\partial u}{\partial x}\right|_{\rho} + u\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \rho}\right) = 0$$

Putting the *y* term back in gives

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial x}\right|_{\rho}\left(u\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial y}\right|_{\rho}\left(v\frac{\partial z}{\partial \rho}\right) = 0$$

This is the flux form of the continuity equation. The tendency of *dz/dρ* is given in terms of its flux along density surfaces. *dz/dρ* is a continuous form but this can be identified with mass conservation in terms of a flux of layer thickness

$$\frac{\partial z}{\partial \rho} = \frac{h}{\Delta \rho}$$

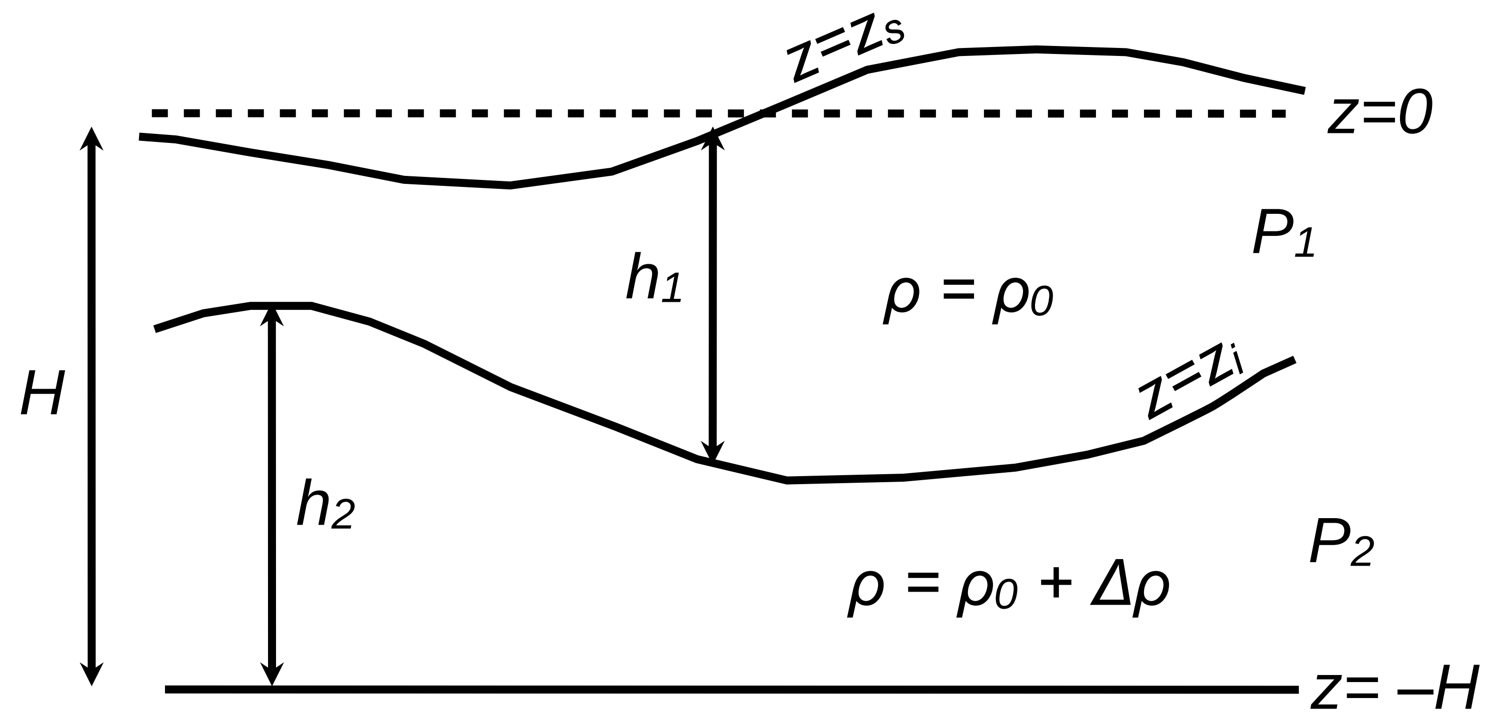
Shallow water layers

Apply the hydrostatic equation across the layer interface z_i (ignoring atmospheric pressure gradients)

$$P_1 = \rho_0 g(-H + h_2 + h_1) \;\; \{+p_a\}$$

$$\frac{\partial P}{\partial \rho} = \frac{\Delta P}{\Delta \rho} = \frac{P_2 - P_1}{\Delta \rho} = g z_i = g(-H + h_2)$$

$$P_2 - P_1 = \Delta \rho g(-H + h_2)$$



Horizontal gradients of P take the following forms (where $D = h_1 + h_2$)

$$\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} = g\frac{\partial D}{\partial x}$$

$$\frac{1}{\rho_0}\frac{\partial P_2}{\partial x} = g\frac{\partial D}{\partial x} + g'\frac{\partial h_2}{\partial x}$$

and in general, for N layers

$$\frac{1}{\rho_0}\frac{\partial \mathbf{P}}{\partial x} = g\frac{\partial D}{\partial x} + g'\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}$$

For two layers

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For the general N-layer case

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & \dots \\ 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \\ & & & N-1 \end{pmatrix}$$

The first term on the right is the "external mode", associated with fast surface waves. The terms involving the matrix ***C*** *are the "internal modes" associated with slow waves on the layer interfaces. We have a set of linear expressions for the horizontal pressure gradient that we can decouple by finding the eigenvectors of* ***C****.*

The shallow water equations

Now we have expressions for the Montgomery potential we can eliminate it, discretize the stratification and write the equation set in terms of *u,v* and *h:* first for two layers, *i=1,2*

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial u_i}{\partial y} - f v_i = -g \frac{\partial}{\partial x}(h_1 + h_2) - g' \frac{\partial h_2}{\partial x}$$

$$\frac{\partial v_i}{\partial t} + u_i \frac{\partial v_i}{\partial x} + v_i \frac{\partial v_i}{\partial y} + f u_i = -g \frac{\partial}{\partial y}(h_1 + h_2) - g' \frac{\partial h_2}{\partial y}$$

this term just for i=2

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x}(u_i h_i) + \frac{\partial}{\partial y}(v_i h_i) = 0$$

And for *N* layers the momentum equations are

$$\frac{D u_i}{D t} - f v_i = -g \frac{\partial D}{\partial x} - g' \left[\mathbf{C} \frac{\partial \mathbf{h}}{\partial x} \right]_i$$

where **h** *is the column vector* $(h_1, h_2, ...)$

$$\frac{D v_i}{D t} + f u_i = -g \frac{\partial D}{\partial y} - g' \left[\mathbf{C} \frac{\partial \mathbf{h}}{\partial y} \right]_i$$

The thermocline and the abyss

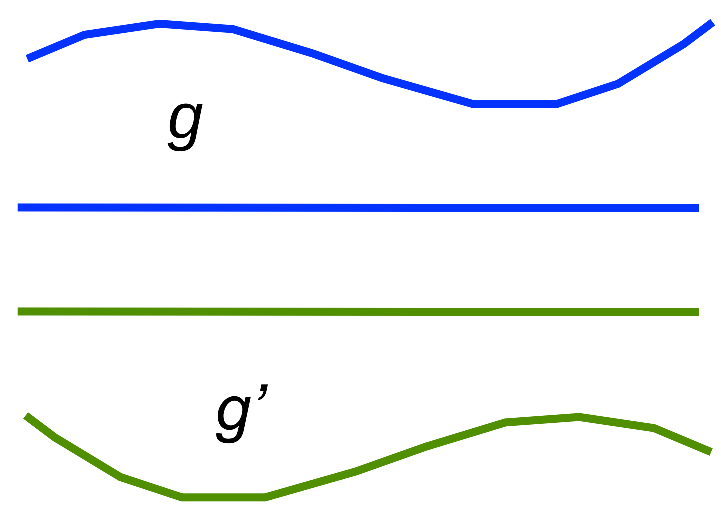
Instead of having a free surface and a flat bottom, we can reconfigure to have a rigid lid and a motionless abyss. This is sometimes called a $1^1/_2$ layer model.

The equations are the same except we replace g with g'

$$\frac{Du}{Dt} - fv = -g^{(\prime)}\frac{\partial h}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -g^{(\prime)}\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$



With a rigid lid we lose the external mode. In the general case (*N* layers) the *x*-momentum equation becomes

$$\frac{Du_i}{Dt} - fv_i = -g'\left[\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}\right]_i$$

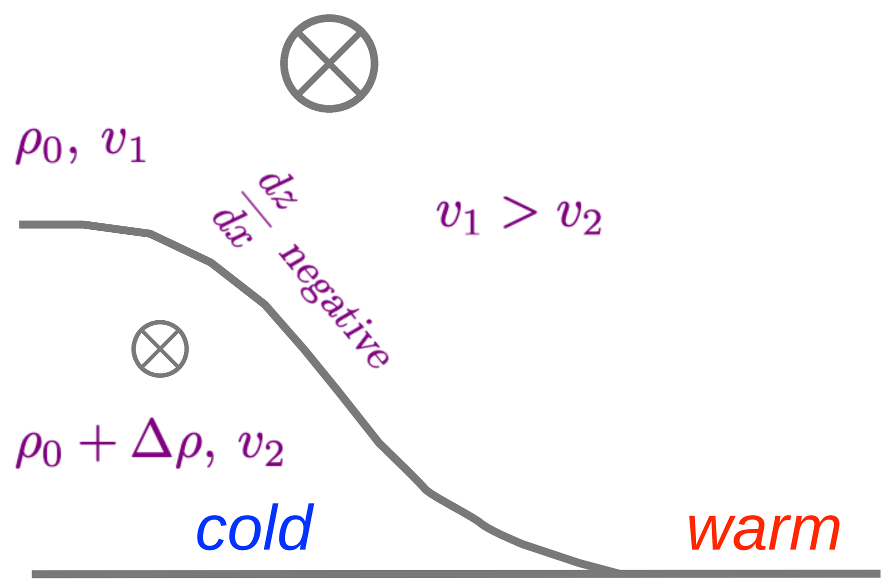
$$\mathbf{C} = \begin{pmatrix} N & N-1 & N-2 & \dots & 1 \\ N-1 & N-1 & N-2 & \dots & 1 \\ N-2 & N-2 & N-2 & \dots & 1 \\ \vdots & \vdots & \vdots & & \\ 1 & 1 & 1 & & 1 \end{pmatrix}$$

Note that ***C*** *has been flipped, and stripped of its zeros. One extra internal mode replaces the external mode associated with the free surface in the previous system. All the gravity waves are slow.*

Thermal wind revisited

Geostrophic hydrostatic balance is quite elegant in density coordinates

$$\frac{\partial^2 P}{\partial x \partial \rho} = \rho_0 f \frac{\partial v}{\partial \rho} = g \left.\frac{\partial z}{\partial x}\right|_\rho$$



Application: fronts in the atmosphere

$$\frac{\partial v}{\partial \rho} = \frac{g}{\rho_0 f} \left.\frac{\partial z}{\partial x}\right|_\rho , \qquad v_1 - v_2 = -\frac{g'}{f} \left.\frac{\partial z}{\partial x}\right|_\rho$$

“Margules relation”, southerlies increase with height

Application: currents in the ocean

When you’re floating on a free surface it’s impossible to measure pressure independently of depth. Since the density of water is 1000x the density of air, pressure surfaces are almost flat, making it very difficult to measure horizontal gradients. You have to make do with temperature and salinity (and thence density).

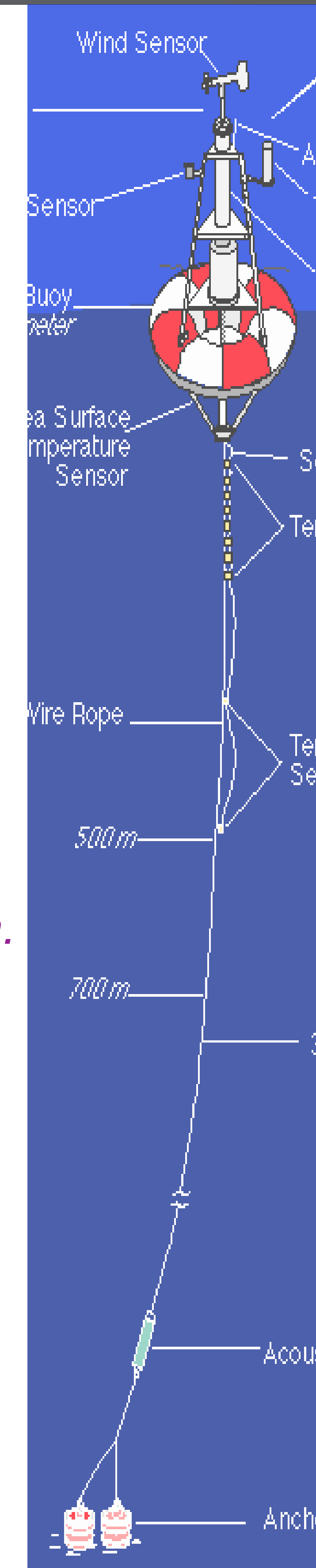
Measure vertical profiles at two points to see how the position of the thermocline varies horizontally.

The slope of the thermocline gives you the difference in current across it.

Sometimes oceanographers call this the “geostrophic current”.

This assumes that the abyssal flow is weak, or that there is a “level of no motion”.

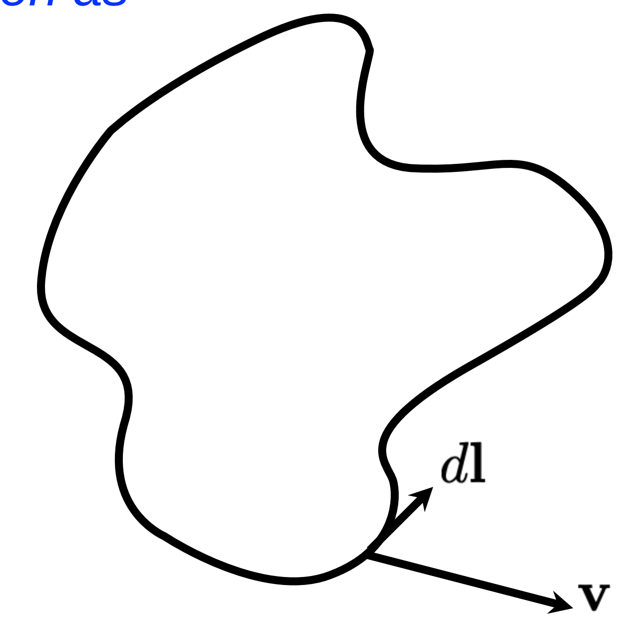
Final note: Thermal wind balance is nothing more than the horizontal component of the ***vorticity equation...***



Circulation and vorticity

Circulation *is the fluid equivalent of angular momentum. It is defined over a region as*

$$C = \oint \mathbf{v}.dl$$



Taking the time derivative gives $$\frac{dC}{dt} = \oint \frac{d\mathbf{v}}{dt}.d\mathbf{l} + \frac{1}{2}\oint d(\mathbf{v}.\mathbf{v})$$

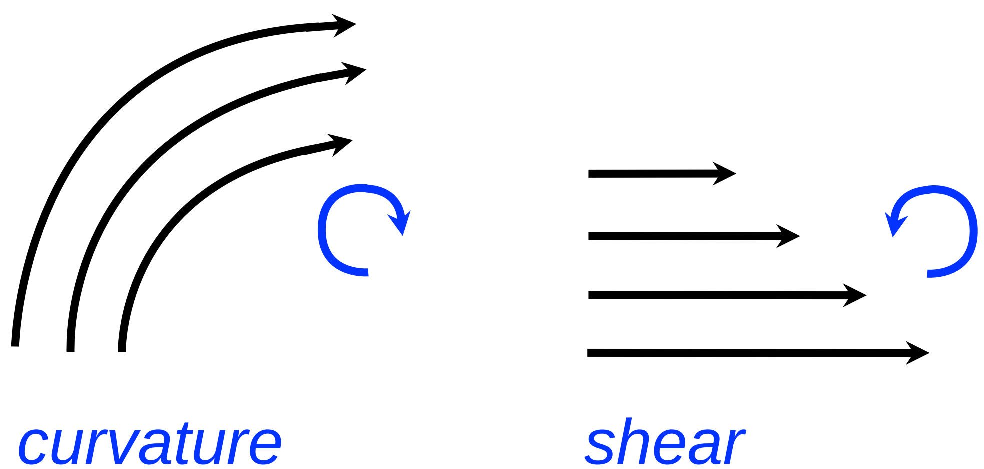
and if $\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p$ *then* $\frac{dC}{dt} = -\oint \frac{dp}{\rho}$

So over a fixed region, circulation can only be generated by baroclinic processes $\rho \neq \rho(p)$

Vorticity *is the point quantity of which circulation is the integral*

$$C = \oint \mathbf{v}.d\mathbf{l} = \iint_A (\nabla_\wedge \mathbf{v}) dA$$

$$\nabla_\wedge \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi = \nabla^2 \psi$$



Positive vorticity is anticlockwise

For solid body rotation $C = 2\pi\Omega r^2, \quad \xi = 2\Omega$

The vorticity equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = \mathcal{F}\left[\frac{\partial \mathbf{h}}{\partial x}\right] + \tau_x - \mathcal{D}_x \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = \mathcal{F}\left[\frac{\partial \mathbf{h}}{\partial y}\right] + \tau_y - \mathcal{D}_y \quad (2)$$

τ and $\mathcal{D}$ are sources and sinks of momentum

d/dx(2) - d/dy(1) =>

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left[f + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = \nabla_\wedge(\tau - \mathcal{D})$$

Sverdrup balance

or to put it another way

$$\left[\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right](f + \xi) = -(f + \xi)\nabla.\mathbf{v} + \nabla_\wedge(\tau - \mathcal{D})$$

This is the barotropic vorticity equation.

If the flow is nondivergent it works in individual layers, and absolute vorticity is conserved.

Vorticity can be generated and dissipated by mechanical stress at the boundaries (this is the basis of ocean circulation theory).

Vorticity can be generated by divergence, and divergence is associated with vertical displacement of layer boundaries - otherwise known as "vortex stretching" - which leads to coupling between the layers.

Vorticity can also be generated by "solenoidal" processes, which we have neglected in our Boussinesq fluid (cf circulation theorem)

$$\left(\frac{1}{\rho_0}\frac{\partial p}{\partial x} \to \frac{1}{\rho}\frac{\partial p}{\partial x}\right) \implies \frac{1}{\rho^2}J(\rho, p)$$

details

The vorticity equation

Effectively take the curl of the momentum equation. We'll do it by components:

$$-\frac{\partial}{\partial y} \quad : \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$+\frac{\partial}{\partial x} \quad : \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Rearrange the left hand side assuming we can swap the order of derivatives where necessary (smooth functions):

$$\begin{aligned}
\rightarrow \frac{\partial}{\partial t}\left[-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) & + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) \\
+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) & - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - w\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) \\
& + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - w\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right) \\
& + \frac{\partial f}{\partial y} + f\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] = RHS
\end{aligned}$$

Terms that cancel have been highlighted, along with necessary swapping of coordinates in the derivatives. This leads to

$$\frac{\partial \xi}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\xi) + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + w\frac{\partial \xi}{\partial z} + v\frac{\partial f}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} = RHS$$

we now evaluate the right hand side

$$\begin{aligned}
RHS = & \frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x} \\
& - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial y}
\end{aligned}$$

$$= \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

Rearranging a bit:

$$\frac{\partial}{\partial t}(f+\xi) + u\frac{\partial}{\partial x}(f+\xi) + v\frac{\partial}{\partial y}(f+\xi) + w\frac{\partial}{\partial z}(f+\xi)$$

$$= -(f+\xi)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

In vector form we take

$$\nabla_\wedge(\text{momentum equation})$$

which gives

$$\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v} - \hat{\mathbf{k}}.\nabla w_\wedge\frac{\partial \mathbf{v}}{\partial z} + \frac{\hat{\mathbf{k}}}{\rho^2}.\nabla\rho_\wedge\nabla p$$

The term on the left is the material tendency of absolute vorticity.
The first term on the right is the divergence (or vortex stretching) term.
The second term on the right is the tilting / twisting term.
The third term on the right is the baroclinic "solenoidal" term.

Associated phenomena

Advection / conservation of absolute vorticity: Planetary waves, large scale ocean circulation.
Divergence term: flow over mountains, ocean topography, tropical atmospheric circulation.
Tilting / twisting term: flow with large vertical motion, convective storms, fronts.
Solenoidal term: flow resulting from local differential heating, sea breeze circulations.

Recall barotropic flow

$$\frac{\partial}{\partial z}(\mathbf{v}, p) = 0$$

leads to

$$\boxed{\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v}}$$

the "Barotropic vorticity equation"

Generation of vorticity by divergence

Let's transform continuity equation from flux form to material tendency form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

So $\frac{Dh}{Dt} = -h\nabla.\mathbf{v}$ *looks remarkably similar to* $\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v}$

Clearly the layer thickness tendency, through mass conservation, is generated by the divergent flow. Similarly, the tendency of absolute vorticity is generated by the divergent flow. If we eliminate the divergence:

$$-\nabla.\mathbf{v} = \frac{1}{h}\frac{Dh}{Dt} = \frac{1}{(f+\xi)}\frac{D}{Dt}(f+\xi)$$

$$\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = (f+\xi)\frac{D}{Dt}\left(\frac{1}{h}\right) + \frac{1}{h}\frac{D}{Dt}(f+\xi)$$
$$= -\frac{(f+\xi)}{h^2}\left(\frac{Dh}{Dt}\right) + \frac{1}{h}\frac{D}{Dt}(f+\xi) = 0$$

we get a new conservation law

$$\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = 0$$

This is the "potential vorticity"

In this form, potential vorticity is conserved on density layers.

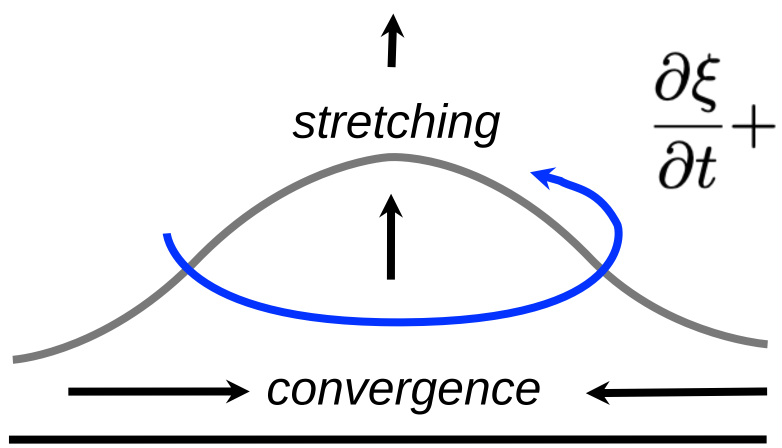
More generally, PV is the ratio of the absolute vorticity to the stratification, $(f+\xi)\frac{\partial\theta}{\partial p}$ *and it is conserved on isentropic surfaces (constant potential temperature).*

It's also a very compact convenient way to express the dynamics

Potential vorticity conservation

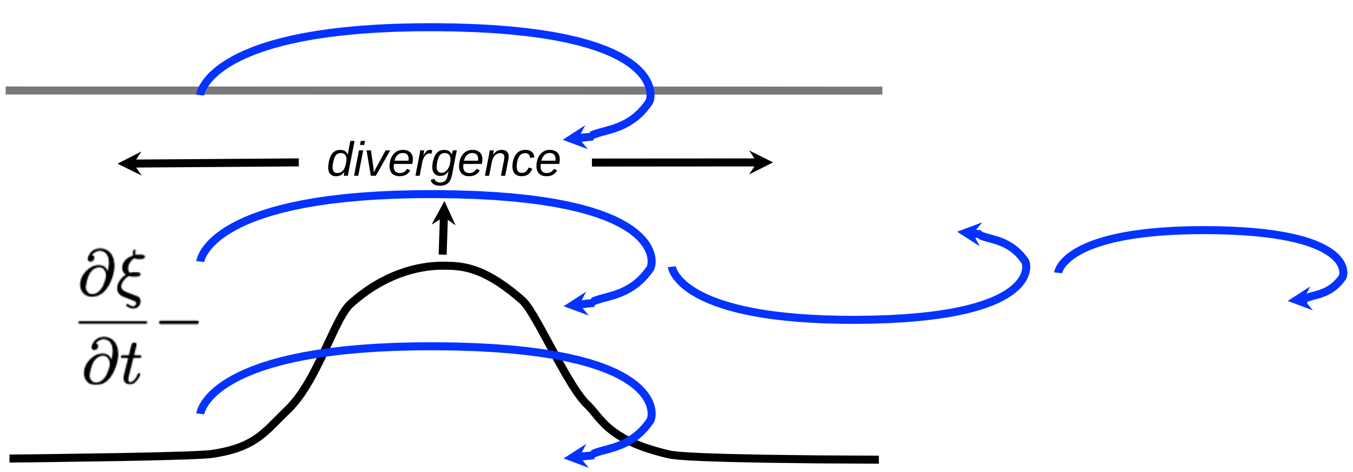
To conserve PV, changes in h are are compensated by changes in either f or ξ. This is another way to understand the link between divergent flow, mass conservation, vortex stretching and the generation of rotational flow.

Example: Cold air mass



$$\frac{f+\xi}{h}, \quad h+ \implies \xi+$$

Example: Mountains, Taylor columns and Rossby waves



$\frac{f}{h}$ constant => flow does not cross h contours

$f = f(y)$ => Rossby waves

Conservation laws and potential quantities

The name “potential” vorticity gives a clue as to why it is conserved.

This is the relative vorticity the fluid parcel **would have** if stretched to the mean layer thickness and brought to the equator.

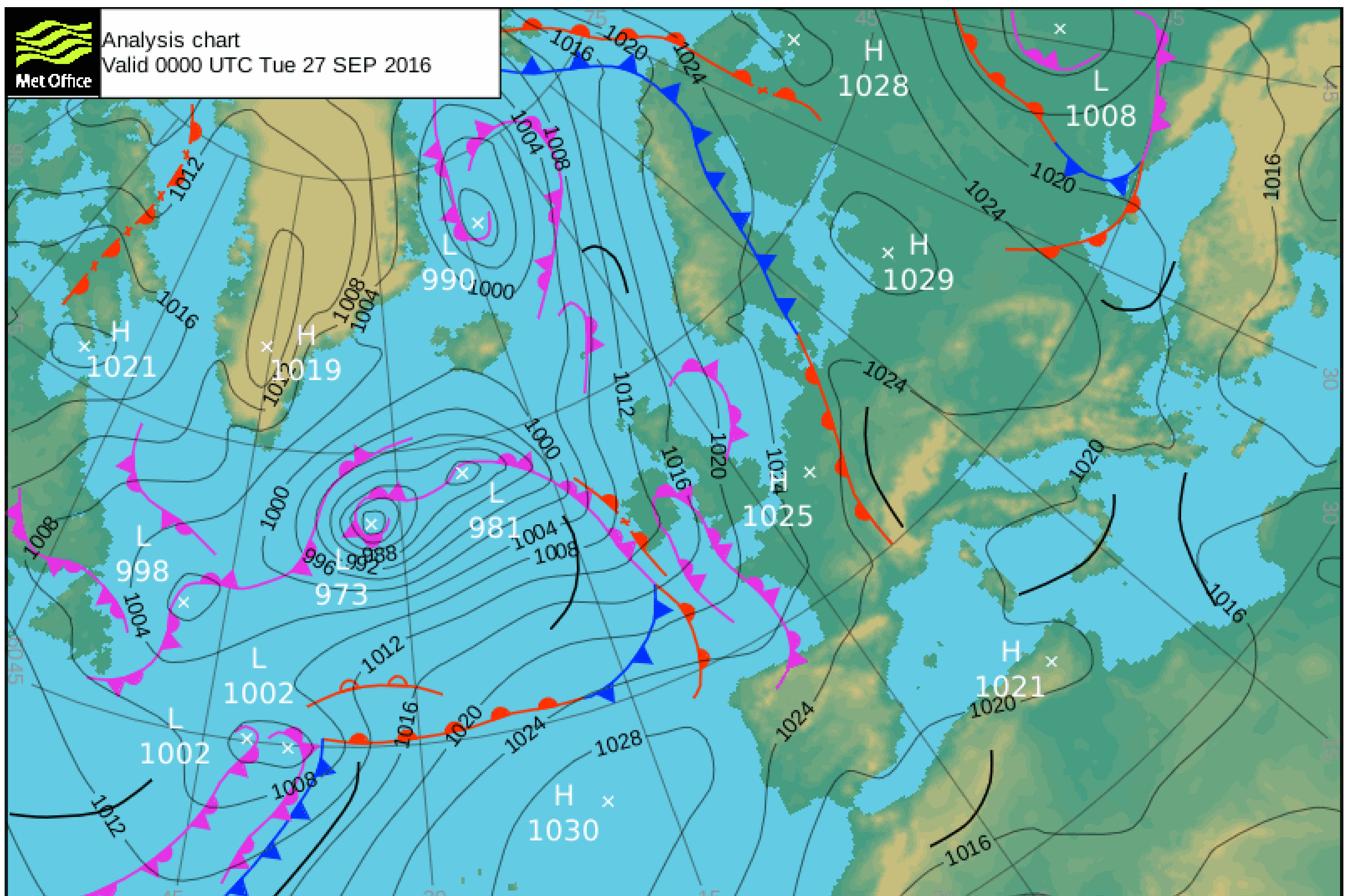
As such, it is like an *address label* that we attach to a parcel of fluid. The label refers to the state a parcel would have in reference conditions.

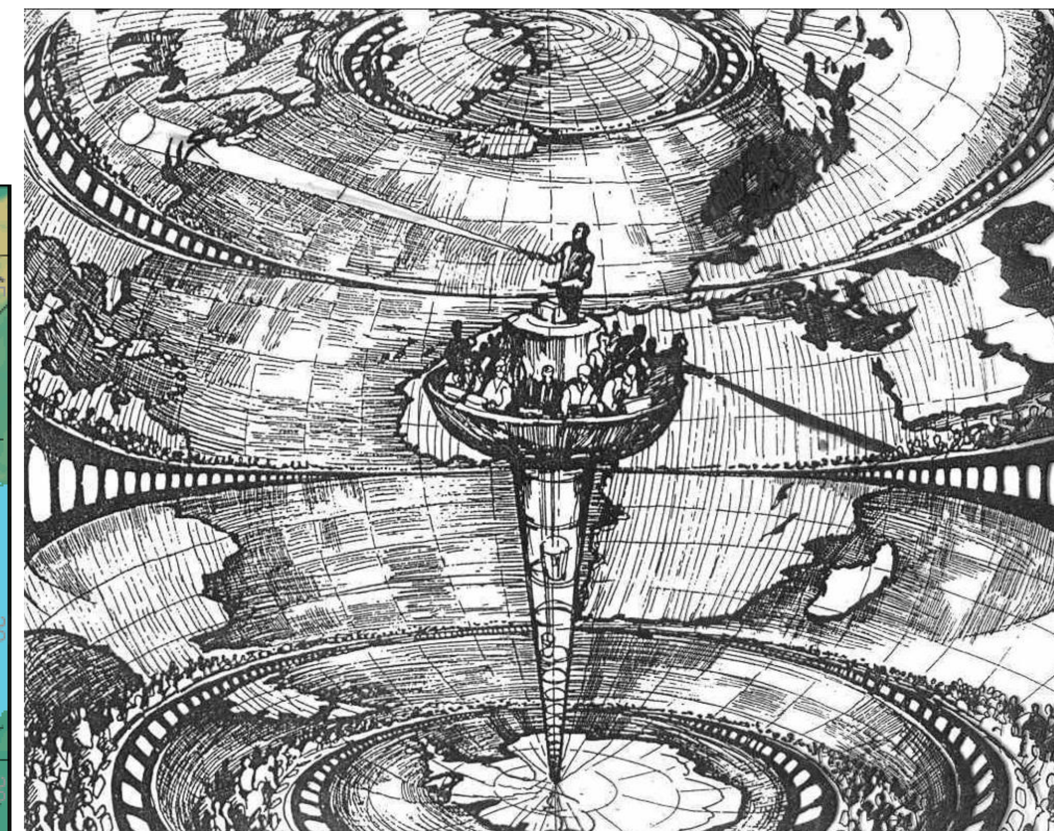
The vorticity of the fluid might change as it shifts latitude or stratification, but this label is a constant reference.

The same principle applies to potential temperature: it’s the temperature a parcel would have if brought adiabatically to 1000mb.

Chapter 2: Quasi-geostrophic theory

⇨ Steady departures from geostrophy: nonlinearity and drag.
Ageostrophy, divergence and potential vorticity.
⇨ *f*-plane quasi-geostrophy in shallow water.
Quasi-geostrophy on a curved planet.
⇨ Continuous stratification
Development and vertical motion.





“Richardson’s dream”

Gradient wind balance

Recall x-momentum equation (for a single layer)

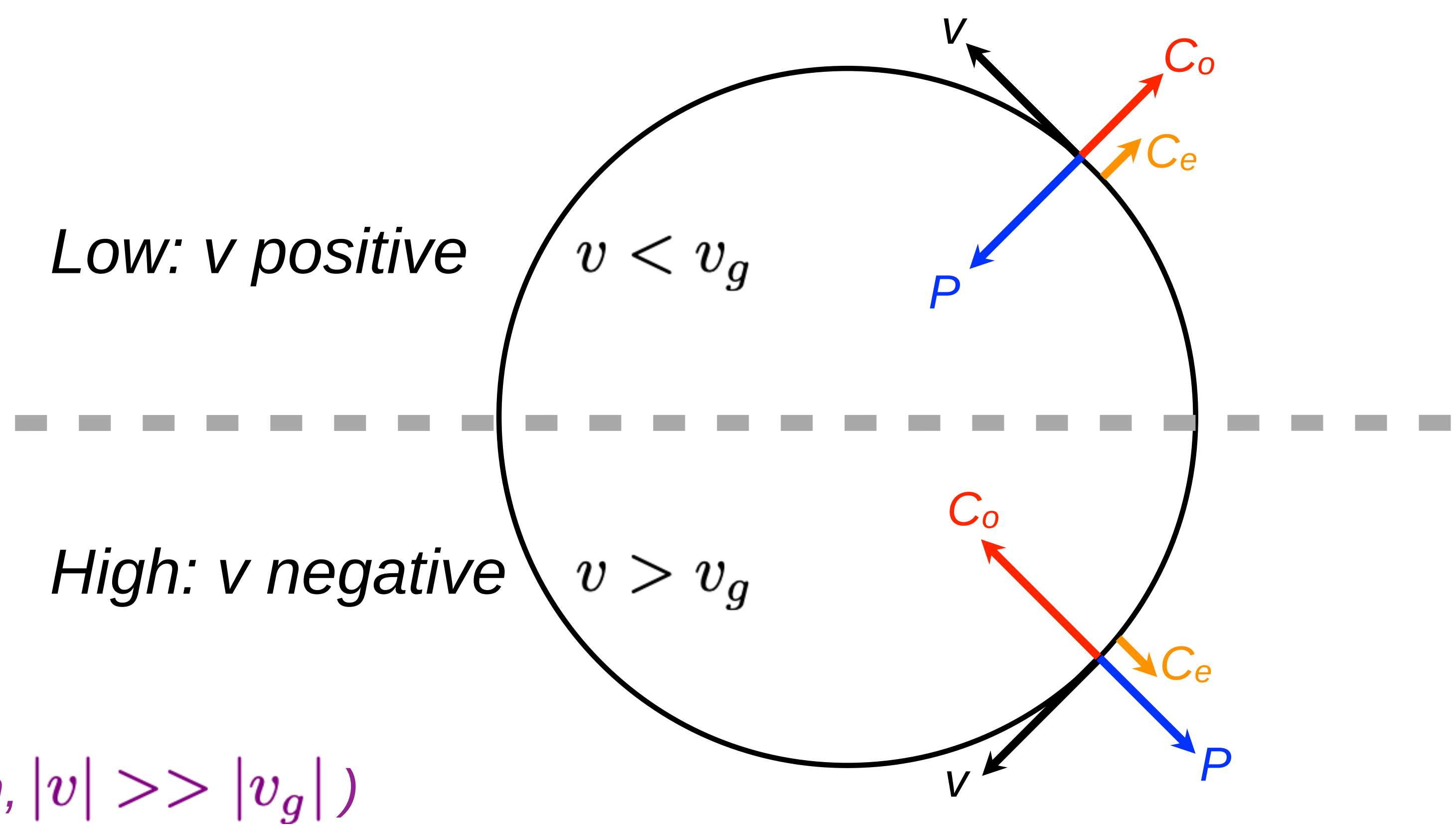
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

Now assume time independent uniform flow in a circle. The nonlinear (advection) terms express centrifugal force - this is gradient wind balance (without the Coriolis force it is “cyclostrophic” balance).

$$fv + \frac{v^2}{r} = g\frac{dh}{dr} = fv_g$$

so $$v\left(1 + \frac{v}{fr}\right) = v_g$$

or $$|v| = \frac{|v_g|}{1 \pm R_o}$$



(“anomalous” cases have v and v_g opposite sign, $|v| >> |v_g|$)

solution for v

$$v = -\frac{fr}{2} \pm \sqrt{\frac{f^2r^2}{4} + rg\frac{dh}{dr}}$$

so flow around a high limited by

$$\left|\frac{dh}{dr}\right| \leq \frac{f^2r}{4g}$$

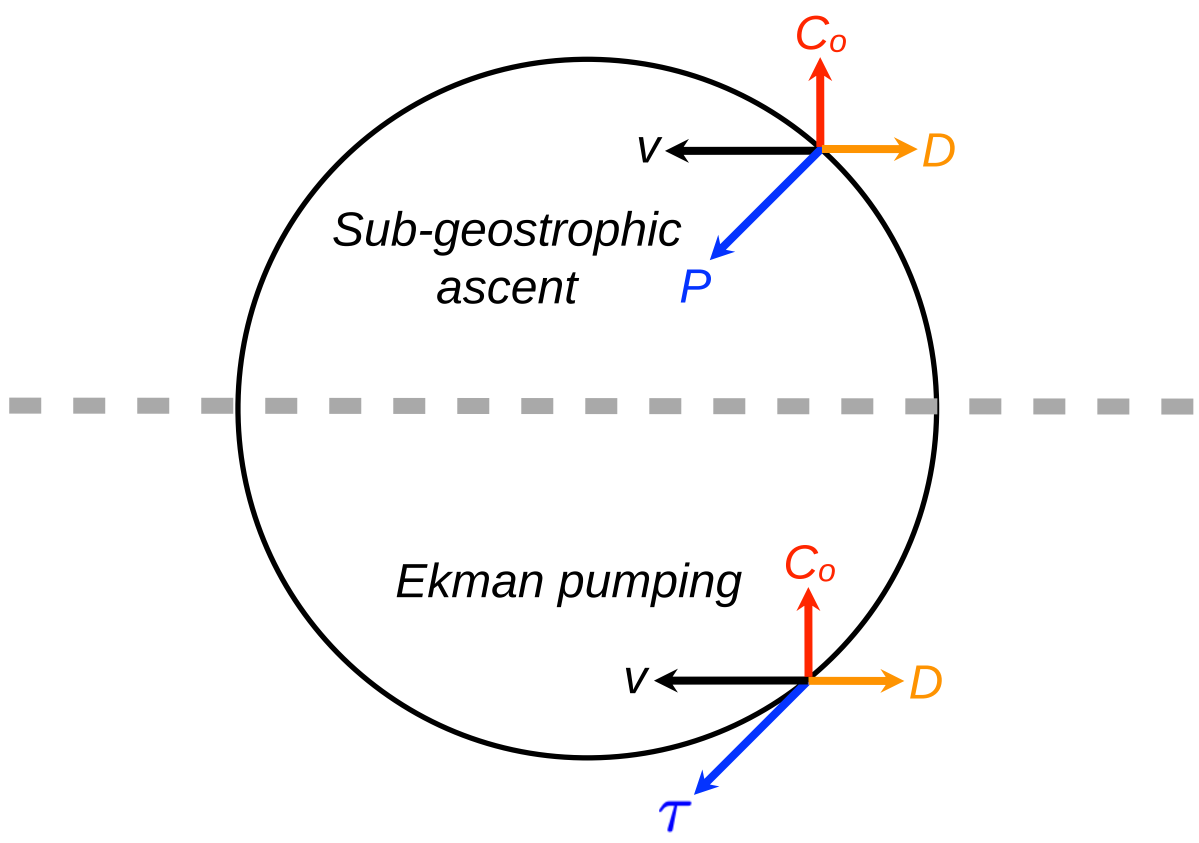
(no such limitation for flow round a low)

Boundary friction

Adding surface stress can alter the balance in a linear framework, leading to convergence or divergence.

This is the reason air ascends in lows (cloudy weather) and descends in highs (clear sky).

It is also the basis of the way the ocean driven the wind through Ekman pumping and Ekman suction.



But now we need to move away from these anecdotal cases, and put together a system with advection and time dependence that is almost, but not quite, geostrophic.

We do this essentially by separating the flow into a geostrophically balanced, nondivergent part, and the ageostrophic plus the divergent parts as a small perturbation. This small perturbation allows prognostic equations that lead to the evolution of the flow.

Ageostrophic perturbation

Start with the shallow water momentum equations in a single layer

$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0$$
$$\frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\left\{ \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right\}$$

What happens if we substitute in geostrophic velocity ? $u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \qquad v_g = \frac{g}{f}\frac{\partial h}{\partial x}$

Redefine the material tendency to advect with the geostrophic wind

$$\frac{D}{Dt} \rightarrow \frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$$

What if we use the geostrophic value in the Coriolis term as well ?
Obviously this can't work because it leads to zero tendency !

$$\frac{D_g}{Dt}u_g - fv_g + g\frac{\partial h}{\partial x} = 0$$

Instead we make sure the equation is linear in the ageostrophic part $v_{ag} = v - v_g$

So the full flow is used in the linear Coriolis terms and we advect with the geostrophic flow.
This is consistent with the idea that the ageostrophic part of the flow is small.

Quasi-geostrophic f-plane vorticity equation

$$\frac{D_g}{Dt}u_g - fv + g\frac{\partial h}{\partial x} = 0 \quad (1)$$
$$\frac{D_g}{Dt}v_g + fu + g\frac{\partial h}{\partial y} = 0 \quad (2)$$

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$$ -> vorticity equation

$$\frac{\partial}{\partial t}\xi_g + u_g\frac{\partial}{\partial x}\xi_g + v_g\frac{\partial}{\partial y}\xi_g + \cancel{\xi_g\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \cancel{v\frac{df}{dy}} = 0$$

If we are on an f-plane the last term disappears and the geostrophic flow is nondivergent

$$\frac{df}{dy} = 0, \quad \nabla.\mathbf{v}_g = 0$$

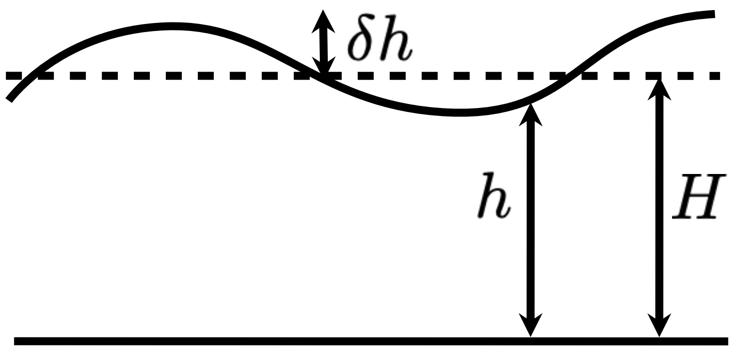
so we can write this as

$$\frac{D_g}{Dt}(f + \xi_g) = -f\nabla.\mathbf{v}$$

Continuity equation

What happens when we try this with the continuity equation ?

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \qquad \Rightarrow \frac{Dh}{Dt} + h\nabla.\mathbf{v} = 0$$



Replace this D/Dt with the geostrophic operator D_g/Dt

$$\rightarrow \frac{\partial h}{\partial t} + \mathbf{v}_g.\nabla h + h\nabla.\mathbf{v} = 0$$

But v depends on h. Terms involving v_{ag} were linear in the momentum equations so they must be linear here too. For consistency we must therefore write:

$$h = H + \delta h, \quad \delta h \ll H$$

$$\rightarrow \frac{\partial h}{\partial t} + \mathbf{v}_g.\nabla h + H\nabla.\mathbf{v} = 0$$

(this is equivalent to the approximation $\mathbf{v}_g h \approx \mathbf{v}_g \delta h + \mathbf{v} H$)

The ageostrophic term is now linear.

$$\rightarrow \frac{\partial}{\partial t}\delta h + \mathbf{v}_g.\nabla \delta h + H\nabla.\mathbf{v} = 0 \quad \textit{or} \quad \frac{D_g}{Dt}\delta h + H\nabla.\mathbf{v} = 0$$

In order to keep the ageostrophic term linear we had to make a strong assumption about the stratification.

Quasi-geostrophic potential vorticity

We can rewrite the continuity equation as $$\frac{f}{H}\frac{D_g}{Dt}\delta h = -f\nabla.\mathbf{v}$$

which as before has the same right hand side as the vorticity equation so we get

$$\frac{D_g}{Dt}\left(f\frac{\delta h}{H}\right) = \frac{D_g}{Dt}(f+\xi_g), \quad \rightarrow \frac{D_g}{Dt}\left\{f+\xi_g - f\frac{\delta h}{H}\right\} = 0$$

This is the conservation law for quasi-geostrophic potential vorticity:

$$\frac{D_g}{Dt}q = 0, \quad q = f + \xi_g - f\frac{\delta h}{H}$$

q is the linearised form of the full "Ertel" potential vorticity (but note that the units are different)

$$\frac{f+\xi}{h} = (f+\xi)(H+\delta h)^{-1} \approx \left(\frac{f+\xi}{H}\right)\left(1-\frac{\delta h}{H}\right) \Rightarrow q = f + \xi - f\frac{\delta h}{H} - \xi\frac{\delta h}{H}$$

Geostrophic scaling

$$Ro \ll 1, \quad \frac{U}{fL} \ll 1 \Rightarrow f \gg \xi, \quad \xi = \xi_g, \quad \Rightarrow \boxed{q = f + \xi_g - f\frac{\delta h}{H}}$$

This linearisation of layer thickness variations is a surprising consequence of our insistence that the flow be close to geostrophic.

In a vertically continuous framework it means that the stratification is uniform in the horizontal

Adding curvature to the earth

That was all pretty straightforward because we assumed that *f* was constant.

But for many important dynamical phenomena the variation of *f* is important (Rossby waves, for example).

On an *f*-plane the geostrophic flow is strictly nondivergent.

If *f* varies then we have to deal with the divergent part of the geostrophic flow as well as the ageostrophic flow.

We will assume that both these components are small compared to the nondivergent part of the geotrophic flow.

To proceed, we must derive the quasi-geostrophic set as an expansion of this perturbation in a small parameter.

We naturally choose the Rossby number for this small parameter.

Derivation of the quasi-geostrophic set for a shallow water layer

Recall full momentum and continuity equations:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} + f\hat{\mathbf{k}}_\wedge\mathbf{v} + g\nabla h = 0, \qquad \frac{\partial h}{\partial t} + \mathbf{v}.\nabla h + h\nabla.\mathbf{v} = 0$$

Introduce scaling for non-dimensionalisation:

$$x' = x/L, \quad u' = u/U, \quad t' = t/T, \quad \eta' = \delta h/\Delta h, \quad (h = H + \delta h)$$

So the full equations become

$$\frac{U}{T}\frac{\partial \mathbf{v}'}{\partial t'} + \frac{U^2}{L}\mathbf{v}'.\nabla\mathbf{v}' + Uf\hat{\mathbf{k}}_\wedge\mathbf{v}' + g\frac{\Delta h}{L}\nabla\eta' = 0 \quad (1)$$

$$\frac{\Delta h}{T}\frac{\partial \eta'}{\partial t'} + \frac{U}{L}\Delta h\mathbf{v}'.\nabla\eta' + \frac{U}{L}(H + \Delta h\eta')\nabla.\mathbf{v}' = 0 \quad (2)$$

If the basic scalings conform to geostrophic balance (f_0 is the value of f at a reference latitude)

$$f\mathbf{v} \sim g\nabla h \quad \rightarrow \quad Uf_0 \sim g\frac{\Delta h}{L} \quad \rightarrow \quad \Delta h = \frac{Uf_0L}{g} \quad (3)$$

Define the Rossby number and temporal Rossby number as $\epsilon = \frac{U}{f_0L}$ (4) *and* $\epsilon_T = \frac{1}{f_0T}$ (5)

Dropping primes, (1) / f_0U, (3), (4) and (5) ->

$$\boxed{\epsilon_T\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + \frac{f}{f_0}\hat{\mathbf{k}}_\wedge\mathbf{v} + \nabla\eta = 0}$$

Quasi-geostrophic continuity equation

Again dropping primes, L/UH x (2) -> $$\epsilon_T \left(\frac{L^2 f_0^2}{gH}\right) \frac{\partial \eta}{\partial t} + \epsilon \left(\frac{L^2 f_0^2}{gH}\right) (\mathbf{v}.\nabla \eta + \eta \nabla . \mathbf{v}) + \nabla . \mathbf{v} = 0$$

The non-dimensional constant that appears in brackets in this equation is B_u^{-1}, or L^2/L_R^2. Call it F. Remember, when F~1, Coriolis and gravity / buoyancy effects are comparable.

So the non-dimensional continuity equation is

$$\epsilon_T F \frac{\partial \eta}{\partial t} + \epsilon F (\mathbf{v}.\nabla \eta + \eta \nabla . \mathbf{v}) + \nabla . \mathbf{v} = 0$$

So far we haven't made any approximations.

But we can already see from these two equations that to zero order in our Rossby number parameters, the flow is geostrophic and nondivergent.

First order terms concern advection, divergence and time development.

Before doing a formal expansion in the Rossby number, we will set out our assumptions in detail.

The assumptions of quasi-geostrophic theory

1) small Rossby number, i.e. close to geostrophy $\epsilon \ll 1$

2) small temporal Rossby number, i.e. timescales slow compared to local rotation rate. $\epsilon_T \ll 1$
- *no fast moving linear waves*
- *nonlinear (advection terms important for time development)*

In fact we assume that $\epsilon_T = \epsilon$

3) buoyancy/gravity - stratification effects as important as Coriolis effects $L \sim L_R, \quad F = O(1)$

A consequence of (3) and (1) is that δh << H
- *linearisation of continuity equation and q.g.p.v. as we saw before.*

In a continuously stratified case this is equivalent to saying that N^2 is a function of z but not of x and y.

4) scales of motion small compared to the radius of the earth $\frac{L}{r_e} \ll 1$

In fact we assume that $\frac{L}{r_e} = \epsilon$

Assumptions (3) and (4) have nothing to do with geostrophy !
They are necessary for our expansion to be self-consistent.

The beta effect

Is a consequence of assumption (4), that the scale of motion is small compared to the radius of the earth.

$$f = 2\Omega \sin\phi$$

Taylor expansion about reference latitude Φ_0 (where $y' = y/r_e$)

$$f = f_0 + \beta_0 y + ... \quad = f_0 + \left.\frac{df}{dy}\right|_0 y + \left.\frac{d^2 f}{dy^2}\right|_0 \frac{y^2}{2} + ... \quad = 2\Omega \sin\phi_0 + \frac{y' L}{r_e} 2\Omega \cos\phi_0 + ...$$

set $\beta' = \cot\phi_0 = \frac{\beta_0 L}{f_0 r_e} \sim 1$ *then provided* $\frac{L}{r_e} = \frac{U}{f_0 L}$

we can write, to first order $\frac{f}{f_0} = 1 + \epsilon\beta' y'$ *(as long as we stay away from the equator where* $\cot\phi_0 \to \infty$ *)*

Henceforth drop primes on nondimensional y' and β'

This is often referred to as the "beta-plane" approximation, because the function f describes a plane in x-y space.

Not to be confused with the actual shape of the surface of the earth !

When we add the beta term, the surface of the earth ceases to be a plane and becomes a curve.

If we approximate the surface of the earth as a plane, then f is constant: the f-plane.

The expansion

Our equations are now non-dimensional.
u, v, η, β, F ~ 1; ε << 1
Expand variables in increasing powers of ε.

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + (1+\epsilon\beta y)\hat{\mathbf{k}}_\wedge\mathbf{v} + \nabla\eta = 0$$

$$\epsilon F\frac{\partial \eta}{\partial t} + \epsilon F(\mathbf{v}.\nabla\eta + \eta\nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + \ldots$$

$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots$$

Substitute this into the equations and compare coefficients of ε^0 (zero order) and ε^1 (first order).

Zero order $\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_0 = 0 \quad (1) \qquad \nabla.\mathbf{v}_0 = 0 \quad (2)$

Note that the curl of (1) gives (2). The two equations are equivalent. No development. Degenerate dynamics. Geostrophic nondivergent flow can only change in time with the help of the first order (divergent) flow.

We can say η_0 acts as a streamfunction for v_0, i.e.

$$v_0 = \frac{\partial \eta_0}{\partial x}, \quad u_0 = -\frac{\partial \eta_0}{\partial y}$$

Note that since v_0 is nondivergent, it is not the total geostrophic flow, just its nondivergent part.
It represents the geostrophic flow on the f-plane at $f = f_0$.

First order in ε

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla\mathbf{v}_0 + \hat{\mathbf{k}}_\wedge\mathbf{v}_1 + \beta y\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_1 = 0 \quad (1)$$

$$F\frac{\partial \eta_0}{\partial t} + F(\mathbf{v}_0.\nabla\eta_0 + \eta_0\nabla.\mathbf{v}_0) + \nabla.\mathbf{v}_1 = 0 \quad (2)$$

*The second term in (2) is zero because **v_0** is perpendicular to $\nabla\eta_0$, and **v_0** is nondivergent.*

$$\rightarrow F\frac{\partial \eta_0}{\partial t} = -\nabla.\mathbf{v}_1$$

The local tendency of zero order height comes from the divergence of the first order flow. Note that this divergence comes from the ageostrophic flow and the divergent part of the geostrophic flow.

Take the curl of (1) to eliminate η_1 and form the first order vorticity equation:

$$\frac{\partial \xi_0}{\partial t} + \xi_0\nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta y\nabla.\mathbf{v}_0 + \beta v_0 = 0$$

*The second and fifth terms are zero (nondivergent **v_0**). Combining this with (2) gives:*

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \beta v_0 = -\nabla.\mathbf{v}_1 = F\frac{\partial \eta_0}{\partial t}$$

then using $\frac{\partial}{\partial t}(\beta y) = 0, \quad \mathbf{v}_0.\nabla(\beta y) = \beta v_0, \quad \mathbf{v}_0.\nabla\eta_0 = 0$ *we can write*

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = F\left[\frac{\partial \eta_0}{\partial t} + \mathbf{v}_0.\nabla\eta_0\right]$$

or

$$\boxed{\left[\frac{\partial}{\partial t} + \mathbf{v}_0.\nabla\right][\beta y + \xi_0 - F\eta_0] = 0}$$

Quasi-geostrophic potential vorticity again

Now, using

$$v_0 = \frac{\partial \eta_0}{\partial x}, \quad u_0 = -\frac{\partial \eta_0}{\partial y}, \quad \xi_0 = \nabla^2 \eta_0$$

$$\rightarrow \mathbf{v}_0.\nabla = \frac{\partial \eta_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial}{\partial x} \quad \rightarrow \mathbf{v}_0.\nabla(q) = J(\eta_0, q)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_0.\nabla\right][\beta y + \xi_0 - F\eta_0] = 0$$

and we can write our prognostic equation as

$$\left[\frac{\partial}{\partial t} + \frac{\partial \eta_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial}{\partial x}\right][\beta y + \nabla^2\eta_0 - F\eta_0] = 0$$

or

$$\frac{\partial q}{\partial t} + J(\eta_0, q) = 0$$

$$q = \beta y + \nabla^2 \eta_0 - F\eta_0$$

One equation, one variable.

q is now the non-dimensional q.g.p.v.

<u>Re-dimonsionalise q:</u>

Using previously defined scalings, working back to dimensional equations leads to

$$q = \beta y + \nabla^2\psi - \frac{f_0}{H}\delta h$$

and if we define the quasi-geostrophic streamfunction

$$\psi = \frac{g}{f_0}\delta h$$

we get $q = \beta y + \nabla^2\psi - \left(\frac{f_0^2}{gH}\right)\psi$ or

$$q = \beta y + \nabla^2\psi - \frac{1}{L_R^2}\psi$$

and

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

Continuously stratified fluid

Up until now we have worked with discrete layers, each of which is homogeneous (constant density). The extension to continuous stratification requires that we abandon this formulation and reintroduce a vertical coordinate. The expansion around small Rossby number is very similar so it is shown in appendix slides. The result is once again a conservation law for potential vorticity, which is defined entirely in terms of a streamfunction, so one equation, one variable.

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$ *where now, ψ is defined as* $$\psi = \frac{p_0}{\rho_s f_0}$$

This is an anelastic fluid, which allows large variations of density with height, accounting for the static compressibility of the atmosphere. In this case the q.g.p.v. is

$$q = \beta y + \nabla^2 \psi + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left(\frac{f_0^2}{N^2} \rho_s \frac{\partial \psi}{\partial z} \right)$$

$$p = p_s(z) + \tilde{p}(x, y, z, t)$$
$$\rho = \rho_s(z) + \tilde{\rho}(x, y, z, t)$$

Only the vortex stretching component has changed.

In a Boussinesq fluid, where ρ_s is a constant (independent of z), this simplifies to

$$q = \beta y + \nabla^2 \psi + \frac{\partial}{\partial z} \left(\frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right)$$

$$p = p_s + \tilde{p}(x, y, z, t)$$
$$\rho = \rho_s + \tilde{\rho}(x, y, z, t)$$

details

IV) EXTENSION TO A CONTINUOUSLY STRATIFIED FLUID
(with non-Boussinesq, static compressibility effects)

Three dimensional scalings for a compressible, baroclinic stratified fluid:

$$x, y \rightarrow L, \quad u, v \rightarrow U, \quad z \rightarrow H, \quad w \rightarrow \frac{UH}{L}, \quad t \rightarrow \frac{L}{U}$$

$$p = p_s(z) + \tilde{p}(x, y, z, t)$$
$$\rho = \rho_s(z) + \tilde{\rho}(x, y, z, t)$$

Geostrophic scaling for pressure

$$fv \sim \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x}$$

so

$$\tilde{p} \rightarrow f_0 U L \rho_s$$

Hydrostatic scaling for density

$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g$$

so

$$\tilde{\rho} \rightarrow \frac{f_0 U \rho_s L}{Hg} = \rho_s \epsilon F$$

so

$$\boxed{\rho = \rho_s(1 + \epsilon F \rho')}$$

recall

$$F = \frac{f_0^2 L^2}{gH}$$
$$\epsilon = \frac{U}{f_0 L}$$

also

$$\frac{f}{f_0} = 1 + \epsilon \beta' y'$$

where

details

$$\beta' = \frac{\beta_0 L^2}{U} = \cot\phi_0$$

as before.

Non-dimensional momentum equation:

$$\frac{\partial \mathbf{v}'}{\partial t}\frac{U^2}{L} + \mathbf{v}'.\nabla\mathbf{v}'\frac{U^2}{L} + w'\frac{\partial \mathbf{v}'}{\partial z}\frac{HU^2}{LH} + fU\hat{\mathbf{k}}_\wedge\mathbf{v}' = -\frac{1}{\rho_s(1+\epsilon F\rho')}\frac{\nabla p'}{L}ULf_0\rho_s$$

$$= -Uf_0\nabla p'(1-\epsilon F\rho')$$

(to first order)

Divide by Uf_0, drop primes

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + \epsilon w\frac{\partial \mathbf{v}}{\partial z} + (1+\epsilon\beta y)\hat{\mathbf{k}}_\wedge\mathbf{v} = -(1-\epsilon F\rho)\nabla p$$

Non-dimensional continuity equation (non-Boussinesq)

$$\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla\rho + w\frac{\partial \rho}{\partial z} + \rho\nabla.\mathbf{v} + \rho\frac{\partial w}{\partial z} = 0$$

$$\rho_s\epsilon F\frac{U}{L}\frac{\partial \rho'}{\partial t'} + \rho_s\epsilon F\frac{U}{L}\mathbf{v}'.\nabla\rho' + \rho_s\epsilon F\frac{UH}{LH}w'\frac{\partial \rho'}{\partial z'}$$

$$+\frac{UH}{L}w'\left[\frac{\partial \rho_s}{\partial z}\right] + \rho_s(1+\epsilon F\rho')\left[\frac{U}{L}\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right)\right] = 0$$

$$\times\frac{L}{\rho_s U} \rightarrow$$

$$\epsilon F\frac{\partial \rho'}{\partial t'} + \epsilon F\mathbf{v}'.\nabla\rho' + \epsilon Fw'\frac{\partial \rho'}{\partial z} + Hw'\left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z}\right] + (1+\epsilon F\rho')\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right) = 0$$

Note that the expression in square brackets resembles N^2, and note that *z* is dimensionless.

$$N^2 = \frac{g}{\theta_s}\frac{\partial \theta_s}{\partial z}$$

Define

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

then the fourth term above becomes

$$\left(\frac{HS^2}{g}\right)w'$$

This is the non-Boussinesq term.

So dropping primes

$$\epsilon F\left(\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla\rho + w\frac{\partial \rho}{\partial z}\right) + \frac{HS^2}{g}w + (1+\epsilon F\rho)\left(\nabla.\mathbf{v} + \frac{\partial w}{\partial z}\right) = 0$$

Expansion of non-dimensional variables

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + ...$$
$$w = w_0 + \epsilon w_1 + ...$$
$$\tilde{p} = p_0 + \epsilon p_1 + ...$$
$$\tilde{\rho} = \rho_0 + \epsilon\rho_1 + ...$$

Momentum equation to zero order

Geostrophic balance

$$\hat{\mathbf{k}}_\wedge\mathbf{v}_0 = -\nabla p_0$$

and

$$\nabla.\mathbf{v}_0 = 0$$

Continuity equation to zero order

$$\frac{HS^2}{g}w_0 + \nabla.\mathbf{v}_0 + \frac{\partial w_0}{\partial z} = 0$$

Therefore we can't generate w_0 in the body of the fluid by horizontal motion. At zero order, vertical motion can only be generated at the boundary.

Assume that the bottom vertical velocity

details

$$w_b = 0 + \epsilon w_{1b} + ...$$

(this is assumption (3): weak orography)
Integrate upwards, this implies

$$w_0 = 0$$

everywhere, so

$$w = \epsilon w_1 + ...$$

Momentum equation to first order

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla \mathbf{v}_0 + \hat{\mathbf{k}}_\wedge \mathbf{v}_1 + \beta y \hat{\mathbf{k}}_\wedge \mathbf{v}_0 + \nabla p_1 - F\rho_0 \nabla p_0 = 0$$

$$\hat{\mathbf{k}}.\nabla_\wedge(\text{this}) \rightarrow$$

vorticity equation:

$$\frac{\partial \xi_0}{\partial t} + \xi_0 \nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla \xi_0 + \nabla.\mathbf{v}_1 + \beta y \nabla.\mathbf{v}_0 + \beta \mathbf{v}_0 - F\left[\frac{\partial \rho_0}{\partial x}\frac{\partial p_0}{\partial y} - \frac{\partial \rho_0}{\partial y}\frac{\partial p_0}{\partial x}\right] = 0$$

Second and fifth terms disappear by nondivergence of the zero order flow, and the last term can be rewritten using geostrophy of the zero order flow to give

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla \xi_0 + \nabla.\mathbf{v}_1 + \beta \mathbf{v}_0 + F\mathbf{v}_0.\nabla \rho_0 = 0$$

Continuity equation to first order

$$F\frac{\partial \rho_0}{\partial t} + F\mathbf{v}_0.\nabla \rho_0 + \left(\frac{HS^2}{g}\right) w_1 + \nabla.\mathbf{v}_1 + \frac{\partial w_1}{\partial z} = 0$$

Note that the second and fourth terms have just appeard in the vorticity equation. So we can eliminate them by combining the continuity and vorticity equations:

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla \xi_0 + \beta \mathbf{v}_0 = -F\mathbf{v}_0.\nabla \rho_0 - \nabla.\mathbf{v}_1$$

$$= F\frac{\partial \rho_0}{\partial t} + \left(\frac{HS^2}{g}\right) w_1 + \frac{\partial w_1}{\partial z}$$

At this stage we note that for synoptic scales $F \sim 0.1$ so we neglect the first term on the right hand side. This is because we have set

$$F = \frac{f_0^2 L^2}{gH} = \frac{L^2}{R^2}$$

(remember, for the atmosphere:

$$R_{ext} \sim \frac{\sqrt{gH}}{f_0} = \frac{\sqrt{10 \times 10^4}}{10^{-4}} \sim 3 \times 10^6 \text{ m} = 3000 \text{ km}$$

$$F = \frac{L^2}{R^2} \sim \frac{1000^2}{3000^2} \sim 10^{-1}$$

)

So the vorticity equation is now

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = \frac{HS^2}{g} w_1 + \frac{\partial w_1}{\partial z}$$

Using

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z_*}$$

the right hand side can be written

$$= \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1)$$

We can evaluate the right hand side using the ...

Thermodynamic equation

$$\frac{D\theta}{Dt} = 0$$

scale

$$\theta = \theta_s(1 + \epsilon F(\theta_0 + ...))$$

as we did for density, so

$$\frac{U}{L}\frac{\partial \theta'}{\partial t'}\epsilon F\theta_s + \frac{U}{L}\mathbf{v}'.\nabla\theta'\epsilon F\theta_s + w'\frac{\partial \theta'}{\partial z'}\epsilon F\theta_s\frac{HU}{LH} + w\frac{\partial \theta_s}{\partial z_*}\frac{UH}{L} = 0$$

drop primes, get

$$\epsilon F\left(\frac{\partial \theta}{\partial t} + \mathbf{v}.\nabla\theta + w\frac{\partial \theta}{\partial z}\right) + w\frac{N^2 H}{g} = 0$$

At zero order we recover

$$w_0 = 0$$

details

At first order:

$$F\left(\frac{\partial\theta_0}{\partial t}+\mathbf{v}_0.\nabla\theta_0\right)+w_1\frac{N^2H}{g}=0$$
$$\rightarrow w_1=-\frac{f_0^2L^2}{N^2H^2}\left(\frac{\partial\theta_0}{\partial t}+\mathbf{v}_0.\nabla\theta_0\right)$$

introduce

$$\frac{D_0}{Dt}=\frac{\partial}{\partial t}+\mathbf{v}_0.\nabla$$

so

$$w_1=-\frac{f_0^2L^2}{N^2H^2}\left[\frac{D_0\theta_0}{Dt}\right]$$

Multiply by ρ_s, take vertical derivative and then divide by ρ_s, and exchange derivatives when possible. This gives

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1)=-\frac{D_0}{Dt}\left[\frac{f_0^2L^2}{H^2\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\theta_0}{N^2}\right)\right]$$

and we can use this to rewrite the vorticity equation as

$$\frac{D_0}{Dt}\left[\beta y+\xi_0+\frac{f_0^2L^2}{H^2\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\theta_0}{N^2}\right)\right]=0$$

Now we have one last thing to do...

Hydrostatic equation

$$\frac{\partial p}{\partial z}=-\rho g$$

$$p=p_s+p_0\rho_s f_0UL$$
$$\rho=\rho_s+\rho_0\rho_s\epsilon F$$
$$\rightarrow\frac{\partial}{\partial z}(p_o\rho_s)=-\rho_0\rho_s$$

or

$$\rho_0=-\frac{1}{\rho_s}\frac{\partial}{\partial z}(p_0\rho_s)$$

Now, define

$$\theta_*=\theta_s(z)(1+\epsilon F\theta)$$

and define

$$\theta_0=-\rho_0+\frac{1}{\gamma}\left(\frac{\rho_s gH}{p_s}\right)p_0$$

ASIDE: So where does this come from ?
Its needed to ensure

$$\frac{d\theta}{\theta}=\frac{1}{\gamma}\frac{dp}{p}-\frac{d\rho}{\rho}$$

(where

$$\gamma=\frac{c_p}{c_v}$$

)

PROOF:
integrate this, gives

$$\log\theta_*=\frac{1}{\gamma}\log p_*-\log\rho_*+\text{const}$$

but

$$\theta_*=\theta_s(1+\epsilon F\theta_0)$$
$$\rho_*=\rho_s(1+\epsilon F\rho_0)$$
$$p_*=p_s+\rho_s f_0ULp_0=p_s\left(1+f_0UL\frac{\rho_s}{p_s}p_0\right)$$
$$=p_s\left(1+\epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0\right)$$

The inner term in brackets is the reference hydrostatic scaling, ~ 1.
Substitute these expressions for ϑ, ρ* and p* into the log expression using the fact that to first order*

$$\log(1+\epsilon x)=\epsilon x$$

$$\rightarrow\epsilon F\theta_0=\frac{1}{\gamma}\epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0-\epsilon F\rho_0$$

$$\rightarrow\theta_0=\frac{1}{\gamma}\left(\frac{gH\rho_s}{p_s}\right)p_0-\rho_0$$

END OF ASIDE

details

Substitute this into the hydrostatic relation to elimintate density

$$\rho_0 = -\theta_0 + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right) p_0 = -\frac{\partial p_0}{\partial z} - \frac{p_0}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

$$\theta_0 = \frac{\partial p_0}{\partial z} + p_0\left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right)\right]$$

From the reference hydrostatic relation, the second term in square brackets can be written

$$= -\frac{1}{\gamma}\frac{\partial p_s}{\partial z}$$

but this is just

$$\theta_0 = \frac{\partial p_0}{\partial z} - p_0\left(\frac{1}{\theta_s}\frac{\partial \theta_s}{\partial z}\right)$$

and the term in brackets

$$= \frac{N^2 H}{g} \sim \frac{g'}{g} \sim \epsilon$$

so we can write the perturbation hydrostatic relation in terms of perturbation potential temperature:

$$\boxed{\theta_0 = \frac{\partial p_0}{\partial z}}$$

... put this back into the vorticity equation:

$$\frac{D_0}{Dt}\left[\beta y + \xi_o + \frac{f_0^2 L^2}{H^2 \rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial p_0}{\partial z}\right)\right] = 0$$

This is the non-d quasi-geostrophic potential vorticity.

Redimensionalise:

$$q = \beta y + \xi_0 + \frac{f_0}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial}{\partial z}\left(\frac{p_0}{\rho_s}\right)\right)$$

introduce a dimensional geostrophic streamfunction

$$\psi = \frac{p_0}{\rho_s f_0}$$

get

$$\boxed{q = \beta y + \nabla^2\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial \psi}{\partial z}\right)}$$

This is the full quasi-geostrophic potential vorticity for a compressible stratified fluid.

Note: for stratified Boussinesq fluids this form reduces to

$$\boxed{q = \beta y + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)}$$

(this is OK for the ocean).

q is conserved following the flow:

$$\boxed{\frac{\partial q}{\partial t} + J(\psi, q) = 0}$$

Everything is represented in terms of one prognostic equation in one variable (the streamfunction).

One variable to rule them all

Since ψ is the only variable in the system, it must be possible to express anything in term of ψ, and is !

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \quad w = \frac{f}{N^2}\left[\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right]\frac{\partial \psi}{\partial z}$$

$$p' = \rho_0 f_0 \psi, \quad \rho' = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}$$

Knowledge of q plus boundary conditions leads to knowledge of ψ, and hence the advecting flow.

Prediction becomes a sequence of operations:

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = 0,$$

1) diagnose q
2) integrate the prognostic equation forward in time to find the next values of q
3) apply boundary conditions and invert the elliptic operator to find ψ
4) rinse and repeat

$$q = \beta y + \nabla^2 \psi + \mathrm{fn}(\psi_z)$$

Development

We approximate the quasi-geostrophic potential vorticity equation as

$$\frac{\partial}{\partial t}\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) + \mathbf{v}.\nabla\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) = 0$$

rearrange the derivatives on the variable ψ:

$$\left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial\psi}{\partial t} = -\mathbf{v}.\nabla\left(\nabla^2\psi + f\right) - \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

Now assume ψ is a wavelike disturbance with a sign change in the vertical (first baroclinic mode)

$$\psi \propto \sin lx\ \sin my\ \cos \pi z/H$$

$$\rightarrow \left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right) = -\left(l^2 + m^2 + \frac{f^2\pi^2}{N^2H^2}\right)$$

so

$$\frac{\partial\psi}{\partial t} \propto +\mathbf{v}.\nabla(\nabla^2\psi + f) + \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

The terms on the right hand side generate the tendency in ψ. So a local rate of change of ψ, or equivalently a change of pressure or geopotential, is proportional to...

Advection of absolute vorticity

$$\frac{\partial \psi}{\partial t} \propto \mathbf{v}.\nabla(\nabla^2\psi + f)$$

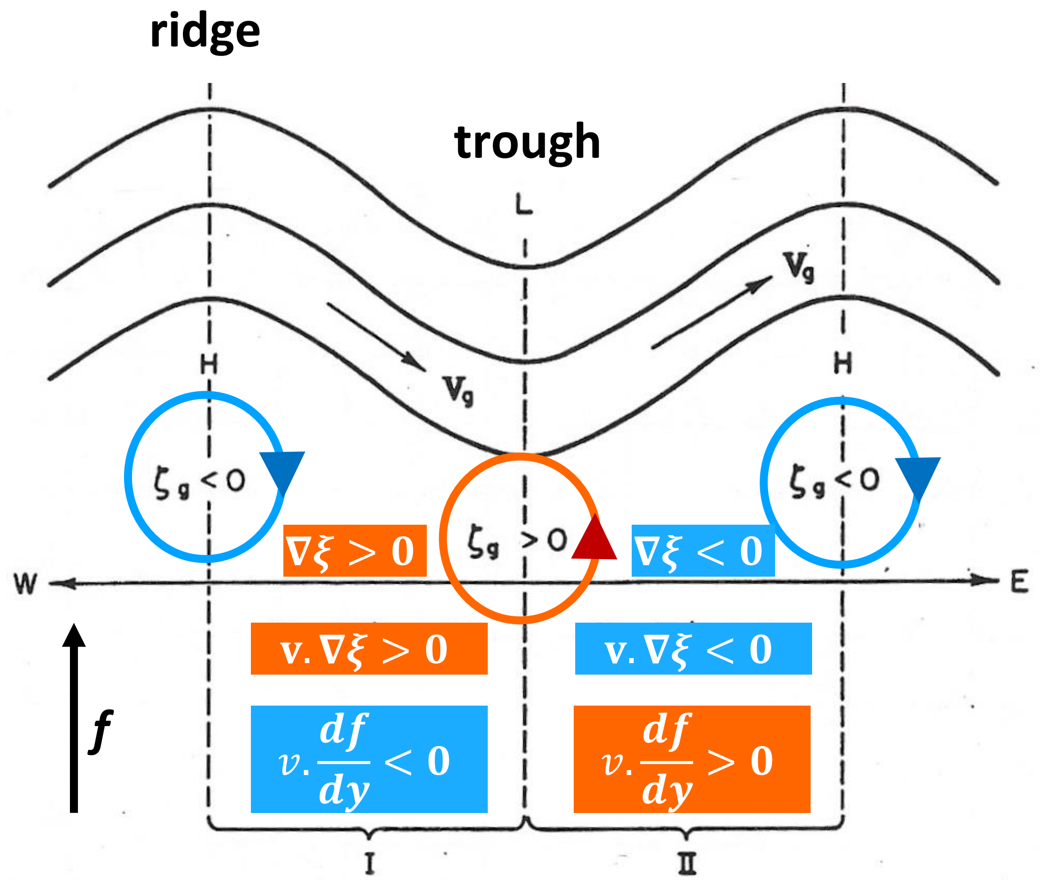
We see from the picture that zonal advection of relative vorticity sends troughs and ridges east. Meridional advection of planetary vorticity sends troughs and ridges west. Which process wins ?

$$\nabla^2\psi = -(l^2 + m^2)\psi$$

Long waves go west, f dominates (Rossby waves).
Short waves go east, ξ dominates

For short waves $+\mathbf{v}.\nabla\xi$ positive $\Rightarrow \frac{\partial \psi}{\partial t}$ positive

so a ridge in region I propagates east. But the tendency is zero at the axes of the ridges and troughs, so no amplification.



Vertical gradient of temperature advection

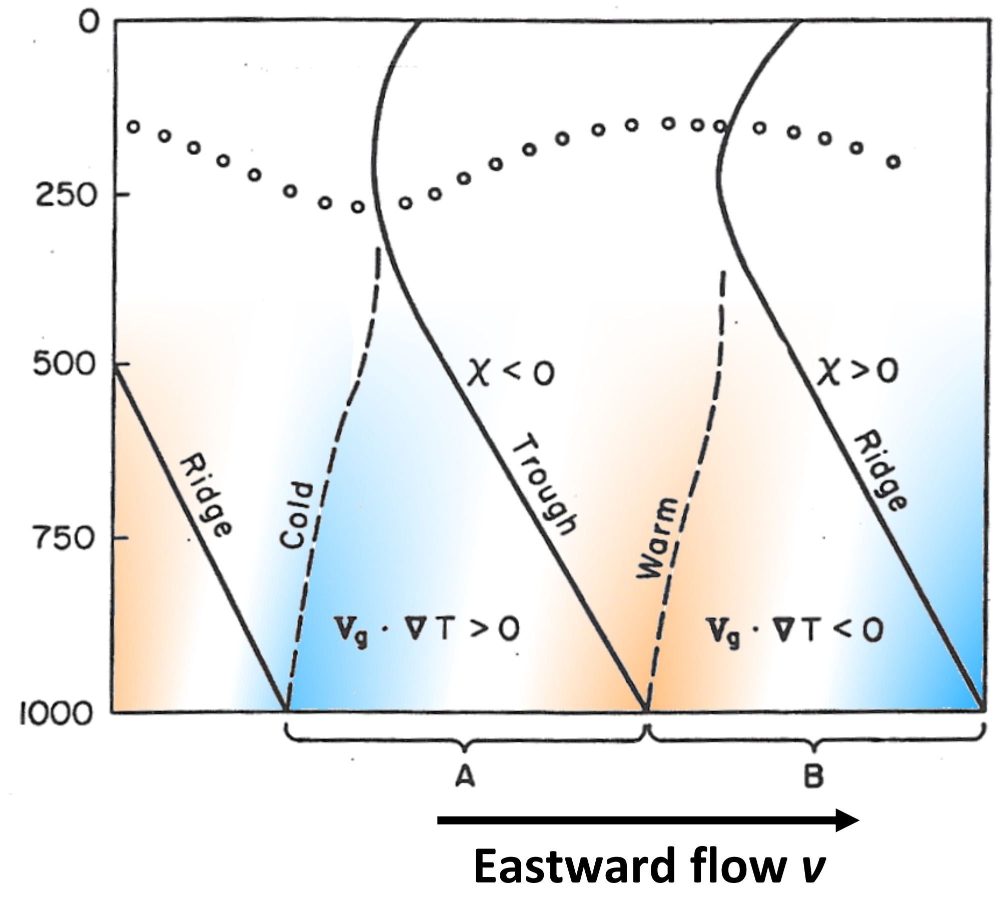
$$\frac{\partial \psi}{\partial t} \propto \frac{\partial}{\partial z}\left(\mathbf{v}.\nabla \frac{\partial \psi}{\partial z}\right) \propto \frac{\partial}{\partial z}\left(\mathbf{v}.\nabla \theta\right) \quad \left[\theta_0 = \frac{\partial p_0}{\partial z}\right]$$

This is sometimes called the "differential thickness advection"

(ever noticed how synopticians love talking in multiple derivatives ?)

If we have warm advection at low levels then this term is positive and a ridge is created.

If we have cold advection at low levels this term is negative and a trough is created.



Vertical velocity

The quasi-geostrophic system allows us to do a more accurate diagnosis than we can do with 3-d nondivergence which suffers from large cancellation.

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

The first order Boussinesq thermodynamic equation yields

$$w = -B_u \left(\frac{\partial}{\partial t}\frac{\partial p_o}{\partial z} + \mathbf{v}_0.\nabla \frac{\partial p_0}{\partial z}\right)$$

Compare the Laplacian of this equation with the vertical derivative of the vorticity equation

$$\nabla^2 \left(\frac{\partial}{\partial z}\frac{\partial p_o}{\partial t}\right) = -\nabla^2 \left(\mathbf{v}_0.\nabla \frac{\partial p_0}{\partial z}\right) - B_u^{-1}\nabla^2 w_1$$

(using $\xi_0 = \nabla^2 p_0$)

$$\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial t}(f+\xi_0) + \mathbf{v}_0.\nabla(f+\xi_0) = \frac{\partial w_1}{\partial z}\right\} \rightarrow \nabla^2\left(\frac{\partial}{\partial z}\frac{\partial p_0}{\partial t}\right) = \frac{\partial^2 w_1}{\partial z^2} - \frac{\partial}{\partial z}\left(\mathbf{v}_0.\nabla(f+\xi_0)\right)$$

equate right hand sides

$$\left(B_u^{-1}\nabla^2 + \frac{\partial^2}{\partial z^2}\right) w_1 = \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f+\xi_0)) - \nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_o}{\partial z}\right)$$

Note that this time we have eliminated the tendency term (rather than the vertical velocity term) between the vorticity and thermodynamic equations and obtained a diagnostic equation for w (rather than a prognostic equation for ψ). It's an elliptic equation for vertical velocity in terms of the geostrophic streamfunction. It's often called the Omega Equation (usually derived in pressure coordinates).

Recap

Tendency equation: $$\left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial \psi}{\partial t} = -\mathbf{v}.\nabla\left(\nabla^2\psi + f\right) - \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial \psi}{\partial z}\right)$$

Geopotential (fall/rise) proportional to:

A) (+/-) vorticity advection
B) rate of decrease with height of (cold/warm) advection.

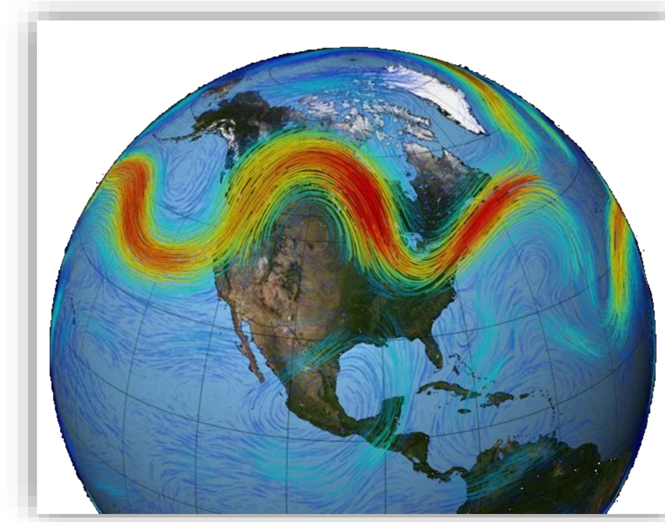
Omega equation: $$\left(B_u^{-1}\nabla^2 + \frac{\partial^2}{\partial z^2}\right)w_1 = \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f + \xi_0)) - \nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_o}{\partial z}\right)$$

(rising/sinking) motion proportional to:

A) rate of increase with height of (+/-) vorticity advection
B) (warm/cold) advection

Chapter 3: Rossby waves and instability

- ⇨ Parcel displacements and the conservation of potential vorticity
- ⇨ The **Rossby wave** dispersion relation
- ⇨ Topographic RW, baroclinic RW and vertical modes



- ⇨ Parcel displacements in shear flow
- ⇨ Barotropic **instability** and the necessary conditions for growth
- ⇨ Scales and structures for baroclinic growth and the Eady problem

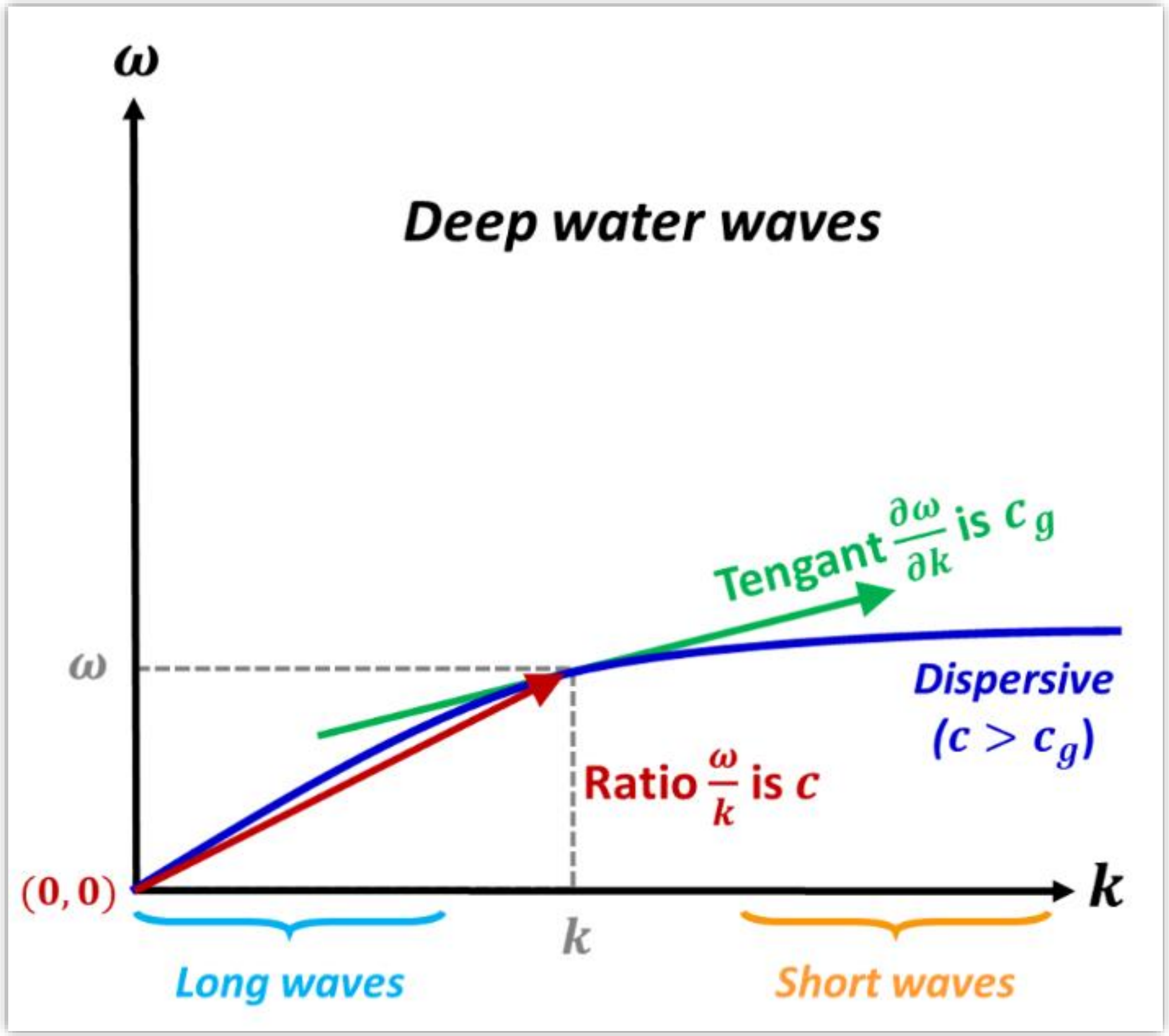
$$e^{i(lx+my-\omega t)}$$

Some Recap

$$e^{i(kx-\omega t)}$$

⇢ **the phase speed** (c) is the arrow that points from the origin toward the curve (the ratio $\frac{\omega}{k}$)

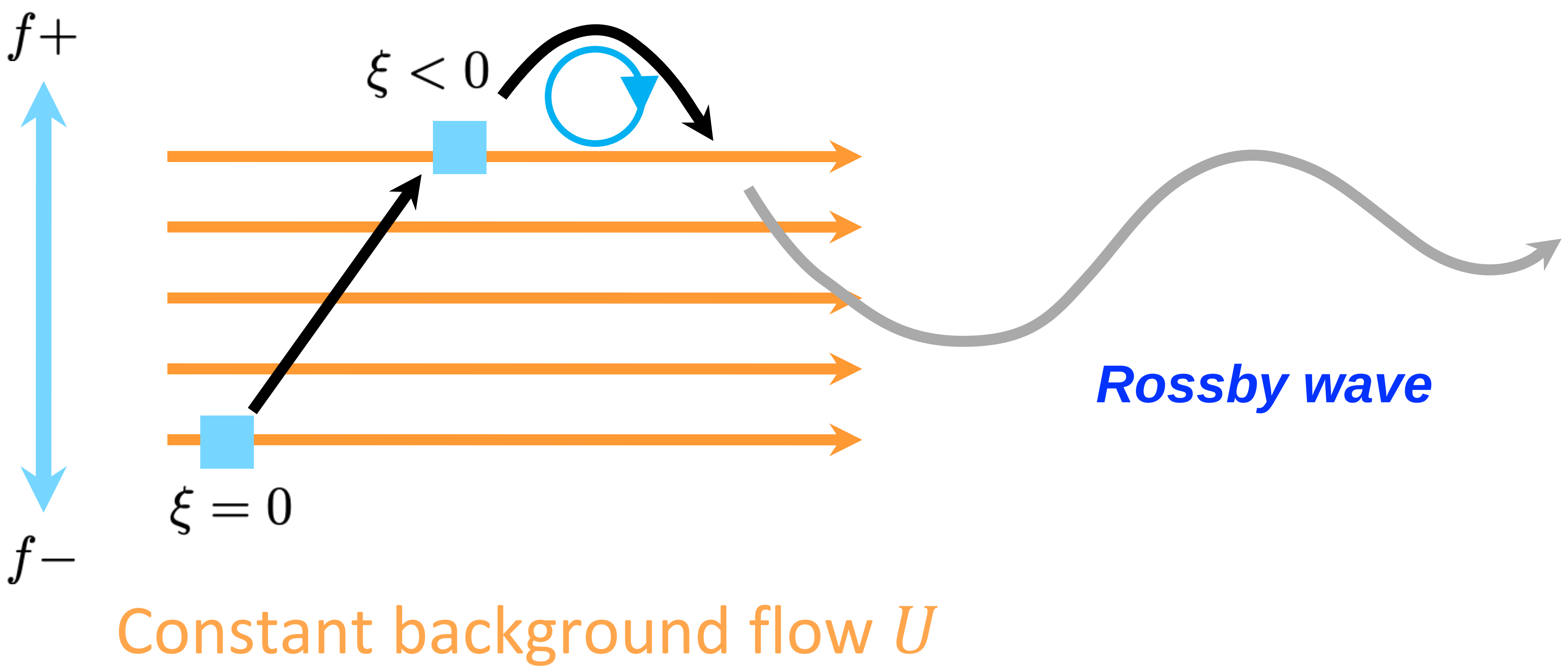
⇢ **the group speed** (c_g) is the tangent to the curve ($\frac{\partial\omega}{\partial k}$)



Parcel displacements in a vorticity gradient

- Consider a parcel of fluid that **conserves** its **absolute vorticity** in a westerly current

$$f + \xi = \text{const}$$



The conservation of vorticity

⇨ *Let's look at* ***various forms*** *of the vorticity equation in a westerly flow* $\dfrac{Dq}{Dt} = 0$

where D/Dt is given by $$\frac{D}{Dt} = \frac{\partial}{\partial t} + (U + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$$ *(prime denotes small perturbation)*

... and ***q can take various forms*** :

1) Nondivergent barotropic

$$q = \beta y + \nabla^2 \psi$$

2) Single layer of variable thickness

$$q = \beta y + \nabla^2 \psi - L_R^{-2} \psi$$

3) Two active quasi-geostrophic layers with a flat bottom and a rigid lid

$$q = \beta y + \nabla^2 \psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial \psi}{\partial z}$$

$$\left(N^2 = \frac{g'}{\overline{\overline{H}}} \qquad L_{1,2} = \frac{\sqrt{g' H_{1,2}}}{f} \right)$$

$\dfrac{\partial \psi}{\partial z} = 0$

H_1 $$q_1 = \beta y + \nabla^2 \psi_1 - L_1^{-2}(\psi_1 - \psi_2)$$

H_2 $$q_2 = \beta y + \nabla^2 \psi_2 + L_2^{-2}(\psi_1 - \psi_2)$$

$\dfrac{\partial \psi}{\partial z} = 0$

1) Nondivergent barotropic case

In the first case we write down the vorticity equation $\frac{Dq}{Dt} = 0$ *as:*

$$q = \beta y + \nabla^2 \psi$$

u=U+u'   *v=v'*

$$\frac{\partial}{\partial t}(\beta y + \nabla^2 \psi) + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}(\beta y + \nabla^2 \psi) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\beta y + \nabla^2 \psi) = 0$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (U + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial t}\nabla^2 \psi + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}\nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2 \psi = 0$$

Note that q is conserved with the flow. $\Psi = \Psi_B + \Psi$, with $\Psi_B = -Uy$. $\nabla^2 \Psi_B = 0$
It is crossed out from the PV equation

*The **linear** equation in perturbations ψ is*

$$\frac{\partial}{\partial t}\nabla^2 \psi + U\frac{\partial}{\partial x}\nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} = 0$$

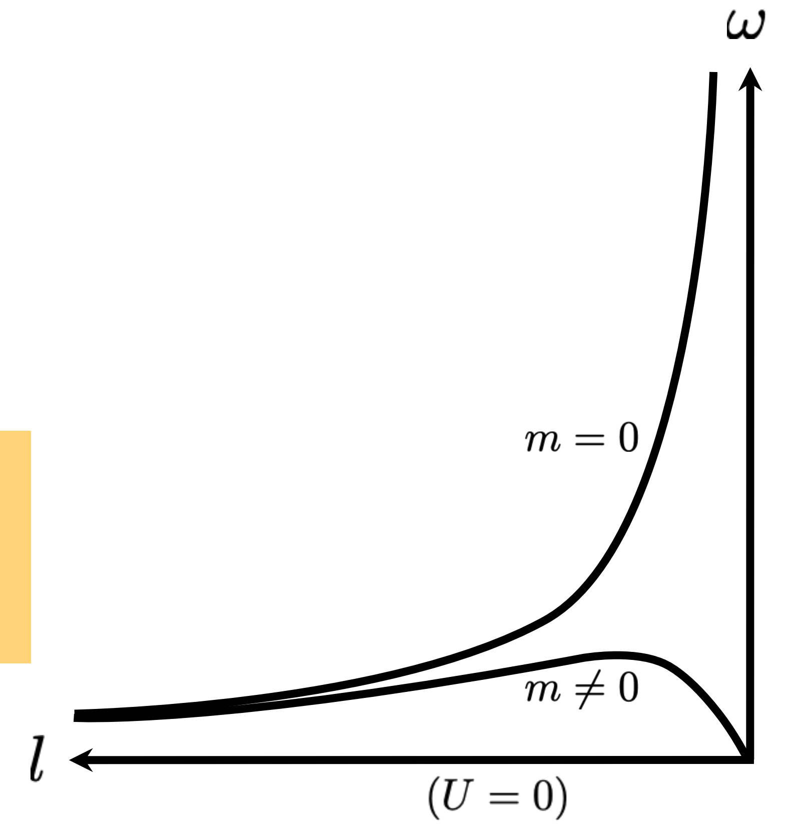
Look for zonal wave solutions of the form $\psi = Re\, \tilde{\psi} e^{i(lx + my - \omega t)}$

Substitution into the derivatives gives algebraic expressions

$$\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \frac{\partial}{x} \rightarrow il \qquad \nabla^2 \rightarrow -(l^2 + m^2)$$

⇨ *Leads to the dispersion relation:*

$$\omega = Ul - \frac{\beta l}{l^2 + m^2}$$



- l is the **zonal wave number** (2π divided by the x-wavelength)
- m is the **meridional wave number** (2π divided by the y-wavelength)
- ω is the **angular frequency** (2π divided by the period)

details

$$\frac{\partial}{\partial t}\nabla^2\psi + U\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0$$

$$\frac{\partial}{\partial t} \to -i\omega \quad \frac{\partial}{x} \to il \quad \nabla^2 \to -(l^2+m^2)$$

$$i\omega(l^2+m^2) - il(l^2+m^2)U + il\beta = 0$$
$$\omega(l^2+m^2) = l(l^2+m^2)U - l\beta$$

$$\omega = lU - \frac{l\beta}{(l^2+m^2)}$$

$$\frac{\omega}{l} = c_x = U - \frac{\beta}{(l^2+m^2)}$$

$$\frac{\partial\omega}{\partial l} = U - \beta\frac{\partial}{\partial l}(l(l^2+m^2)^{-1})$$

$$\frac{1}{(l^2+m^2)} + l(-(l^2+m^2)^{-1}\times 2l) = \frac{(l^2+m^2)-2l^2}{(l^2+m^2)^2} = -\frac{l^2-m^2}{(l^2+m^2)^2}$$

$$\frac{\partial\omega}{\partial l} = U + \beta\frac{l^2-m^2}{(l^2+m^2)^2}$$

Rossby wave dispersion

Dispersion relation

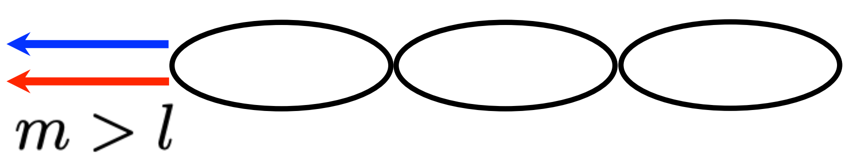
$$\omega = Ul - \frac{\beta l}{l^2 + m^2}$$

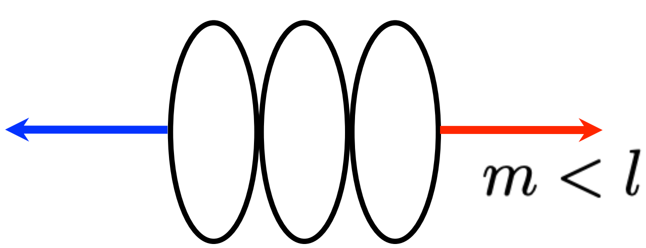
The phase speed and group speed in the x direction are given by

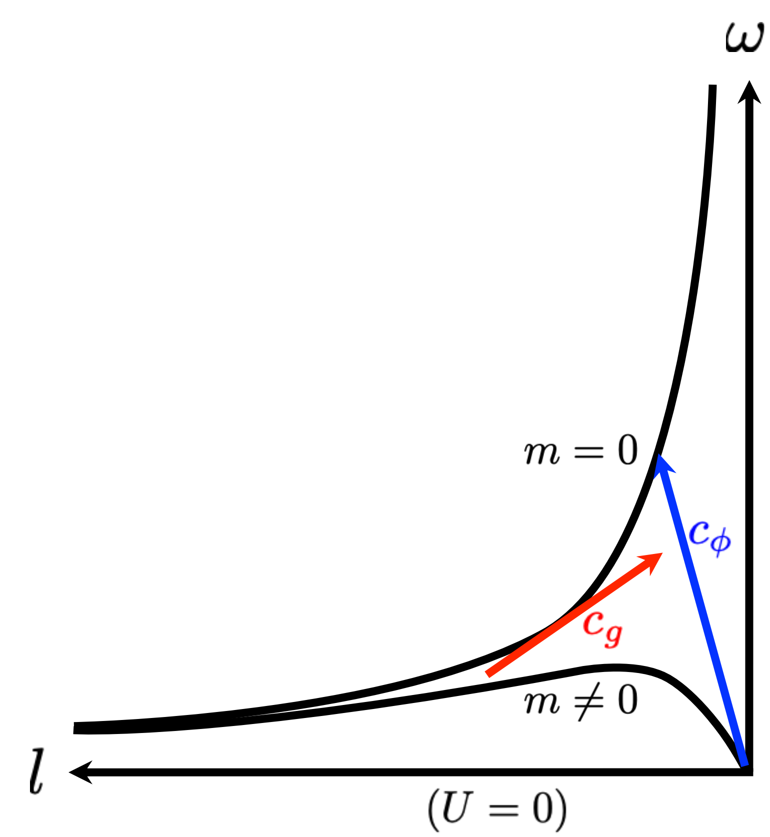
$$c = \frac{\omega}{l} = U - \frac{\beta}{l^2 + m^2} \qquad c_g = \frac{\partial \omega}{\partial l} = U + \frac{\beta(l^2 - m^2)}{(l^2 + m^2)^2}$$

The phase speed is westwards relative to the mean flow.

The group speed depends on the zonal and meridional scale of the wave.



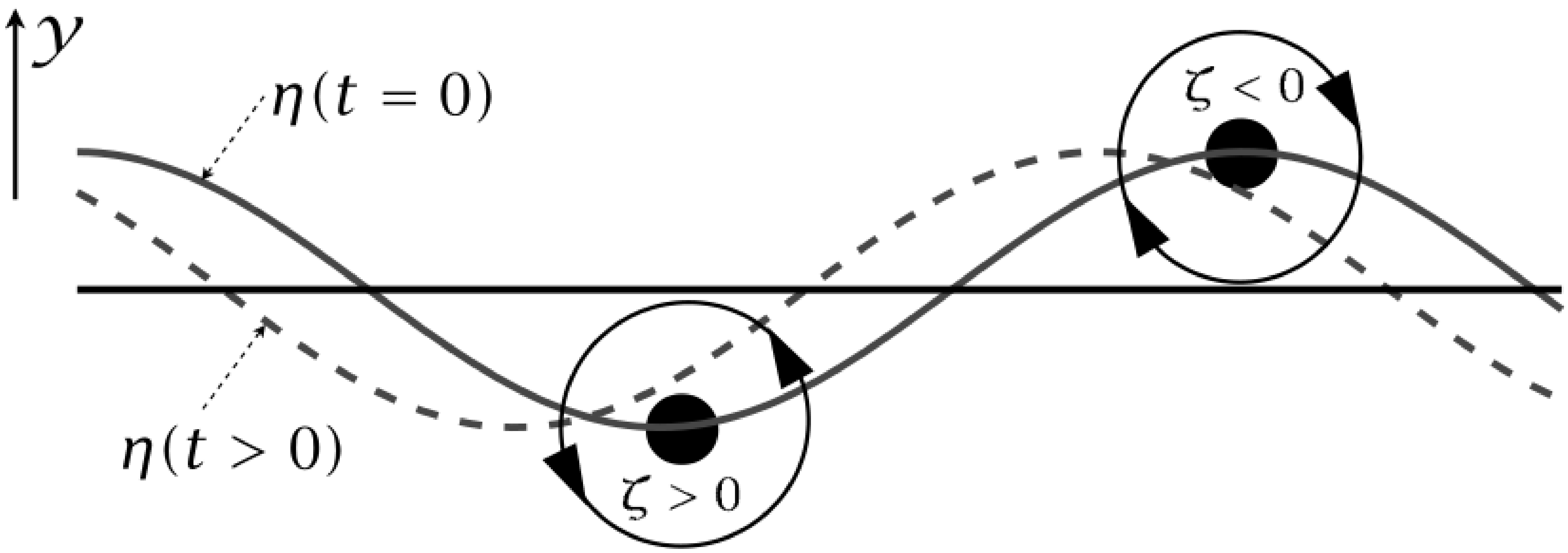




Longer waves (smaller k^2) travel faster.
Waves closer to the equator (bigger β) travel faster.

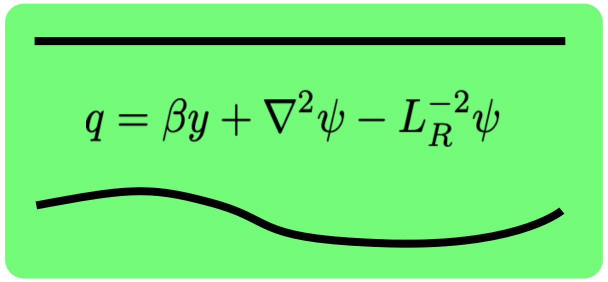
Rossby wave propagation mechanism

Can be understood in terms of the conservation of potential vorticity. When a parcel of fluid changes latitude, to compensate for its changing planetary vorticity, it must acquire either positive or negative relative vorticity. This induces a circulation that leads to the westward propagation of the disturbance.



2) Divergent case (variable layer thickness)

If we allow some vortex stretching in the conservation law, there is some modification of the Rossby wave characteristics. We linearize



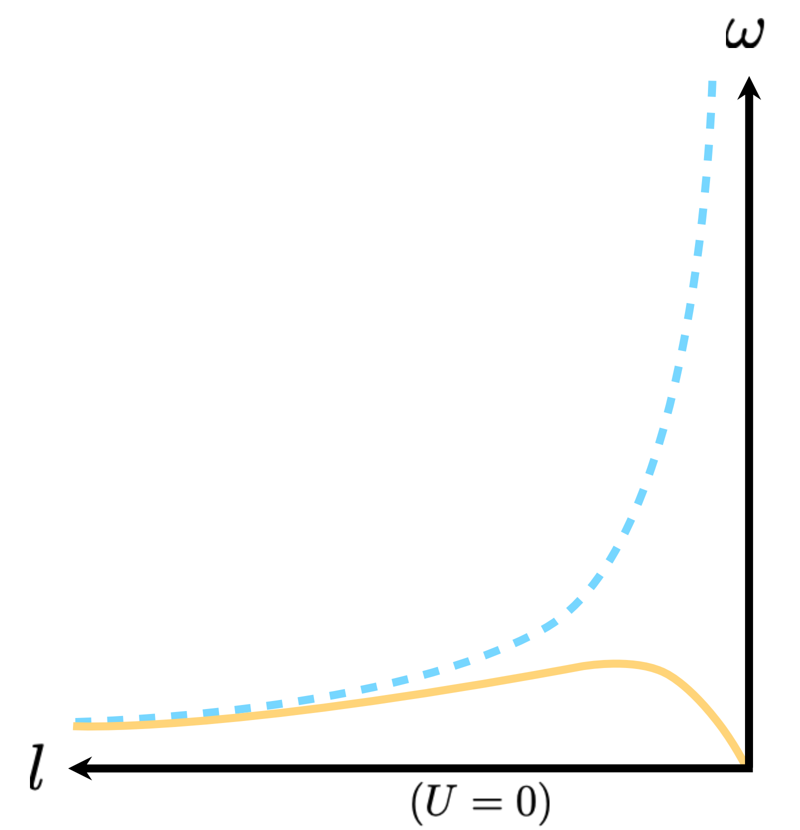
- In the PV conservation equation, the stream function is the summed-up contribution of:
  - the stream function associated with the perturbation ψ
  - the **background flow stream function** ψ_B=-Uy

$$\left(\frac{\partial}{\partial t} + (U+u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\left(\beta y + \nabla^2\psi - L_R^{-2}(\psi - Uy)\right) = 0$$

so $$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\nabla^2\psi - L_R^{-2}\psi) + (\beta + L_R^{-2}U)\frac{\partial\psi}{\partial x} = 0$$

material tendency of perturbation relative vorticity and vortex stretching term

perturbation advection of planetary vorticity and basic state stretching term



$\tilde{\psi}e^{i(lx+my-\omega t)}$ ⇨ *The dispersion relation is now* $\omega = Ul - l\dfrac{\beta + L_R^{-2}U}{l^2 + m^2 + L_R^{-2}}$

The current no longer just provides a simple doppler shift, but actively changes the basic state PV gradient, altering the propagation speed of the waves.

Note also that the denominator does not go to zero, so the phase speed is bounded and long waves are much less dispersive, with group speed to the west, even when m=0.

Topographic Rossby waves

Quasi-geostrophic potential vorticity

$$q = f_0 + \beta y + \xi - \frac{f_0}{H}\delta h$$

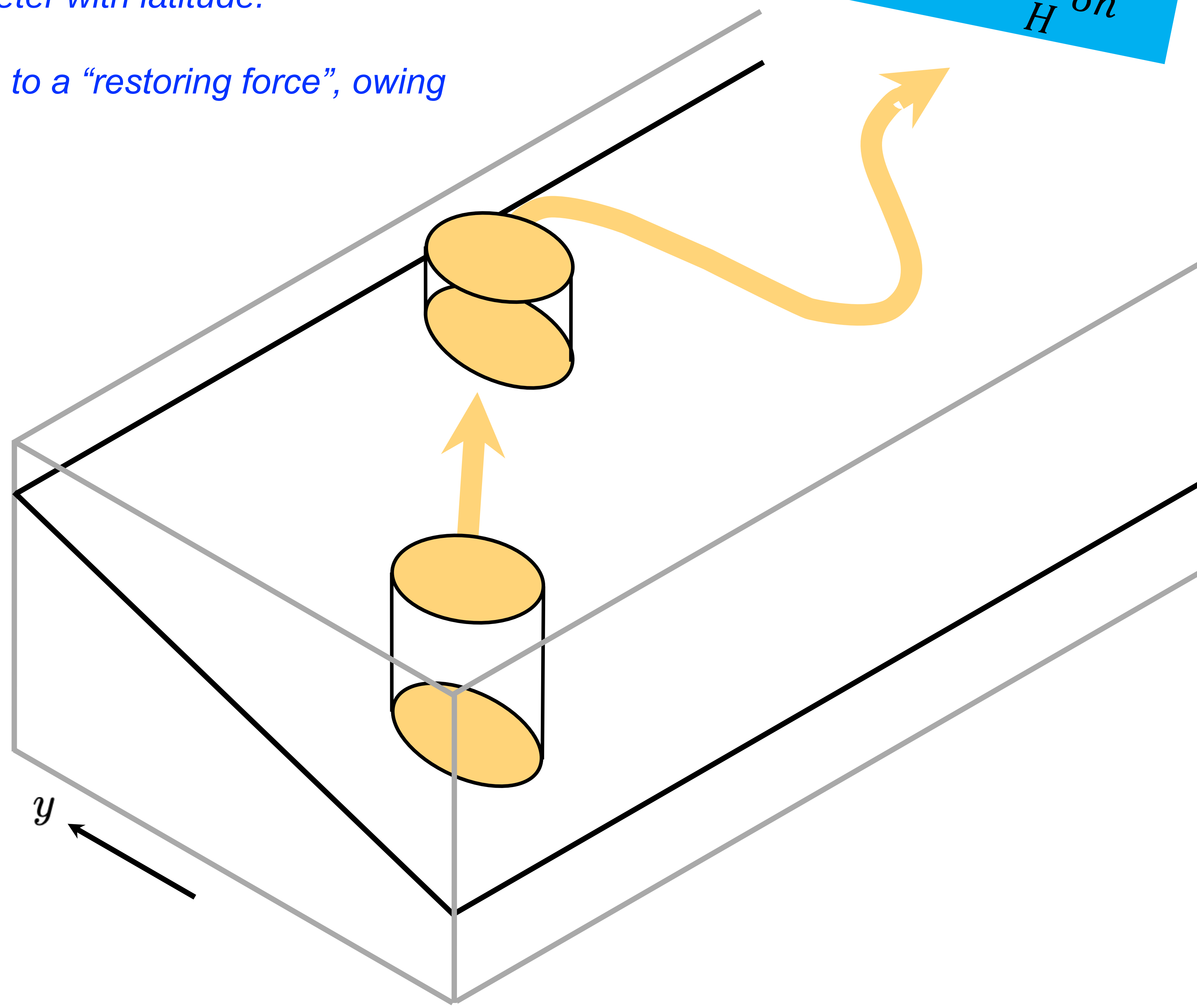
Vortex stretching can be important for Rossby waves in situations where there is a sloping bottom ($h_{OC} = \alpha y$). It is analogous to the effect of changing the Coriolis parameter with latitude.

Both are geometric effects giving rise to a “restoring force”, owing to the generation of relative vorticity.

$$q = f_0 + \beta y + \xi - \frac{f_0}{H}(\alpha y + \eta)$$

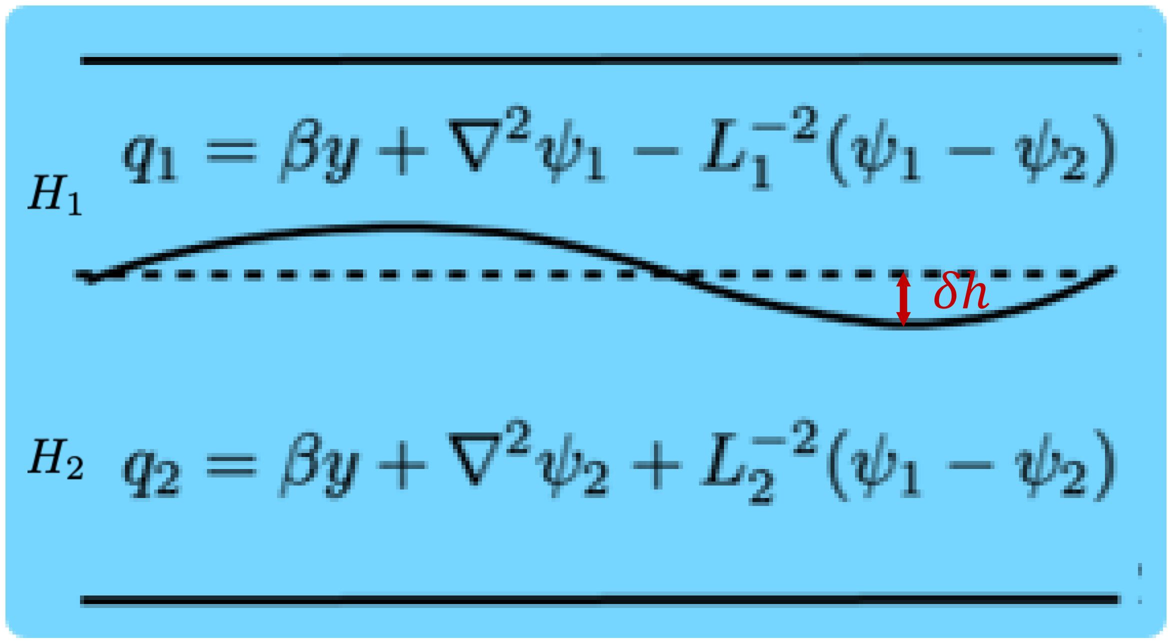
In the northern hemisphere, an ocean floor that is shallowing to the north will have the same effect as beta.

In the southern hemisphere the ocean floor must shallow to the south.



3) Two active layers

Two active quasi-geostrophic layers with a flat bottom and a rigid lid. Ignore advecting current for simplicity.



⇨ In this framework, we have two active layers in which the quasi-geostrophic potential vorticity is conserved (cf. **#GFD2.3h**):

$$q_1 = \frac{f + \xi_1}{h_1} \approx f + \xi_1 - \frac{f}{H_1}\delta h$$
$$q_2 = \frac{f + \xi_2}{h_2} \approx f + \xi_2 + \frac{f}{H_2}\delta h$$

⇨ We retrieve **geostrophic stream functions** for each layer (cf. **#GFD1.4e**):

$$\boldsymbol{f}_0 \times \boldsymbol{u}_1 = -\frac{1}{\rho_0}\nabla P_1 = -g\nabla(h_1 + h_2) \quad \text{and} \quad \boldsymbol{f}_0 \times \boldsymbol{u}_2 = -\frac{1}{\rho_0}\nabla P_2 = -g\nabla(h_1 + h_2) - g'\nabla h_2$$

$$\psi_1 = \frac{g}{f_0}(h_1 + h_2) \quad \text{and} \quad \psi_2 = \frac{g}{f_0}(h_1 + h_2) + \frac{g'}{f_0}h_2$$

⇨ The interface displacements (from the rigid lid) are $\delta h = -h_2 = \frac{f_0}{g'}(\psi_1 - \psi_2)$. Therefore, **the vortex stretching term** is a **coupled term** defined in terms of the **difference between both stream functions**.

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 - L_1^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_1}{\partial x} = 0$$
$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 + L_2^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_2}{\partial x} = 0$$

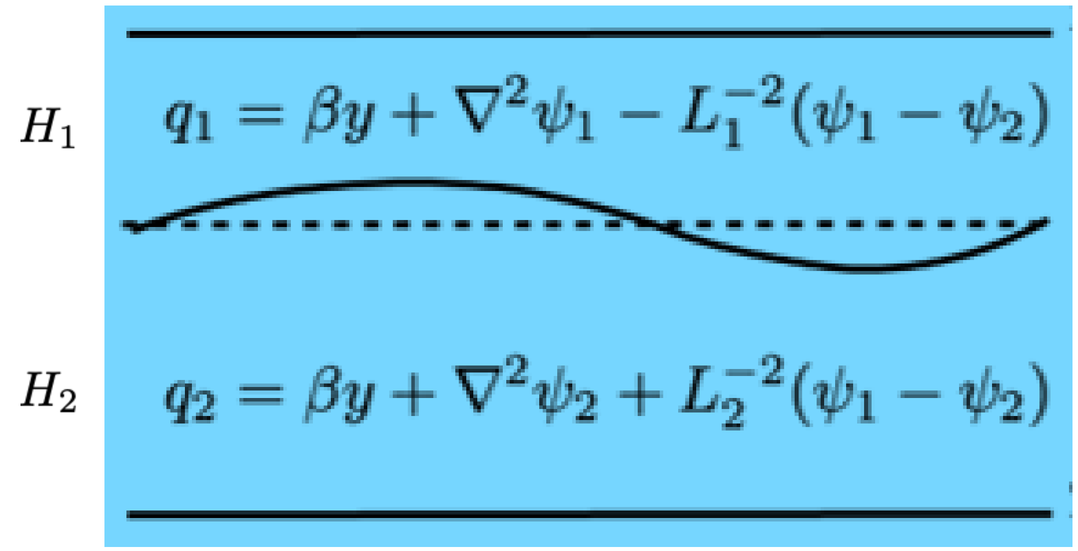
3) Two active layers

Two active quasi-geostrophic layers with a flat bottom and a rigid lid. Ignore advecting current for simplicity.

$$q = \beta y + \nabla^2\psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial \psi}{\partial z}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 - L_1^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_1}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 + L_2^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_2}{\partial x} = 0$$



We can uncouple these equations by subtraction and addition to find the "normal modes". We can then find independent solutions for the two modes.

$$\left(N^2 = \frac{g'}{\overline{\overline{H}}} \quad L_{1,2} = \frac{\sqrt{g' H_{1,2}}}{f}\right)$$

$$L_R^{-2} = L_1^{-2} + L_2^{-2}$$

$$\left(L_R = \frac{\sqrt{g'\hat{H}}}{f}, \quad \hat{H} = \frac{H_1 H_2}{H_1 + H_2}\right)$$

$$\overline{\psi} = \frac{L_2^{-2}\psi_1 + L_1^{-2}\psi_2}{L_1^{-2} + L_2^{-2}} = \frac{H_1\psi_1 + H_2\psi_2}{H_1 + H_2}$$

Barotropic Mode

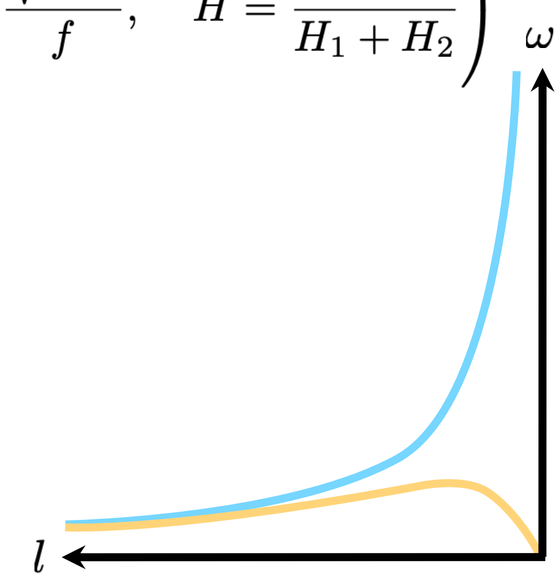
$$\hat{\psi} = \psi_1 - \psi_2$$

Baroclinic Mode

The equations become

$$\frac{\partial}{\partial t}\nabla^2\overline{\psi} + \beta\frac{\partial \overline{\psi}}{\partial x} = 0 \quad \rightarrow \quad \omega = -\frac{\beta l}{l^2 + m^2}$$

$$\frac{\partial}{\partial t}\left[(\nabla^2 - L_R^{-2})\hat{\psi}\right] + \beta\frac{\partial \hat{\psi}}{\partial x} = 0 \quad \rightarrow \quad \omega = -\frac{\beta l}{l^2 + m^2 + L_R^{-2}}$$



Extension to the vertical continuum

Consider quasi-geostrophic fluid bounded at top and bottom by rigid flat surfac
For simplicity we assume constant basic state stratification.

Quasi-geostrophic potential vorticity

$$q = \beta y + \nabla^2\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial\psi}{\partial z}\right)$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \left(\frac{f^2}{N^2}\right)\frac{\partial^2\psi}{\partial z^2}\right] + \beta\frac{\partial\psi}{\partial x} = 0$$ *with boundary condition* $$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial z}\right) = 0$$

seek wave solutions $\psi = Re\,\tilde{\psi}(z)e^{i(lx+my-\omega t)}$

with separable vertical dependence $\left(\frac{f^2}{N^2}\right)\frac{d^2\tilde{\psi}}{dz^2} = -\Gamma\tilde{\psi}$

In general, leads to dispersion relation

$$\omega = -\frac{\beta l}{k^2+\Gamma}$$

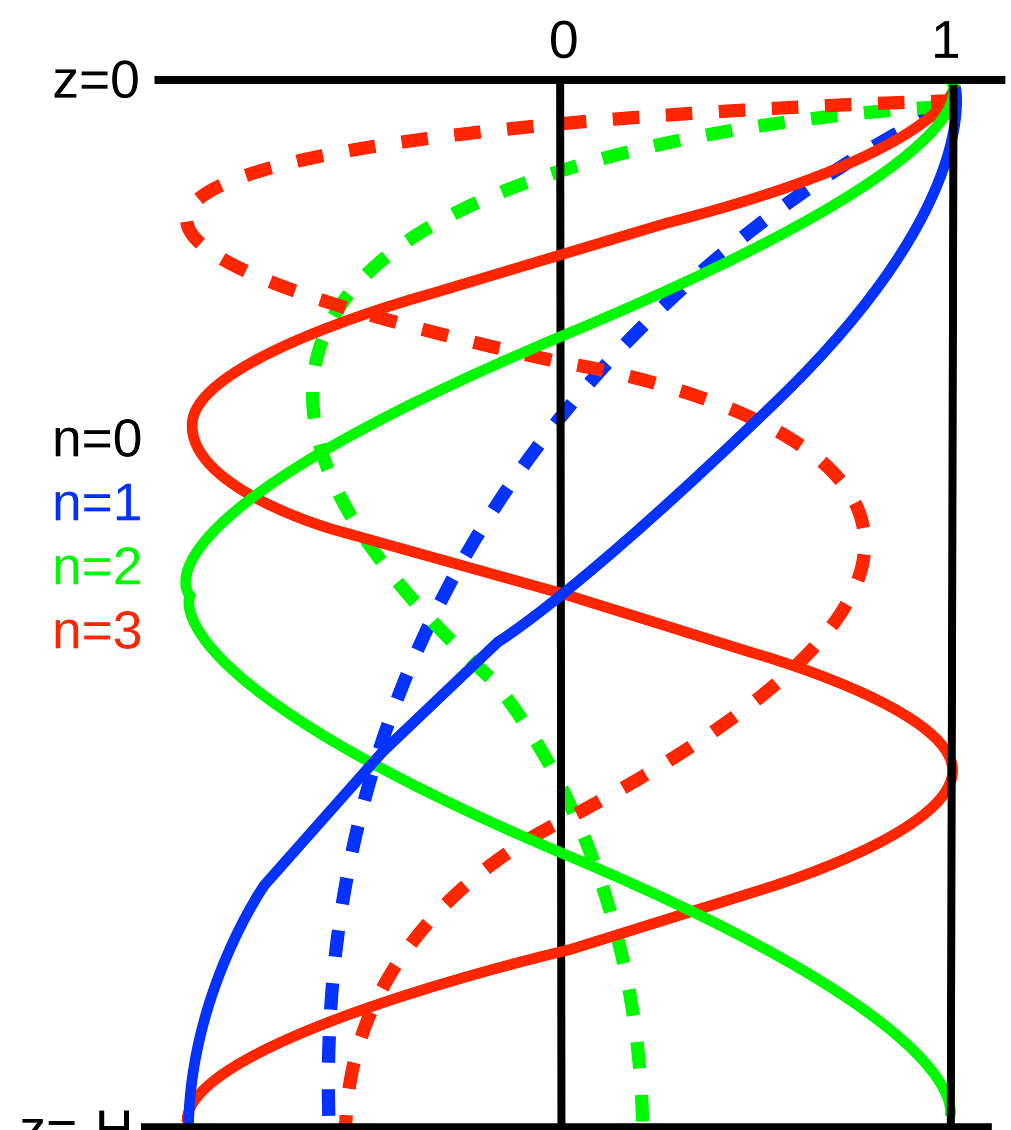
In this simple case the eigensolutions are cosines

$$\tilde{\psi}_n(z) = \cos(n\pi z/H), \quad n = 1, 2..., \quad k_v = n\pi/H$$

$$\omega = -\frac{\beta l}{k^2 + \underbrace{(f^2/N^2)k_v^2}_{\Gamma_n}}$$

$$L_n = \frac{NH}{n\pi f} = \frac{N}{f k_v} = \frac{c_n}{f}$$

$$\Rightarrow \quad c_n = \frac{N}{k_v}$$



Vertically propagating Rossby waves

Consider the vertical wavenumber for each mode

$$k_{vn} = n\pi/H = N/c_n$$

Remember c_n is the gravity wave speed associated with the vertical mode, not the phase speed of the Rossby wave !

Dispersion relation for long Rossby waves

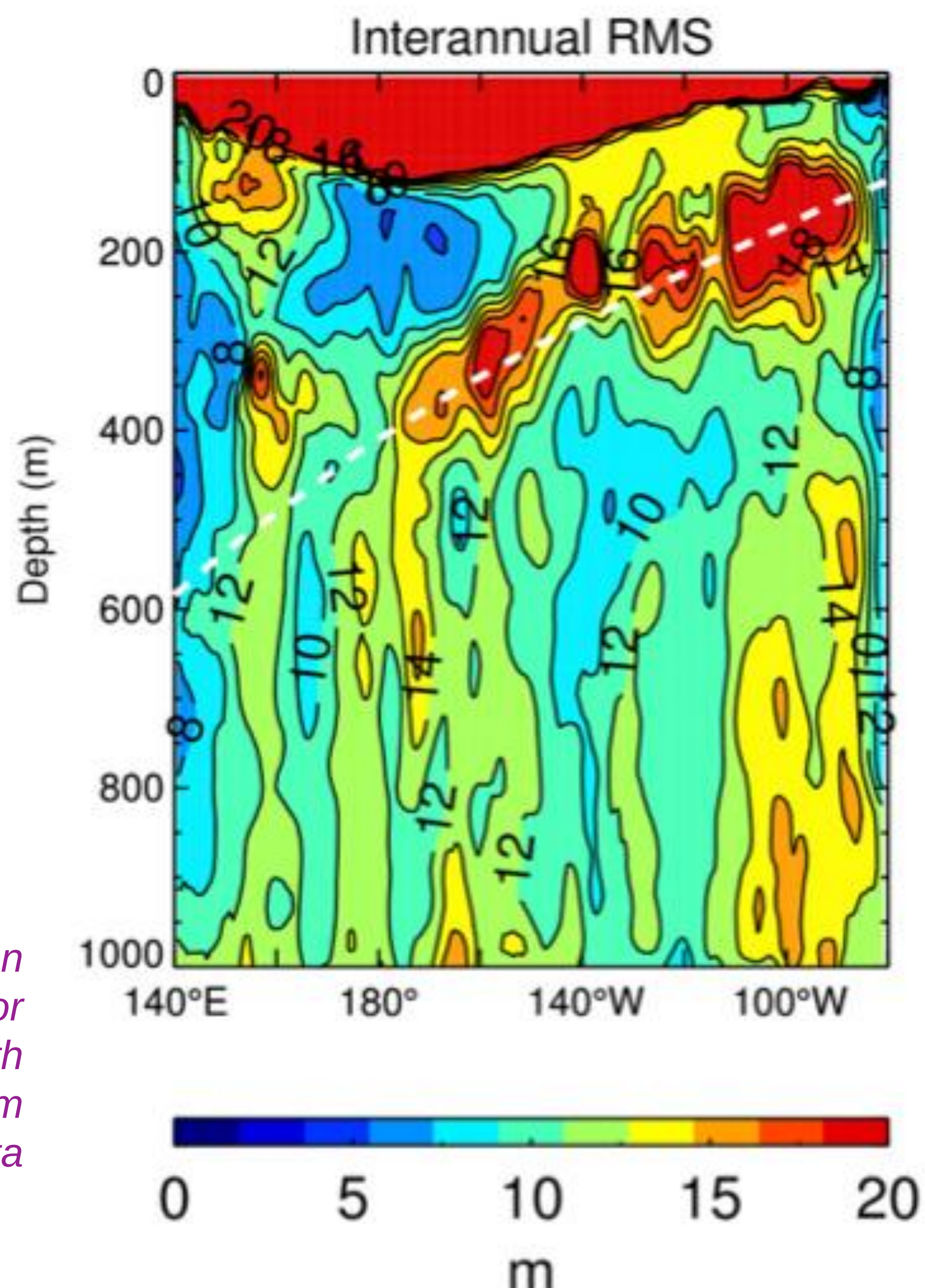
$$\omega = -\frac{\beta l}{k^2 + (f^2/N^2)k_v^2} \approx -\frac{\beta l N^2}{f^2 k_v^2} = -\frac{\beta l c_n^2}{f^2}$$

We can trace the signal path associated with vertical propagation in the x-z plane by calculating the ratio of components of the group velocity

$$\frac{\partial \omega}{\partial l} = -\frac{\beta N^2}{f^2 k_v^2} = -\frac{\beta c_n^2}{f^2}$$

$$\frac{\partial \omega}{\partial k_v} = \frac{2\beta l N^2}{f^2 k_v^3} = \frac{2\beta l c_n^3}{f^2 N}$$

So $$\frac{dz}{dx} = \frac{c_g^z}{c_g^x} = -\frac{2 l c_n}{N} = \frac{2 f^2 \omega}{\beta N c_n}$$

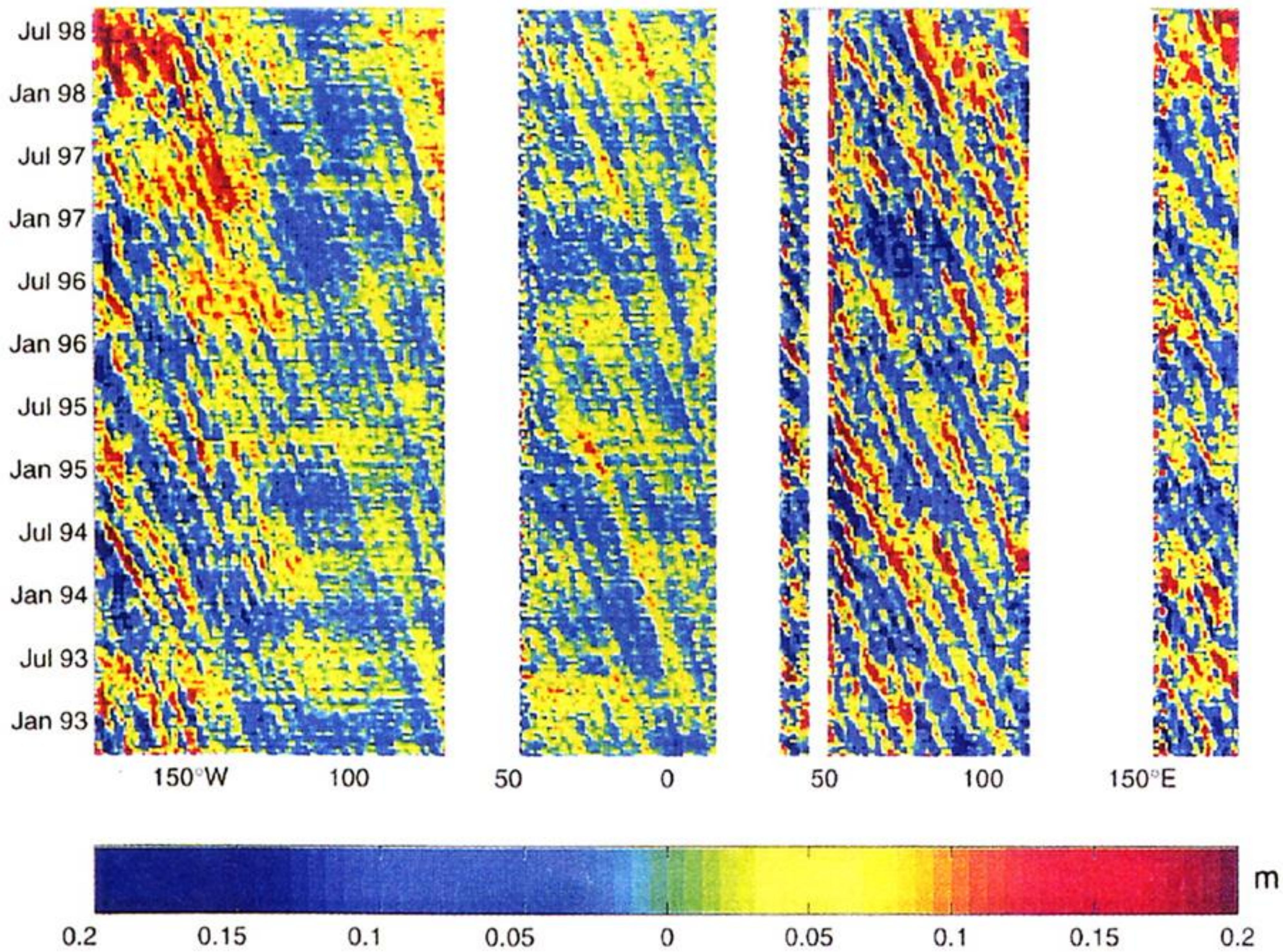


Propagation pathway for isotherm depth variability from ARGO data

(Vergara 2017)

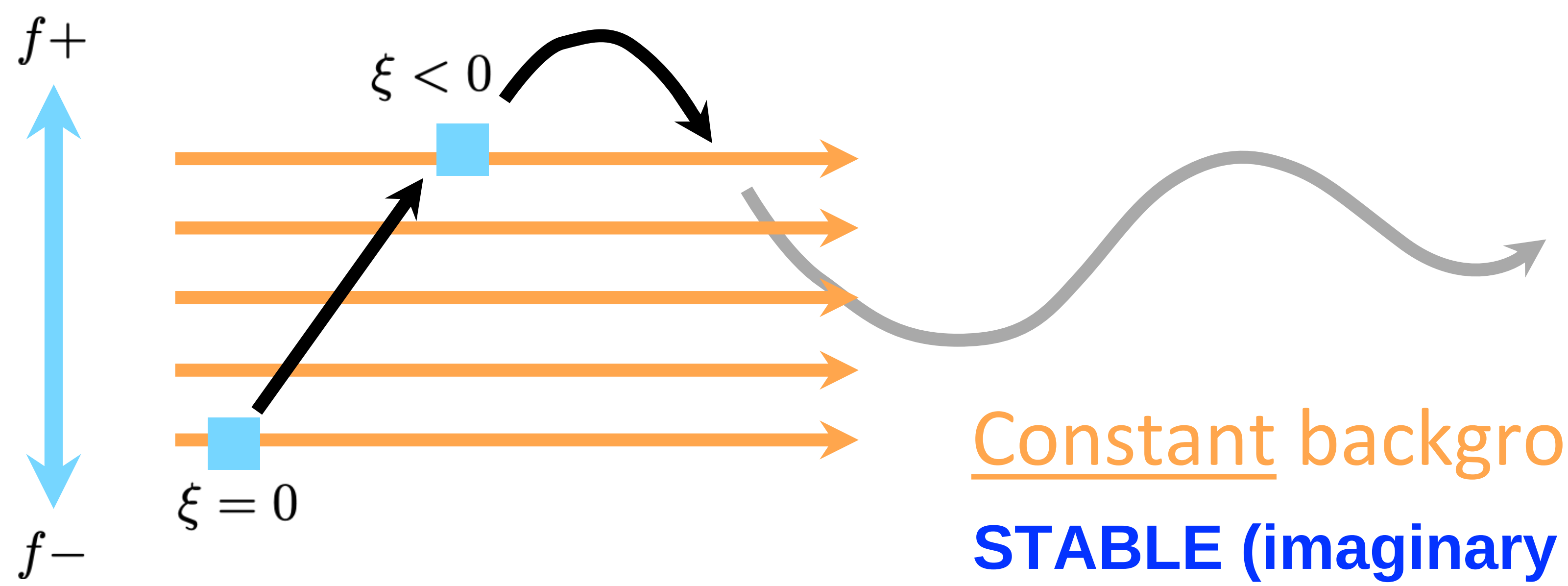
Observations

Evidence of Rossby wave propagation in satellite altimetry of the sea surface ?



Growing Rossby waves ?

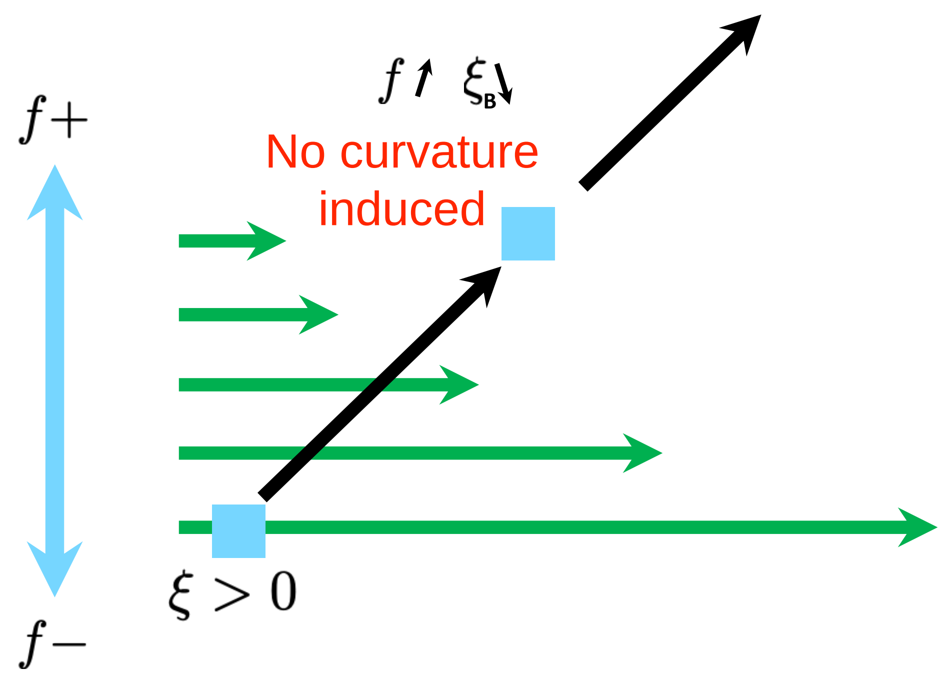
Let's revisit the mechanism for Rossby waves, but this time with horizontal shear



Constant background flow U

STABLE (imaginary exponential)

$$f + \xi = \text{const}$$

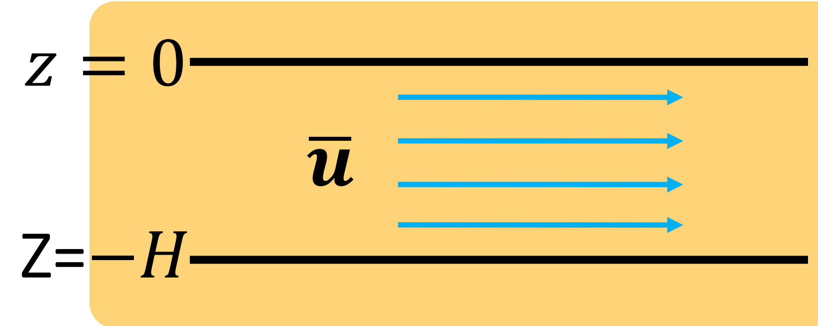


Shear background flow U

UNSTABLE (real exponential)

Perturbations on a shear flow

⇨ ***Barotropic nondivergent flow**: uniform in the vertical*

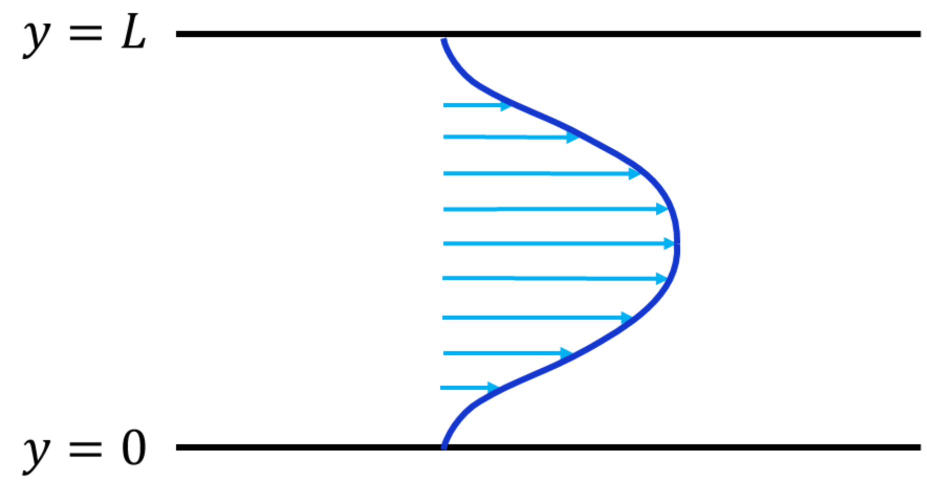


$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

choose a background flow that is a solution of the equations: $u = \overline{u}(y) = \frac{1}{\rho f}\frac{d\overline{p}}{dy}$

add perturbations: $u' = -\frac{\partial \psi}{\partial y}, \;\; v' = \frac{\partial \psi}{\partial x}$



⇨ *Leads to the perturbation barotropic vorticity equation:*

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0$$

advection of perturbation vorticity by basic state winds

advection of basic state absolute vorticity by perturbation winds

details

$$u'_t + \overline{u}u'_x + v'\overline{u}_y - fv' = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad -\psi_{yt} - \overline{u}\psi_{xy} + \overline{u}_y\psi_x - f\psi_x = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$v'_t + \overline{u}v'_x + \cancel{v'\overline{v}_y} + fu' = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad -\psi_{xt} - \overline{u}\psi_{xx} \qquad - f\psi_y = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-\psi_{yyt} - \overline{u}\psi_{xyy} - \cancel{\overline{u}_y\psi_{xy}} + \overline{u}_{yy}\psi_x + \cancel{\overline{u}_y\psi_{xy}} - \beta\psi_x - \cancel{f\psi_{xy}} - \psi_{xxt} - \overline{u}\psi_{xxx} + \cancel{f\psi_{xy}} = 0$$

$$-\frac{\partial}{\partial t}\nabla^2\psi - \overline{u}\frac{\partial}{\partial x}\nabla^2\psi + (\overline{u}_{yy} - \beta)\,\psi_x = 0$$

Stationary Rossby waves

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial\psi}{\partial x} = 0$$

As before, we can derive a dispersion relation for Barotropic Rossby waves, this time on a shear flow, by introducing solutions of the form $\psi = Re\,\tilde{\psi}e^{i(lx+my-\omega t)}$

$$\omega = Ul - \frac{(\beta - U_{yy})l}{l^2 + m^2}$$

Consider stationary waves: $\omega = 0$

$$\Rightarrow U(l^2 + m^2) = (\beta - U_{yy})$$

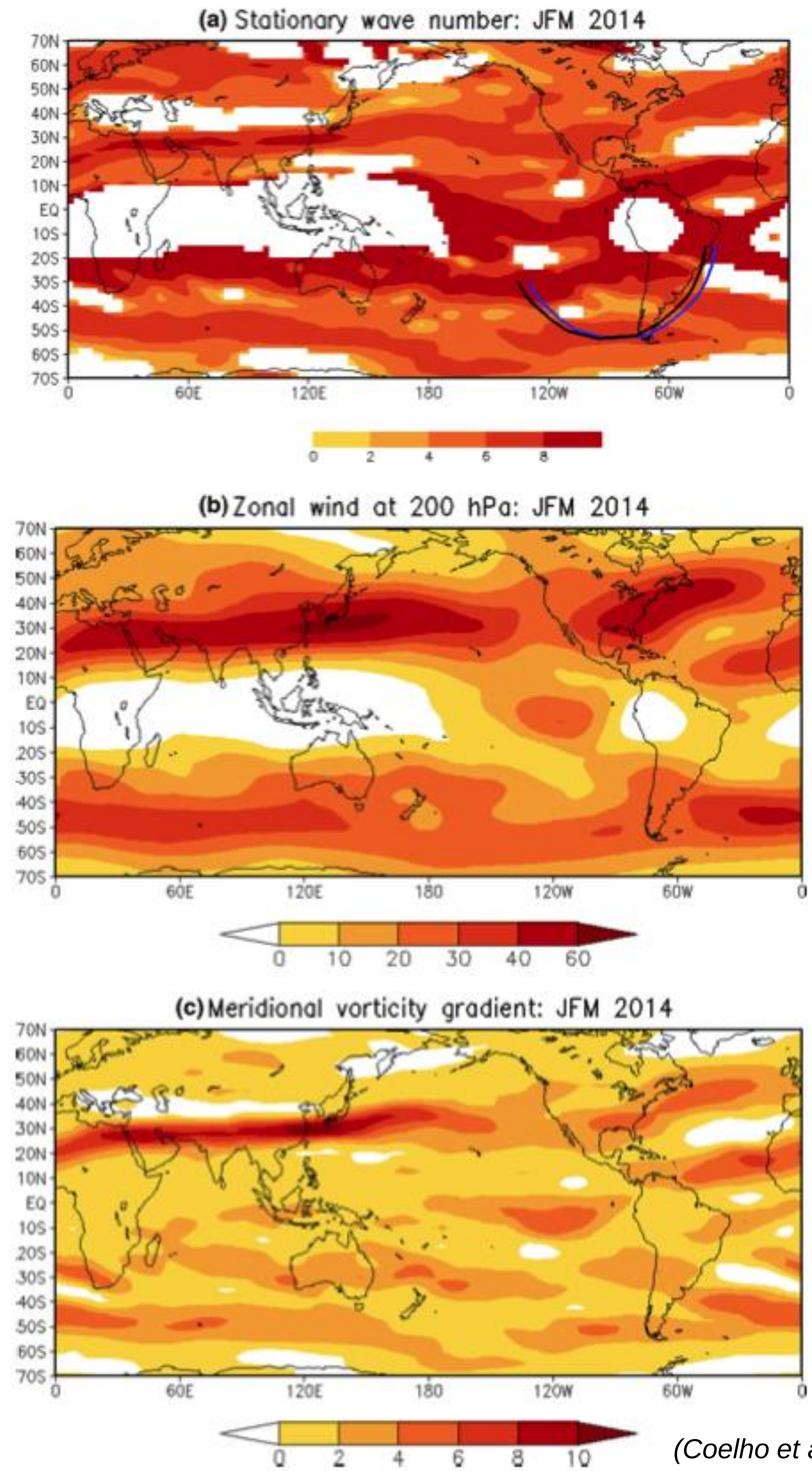
And the stationary wavenumber $k_s = \sqrt{(\beta - U_{yy})/U}$

For stationary Rossby waves to exist, $(\beta - U_{yy})$ *must have the same sign as* U *(which usually means both must be positive).*

Ray paths can be calculated as before from the ratio of components of the group velocity

$$\mathbf{c}_g = \left(U + \frac{\beta_*(l^2 - m^2)}{k^4}, -\frac{2\beta_* lm}{k^4}\right)$$

$(\beta_* = \beta - U_{yy},\ k^2 = l^2 + m^2)$



(Coelho et al 2016)

Growing solutions

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi+\left(\beta-\frac{d^2\overline{u}}{dy^2}\right)\frac{\partial\psi}{\partial x}=0$$

Now let's seek solutions in form of zonal wave with coefficients that depend on y

$$\psi(x,y,t)=\phi(y)e^{i(lx-\omega t)}$$

Substitute in, get

$$\frac{d^2\phi}{dy^2}-l^2\phi+\frac{\beta-d^2\overline{u}/dy^2}{\overline{u}(y)-c}\phi=0$$

the "Rayleigh equation" (where c = ω / l). If we add channel boundary conditions Φ = 0 at y = 0, L, in general we get a set of solutions for Φ associated with complex conjugate pairs of values for c (or ω). The imaginary part of the solution is associated with growth or decay. The growth rate is the imaginary part of ω.

$$c=c_r+ic_i,\;\; c^*=c_r-ic_i$$

$$\omega=\omega_r+i\omega_i,\;\; \omega^*=\omega_r-i\omega_i$$

(note that the wavenumber l is real)

details

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi+\left(\beta-\frac{d^2\overline{u}}{dy^2}\right)\frac{\partial\psi}{\partial x}=0 \qquad \psi=\phi(y)e^{i(lx-\omega t)}$$

$$\nabla^2\psi=\psi_{xx}+\psi_{yy}=\frac{\partial}{\partial x}\left(\phi\, il\, e^{<>}\right)+\frac{\partial}{\partial y}\left(\phi_y e^{<>}\right)=\left(-\phi l^2+\phi_{yy}\right)e^{<>}$$

$$\psi_x=\phi\, il\, e^{<>}$$

$$-\cancel{i}\omega(-\phi\, l^2+\phi_{yy})+\cancel{i}l\,\overline{u}(-\phi\, l^2+\phi_{yy})+(\beta-\overline{u}_{yy})\phi\,\cancel{i}l=0$$

$$-\frac{\omega}{l}(\phi_{yy}-\phi\, l^2)+\overline{u}(\phi_{yy}-\phi\, l^2)+(\beta-\overline{u}_{yy})\phi=0$$

$$-\left(\frac{\omega}{l}-\overline{u}\right)(\phi_{yy}-\phi\, l^2)+(\beta-\overline{u}_{yy})\phi=0$$

$$(\overline{u}-c)(\phi_{yy}-\phi\, l^2)+(\beta-\overline{u}_{yy})\phi=0$$

$$\phi_{yy}-l^2\phi+\left(\frac{\beta-\overline{u}_{yy}}{\overline{u}-c}\right)\phi=0$$

Conditions for growth: the Rayleigh criterion

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta - d^2\overline{u}/dy^2}{\overline{u}(y) - c}\phi = 0$$

Multiply the Rayleigh equation by Φ and integrate across the domain:* (integrate by parts and apply boundary conditions)

$$-\int_0^L \left(\left| \frac{d\phi}{dy} \right|^2 + l^2|\phi|^2 \right) dy + \int_0^L \frac{\beta - d^2\overline{u}/dy^2}{\overline{u} - c}|\phi|^2 dy = 0$$

The term on the left is real. If c is complex, and we multiply top and bottom by $(\overline{u} - c)^*$ *we can isolate the imaginary part:*

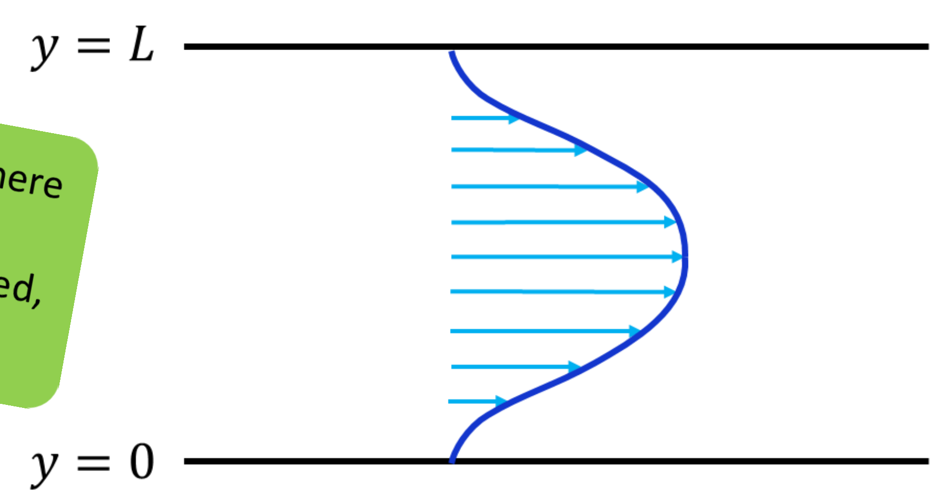
$$c_i \int_0^L \left(\beta - \frac{d^2\overline{u}}{dy^2} \right) \frac{|\phi|^2}{|\overline{u} - c|^2} dy = 0$$

*If $c_i \neq 0$ then we have growth. So a **necessary condition** for growth is that the integral is zero.*
This means that $\beta - \overline{u}_{yy}$ must change sign between y = 0 and y = L.
To put it another way, the gradient of absolute vorticity of the background flow:

$$\frac{d}{dy}(f_0 + \beta y - \overline{u}_y)$$

must change sign in the domain.

$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
If the Rayleigh criterion is satisfied, we might have an instability.



So we require an extremum in absolute vorticity.

derivation: integrating by parts

$$\frac{d}{dy}(\phi\phi_y) = \phi_y^2 + \phi\phi_{yy} \quad \Longrightarrow \quad d(\phi\phi_y) = \phi_y^2\, dy + \phi\phi_{yy}\, dy$$

$$\int \phi(\phi_{yy} - l^2\phi)\, dy = \int (\phi\phi_{yy} - l^2\phi^2)\, dy = \int_0^L d(\phi\phi_y) - \int_0^L (\phi_y)^2\, dy - \int_0^L l^2\phi^2\, dy$$

$$= [\phi\phi_y]_0^L - \int_0^L |\phi_y|^2 + l^2\phi^2\, dy$$

More conditions for growth: the Fjørtoft criterion

The real part of the integral must also be zero. By the same manipulation as before this gives

$$-\int_0^L \left(\left| \frac{d\phi}{dy} \right|^2 + l^2 |\phi|^2 \right) + \int_0^L (\overline{u} - c_r) \left(\beta - \frac{d^2 \overline{u}}{dy^2} \right) \frac{|\phi|^2}{|\overline{u} - c|^2} = 0$$

thus $\int_0^L (\overline{u} - c_r) \left(\beta - \frac{d^2 \overline{u}}{dy^2} \right) \frac{|\phi|^2}{|\overline{u} - c|^2} > 0$ $\quad \int A(u-c) > 0$

as we already know

$$\int_0^L \left(\beta - \frac{d^2 \overline{u}}{dy^2} \right) \frac{|\phi|^2}{|\overline{u} - c|^2} = 0 \qquad \int A = 0$$

Fjørtoft Logic

$\int A = 0$

$\int A(u-c) > 0$

$\int A(u-u_0) = \int A(u-c) + \int A(c-u_0) > 0$ (constant)

$\int A(u-u_0) > 0$

we can deduce that $(\overline{u} - u_0)(\beta - \overline{u}_{yy})$ must be positive somewhere in the domain for any u_0 including the value for the background flow at the extremum.

The expression is obviously zero at the extremum but must be positive somewhere in the domain. The choice of the value of u_0 at the latitude of the vorticity extremum makes this criterion as stringent as possible.

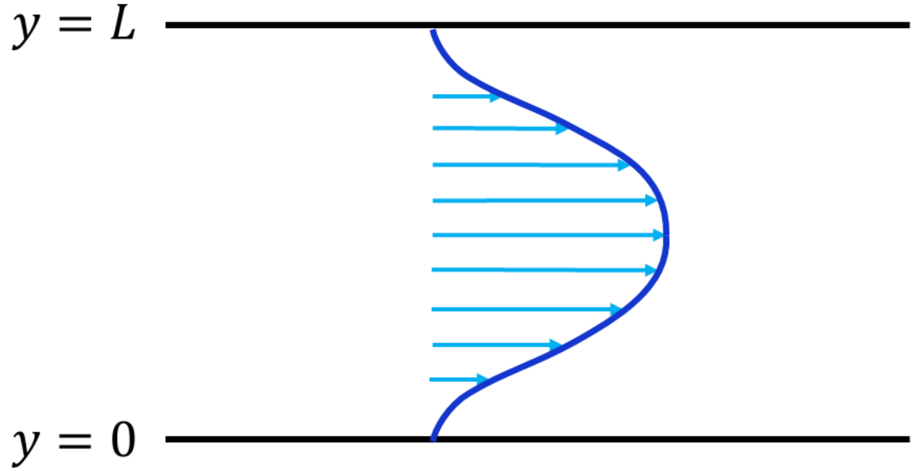
Note: a the non-satisfaction of a necessary condition for instability can also be seen as a sufficient condition for stability

$\forall u_0, (\bar{u} - u_0)(\beta - \bar{u}_{yy})$ must be positive somewhere in the domain.
⇨ If the Fjørtoft criterion is satisfied, we might have an instability.

Conditions for growth

$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
➯ If the Rayleigh criterion is satisfied, we might have an instability.

$\forall u_0, (\bar{u} - u_0)(\beta - \bar{u}_{yy})$ must be positive somewhere in the domain.
➯ If the Fjørtoft criterion is satisfied, we might have an instability.

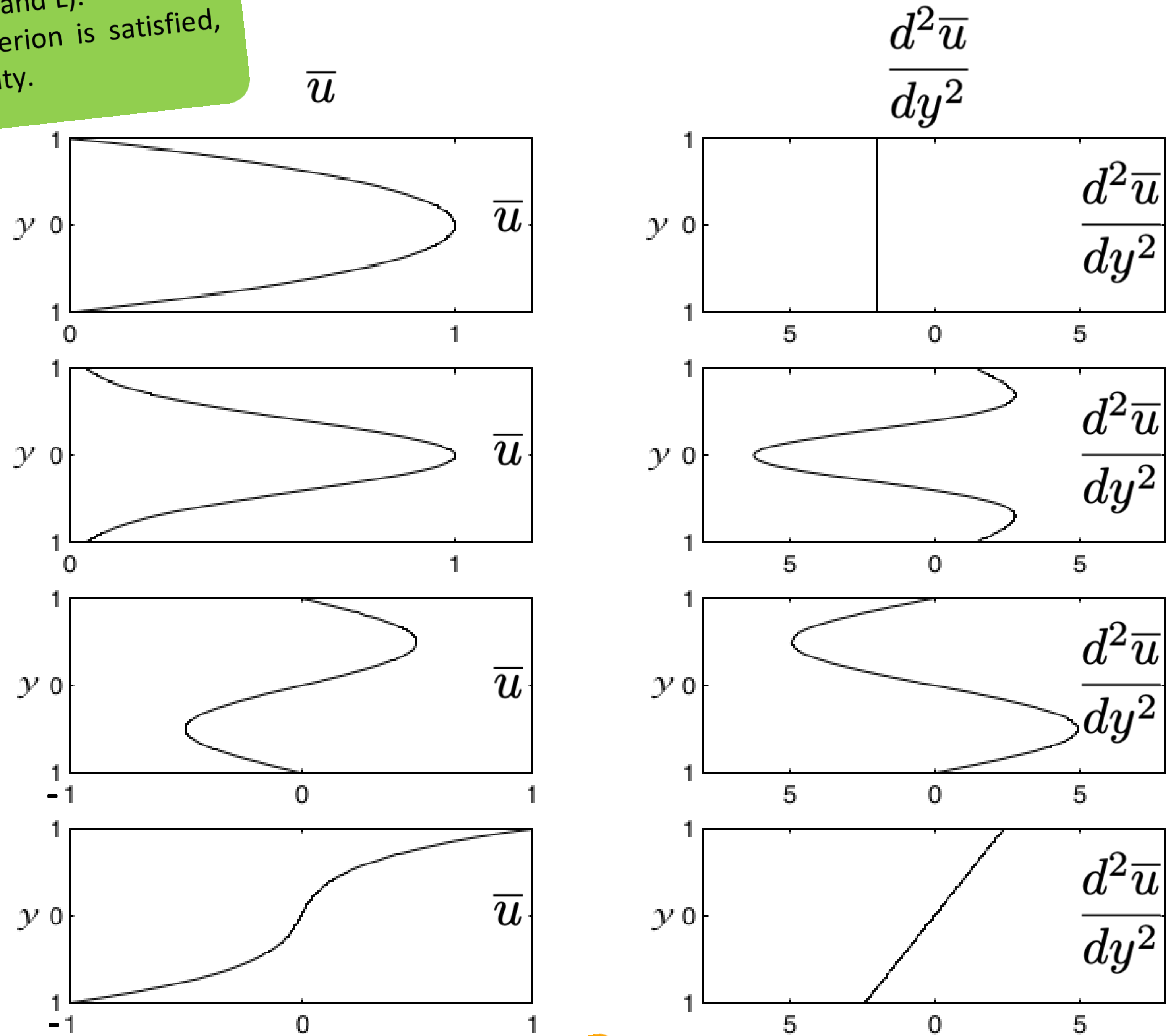


Both Rayleigh and Fjørtoft criteria are just ***necessary conditions***. They are not sufficient conditions. This means that, when analyzing a potential vorticity map, if one of these conditions is satisfied, it does not mean that the flow is unstable, it means that ***it is possible for the flow to be unstable***.

On the other hand, the non-satisfaction of a necessary condition is a sufficient condition, which means that ***if the Rayleigh or the Fjørtoft condition is not satisfied then the flow is stable***.

Stable and unstable profiles

$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
⇨ If the Rayleigh criterion is satisfied, we might have an instability.



Poiseuille Flow ($u = 1 - y^2$) — stable (Rayleigh) - no change of sign

Gaussian jet — possibly unstable – change of sign

sinusoidal — possibly unstable – change of sign

polynomial (boundary extrema) — stable (Fjørtoft) – vorticity extrema at the boundaries

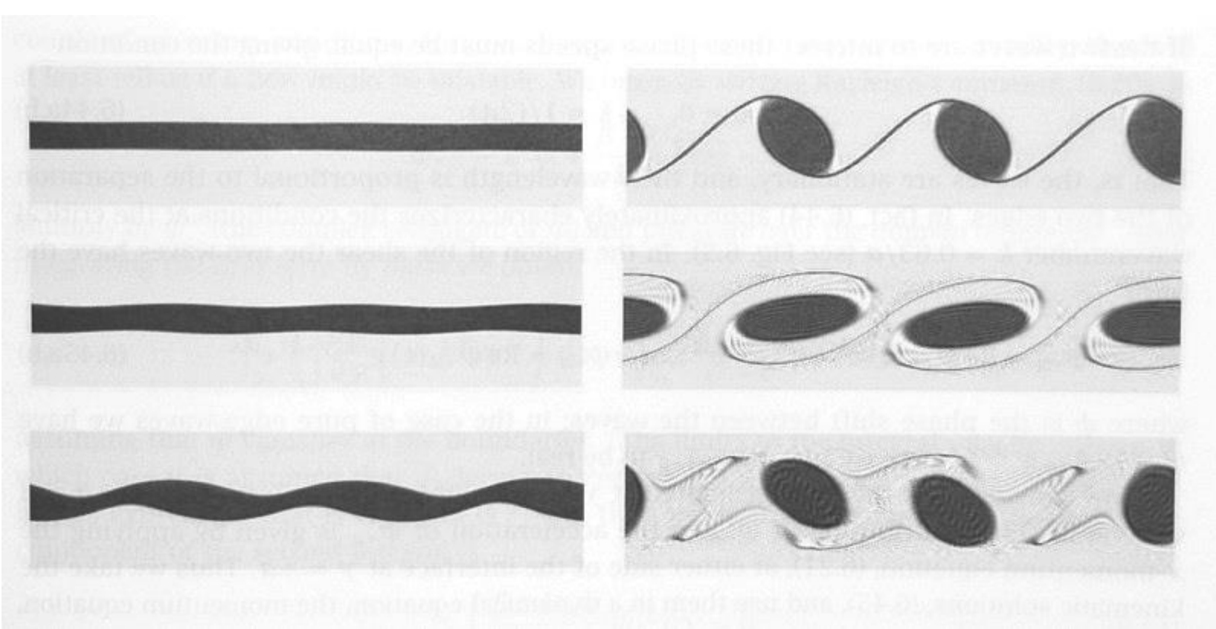
$\forall u_0, (\bar{u} - u_0)(\beta - \bar{u}_{yy})$ must be positive somewhere in the domain.
⇨ If the Fjørtoft criterion is satisfied, we might have an instability.

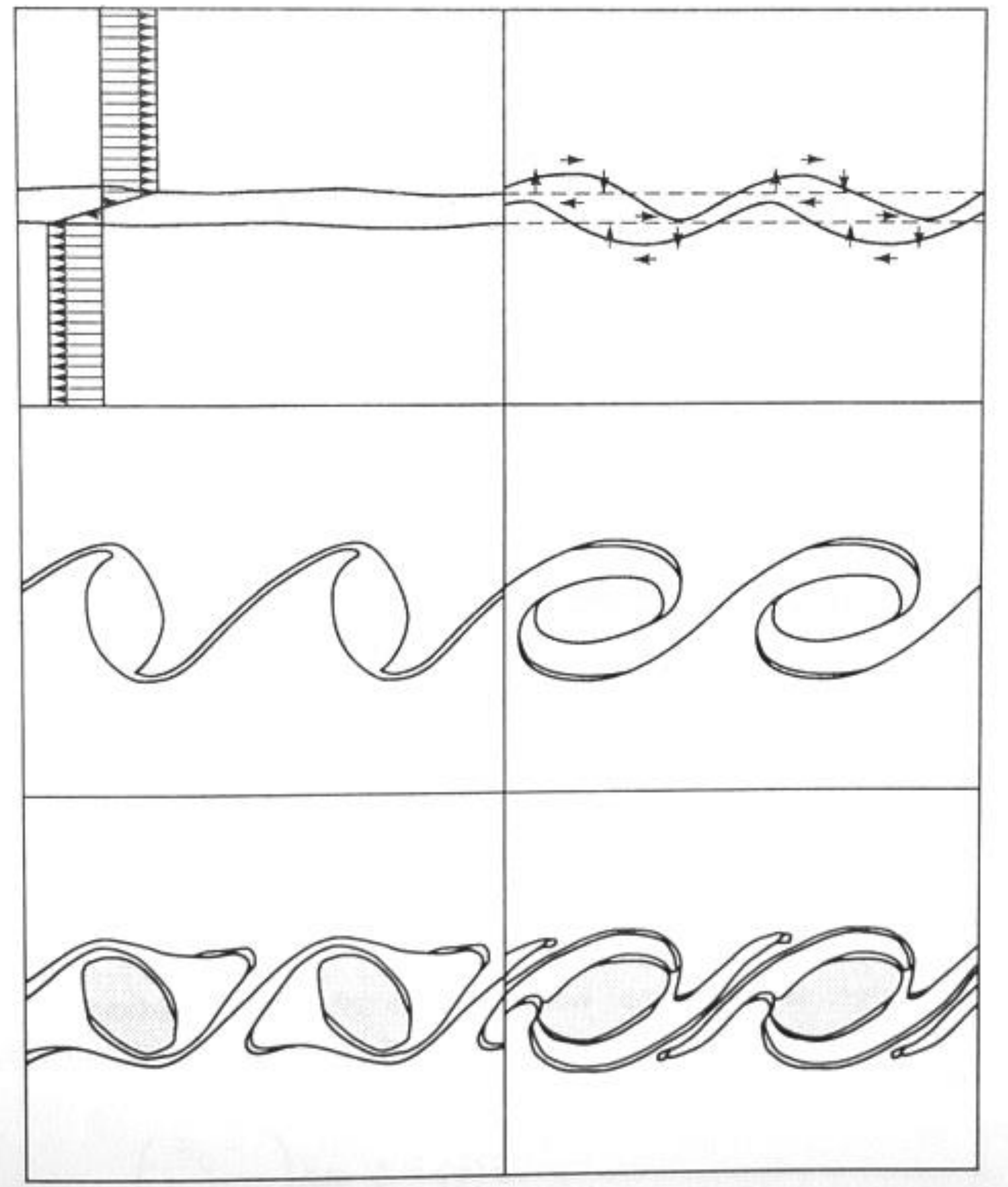
boundary extrema but u always has opposite sign to vorticity gradient (f plane example, beta might change this) so product always negative.

- Remember: Fjortoft criterion must work for any u_0

Physical mechanism

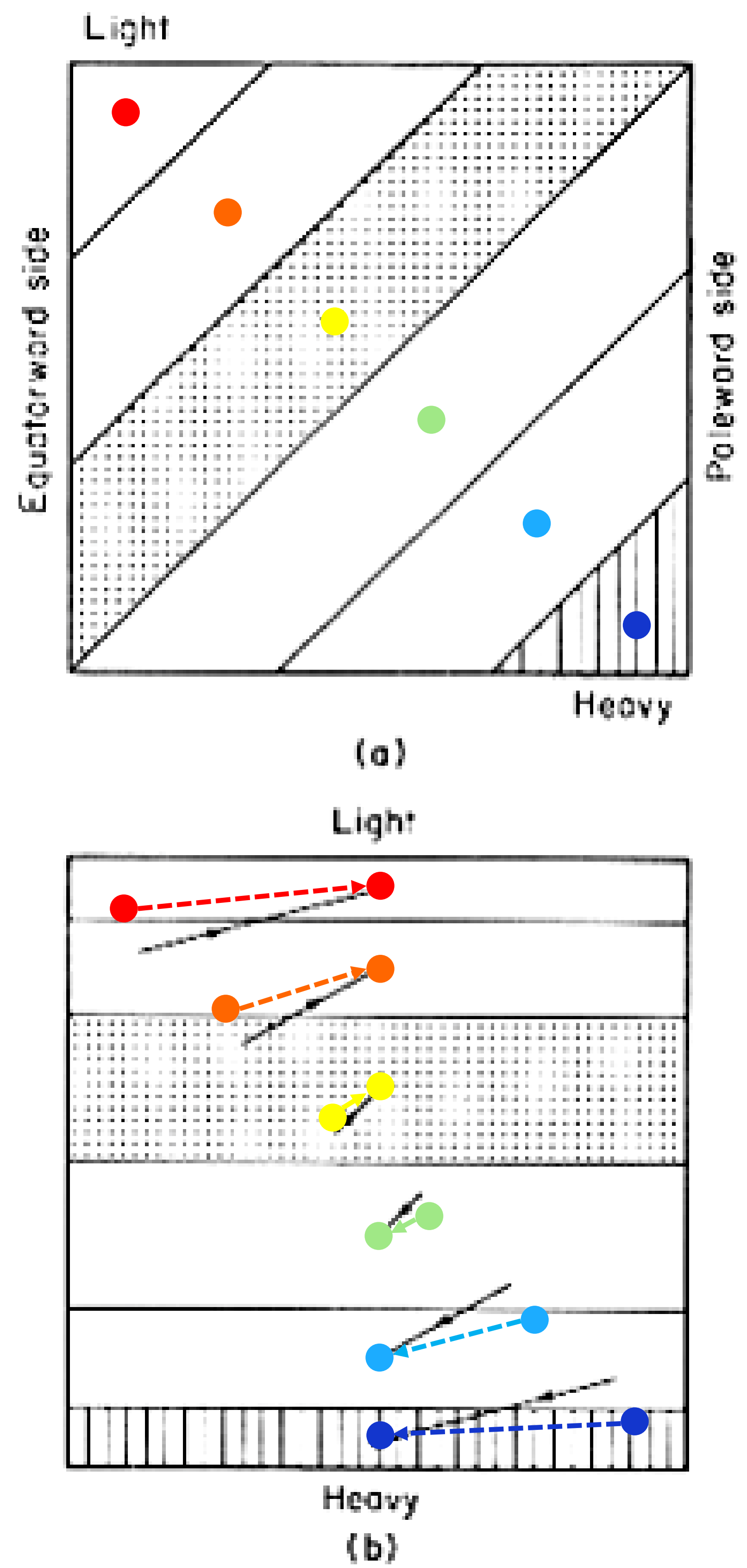
Take the example of an isolated shear layer. It has negative (clockwise) vorticity and is embedded in a flow that has no vorticity. So it represents an extremum. The perturbation meridional flow can export this vorticity into a region where there is none. At the same time, on the other side of the vorticity strip, but just out of phase, the same thing happens. The induced flow deforms the vorticity strip so that the situation amplifies and the deformation continues.



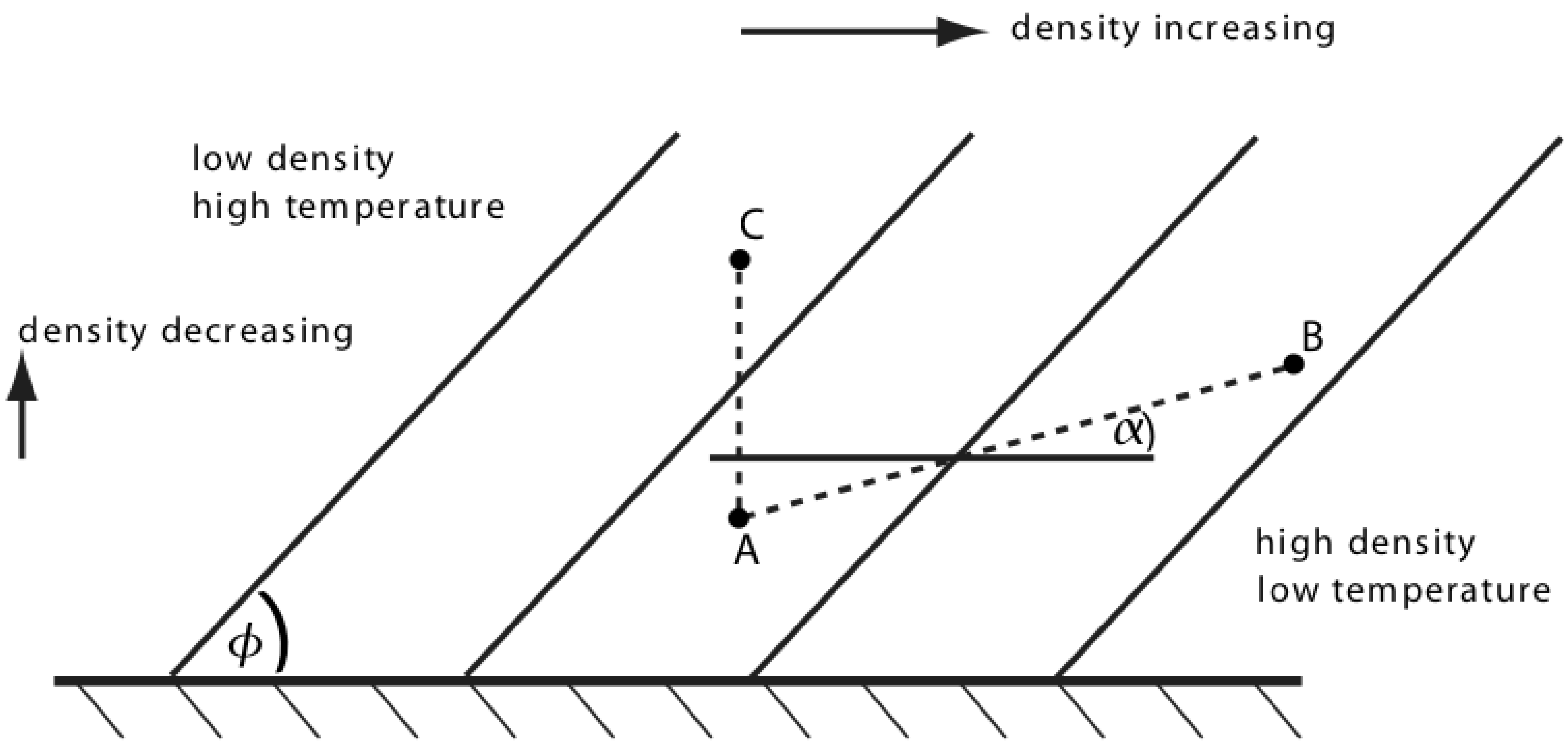


Baroclinic instability

Now we turn to a mechanism that can liberate stored potential energy in a system that may be barotropically (and statically) stable. Ultimately, work is done by gravity to provide growing kinetic energy. The perturbation must have the right structure to make the necessary rearrangements to tap this source of energy.



Sloping convection



In a rotating system we can imagine a steady basic state with inclined density contours (we need rotation to balance the pressure gradient forces). It can be statically stable. But a sloping parcel displacement can still leave a parcel in a situation where it is more buoyant. The displacement A-C is stable. But the exchange of the two parcels A and B will release energy stored in the density structure.

Optimal scales for growth

The mechanism relies on horizontal variations of density, and on a perturbation that has the right phase arrangement to amplify by vortex stretching. Consider the following scaling for the quasigeostrophic potential vorticity (on an f plane):

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right)$$

If distance scales as L, then $\nabla^2\psi \sim \frac{\Psi}{L^2}$

and if height scales as H then the vortex stretching term scales as $\sim \frac{f^2\Psi}{N^2H^2} = \frac{\Psi}{{L_R}^2}$

If L >> L_R then the relative vorticity is unable to balance stetching, so stretching is inhibited and we have vertically uniform disturbances - this is the barotropic limit.

If L<< L_R then relative vorticity dominates, and the layers become uncoupled, and thus unable to cooperate to produce the necessary structures to liberate the potential energy stored in the horizontal variations of stratification, or the vertical shear of the wind.

The optimal scale is thus the Rossby (internal) radius of deformation L_R = NH/f
Growing disturbances of this scale will be ***selected***

Physical mechanism

Consider a two-layered shear flow in thermal wind balance.

Introduce a positive PV anomaly in the upper layer, with associated cyclonic flow.

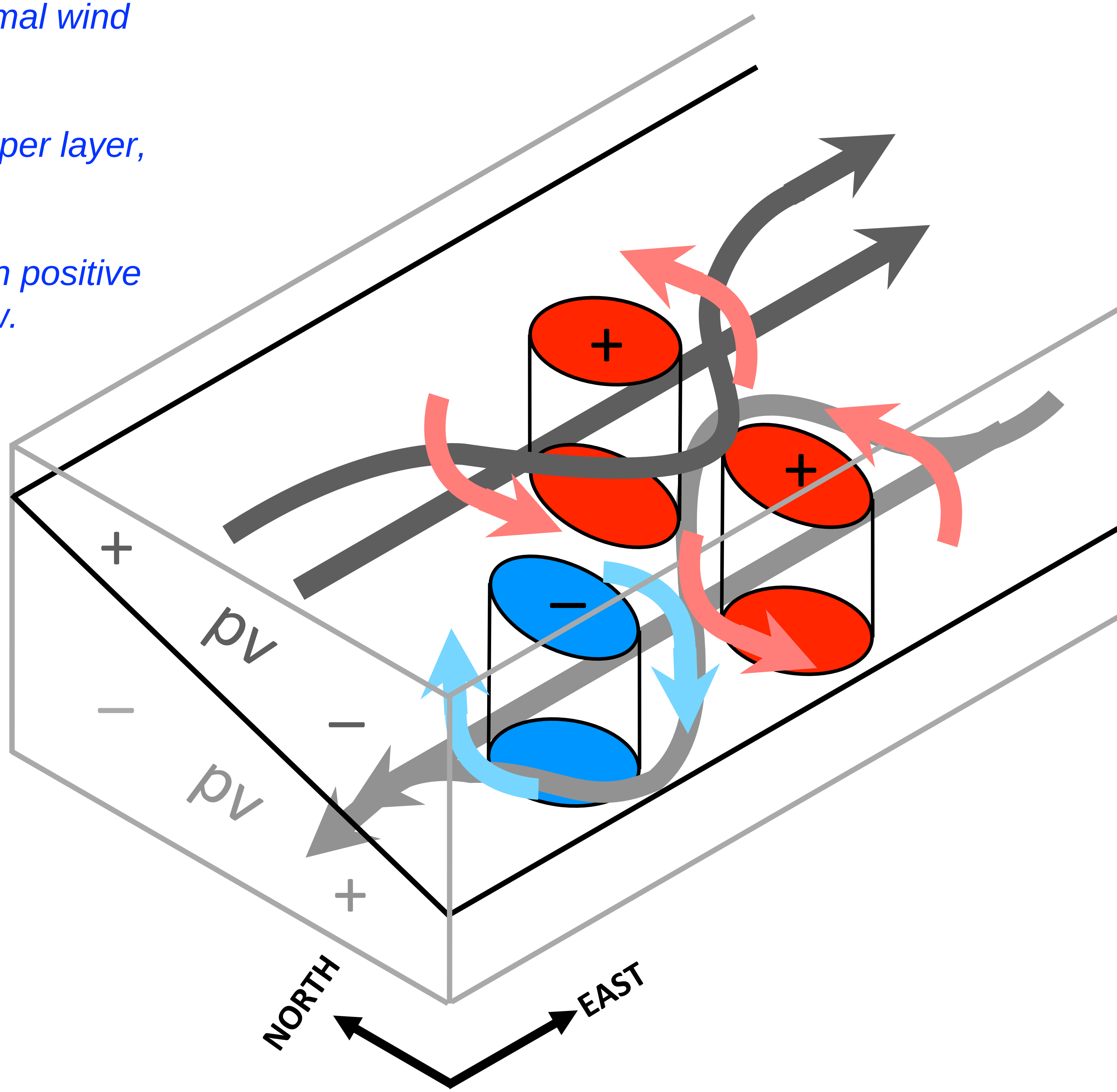
Positive relative vorticity is associated with positive layer thickness, squeezing the layer below.

Induced advection in the lower layer creates a dipole of PV anomalies with associated circulation patterns

This induces southward advection of more positive PV in the layer above, amplifying the original perturbation.

Note the westward tilt with height of the PV perturbations.

At the same time, due to the upper level PV gradient and the gradient of f, the entire structure propagates westwards as a Rossby wave.



Modal solutions

The linear perturbation potential vorticity equation is

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial Q}{\partial y} = 0$$

and as usual we seek wavelike solutions in x

$$\psi' = \tilde{\psi}(y,z)e^{i(lx-\omega t)}$$

substitution leads to the equation

$$(U-c)(\tilde{\psi}_{yy} + \frac{\partial}{\partial z}\frac{{f_0}^2}{N^2}\tilde{\psi}_z - l^2\tilde{\psi}) + Q_y\tilde{\psi} = 0$$

with boundary conditions at the top and bottom (z = 0, H)

$$(U-c)\tilde{\psi}_z - U_z\tilde{\psi} = 0$$

↳ *These are analogous to the Rayleigh equation for barotropic (shear) instability.*

Conditions for growth

We go through the same procedure as before with these equations: multiply by the complex conjugate and integrate over the domain. The "domain" is now in y and z.

This eventually leads to

$$\int_0^L \int_0^H |\Psi_y|^2 + f_0^2/N^2|\tilde{\psi}_z|^2 + l^2|\tilde{\psi}|^2 dzdy - \int_0^L \left\{ \int_0^H \frac{Q_y}{U-c}|\tilde{\psi}|^2 dz + \left[\frac{f_0^2/N^2 U_z |\tilde{\psi}|^2}{U-c} \right]_0^H \right\} dy = 0$$

the imaginary part of which is

$$-c_i \int_0^L \left\{ \int_0^H \frac{Q_y}{|U-c|^2}|\tilde{\psi}|^2 dz + \left[\frac{f_0^2/N^2 U_z |\tilde{\psi}|^2}{|U-c|^2} \right]_0^H \right\} dy = 0$$

If $c_i \neq 0$ then the integral must be zero. This means that at least one of the following conditions must be met (the "Charney - Stern - Pedlosky criteria")

- *Q_y changes sign in the domain (there is a PV extremum)*
- *Q_y has the opposite sign to U_z at $z = H$*
- *Q_y has the same sign as U_z at $z = 0$*
- *if $Q_y = 0$, U_z has the same sign at $z = 0$ and $z = H$*

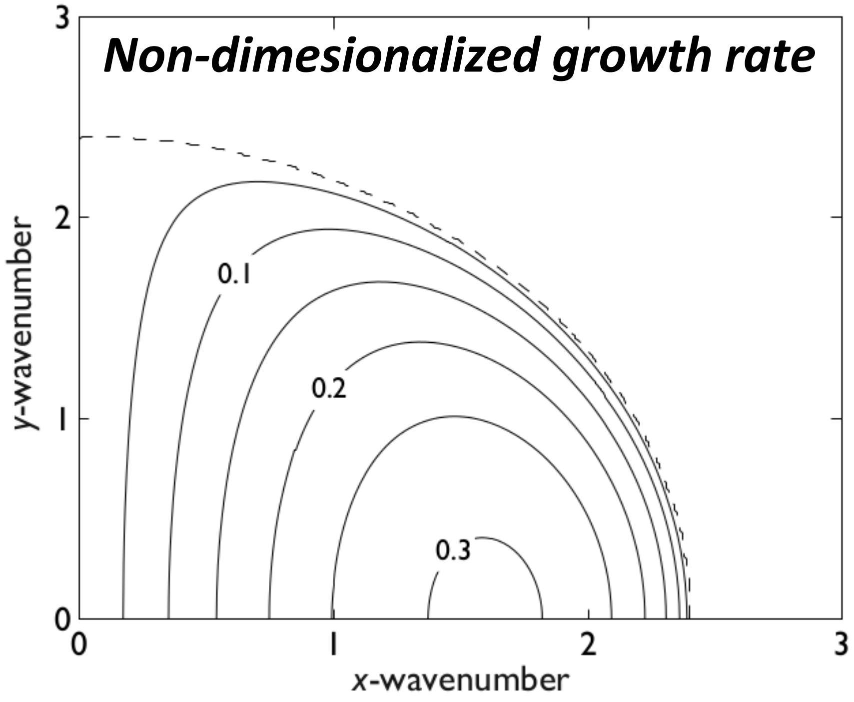
Note these are just necessary conditions for the integral to vanish.
Note that U_z is directly related to the basic state meridional temperature or density gradient.

- Waves can grow in the interior of the fluid (on PV extrema) or as boundary phenomena (on boundary temperature gradients).

The Eady problem

Simplest archetype of baroclinic instability
- f plane
- N^2 constant
- constant vertical shear $U(z) = U z/H$
- motion is between two rigid flat surfaces at $z = 0, H$

- Uniform vertical shear means the basic state PV = 0.
- The procedure for solving the problem is the same as before - substitute wave functions into the PV equation to produce a Rayleigh-type equation, and apply the boundary conditions $w = 0$ at $z = 0, H$.
- Instability arises from boundary temperature gradients.

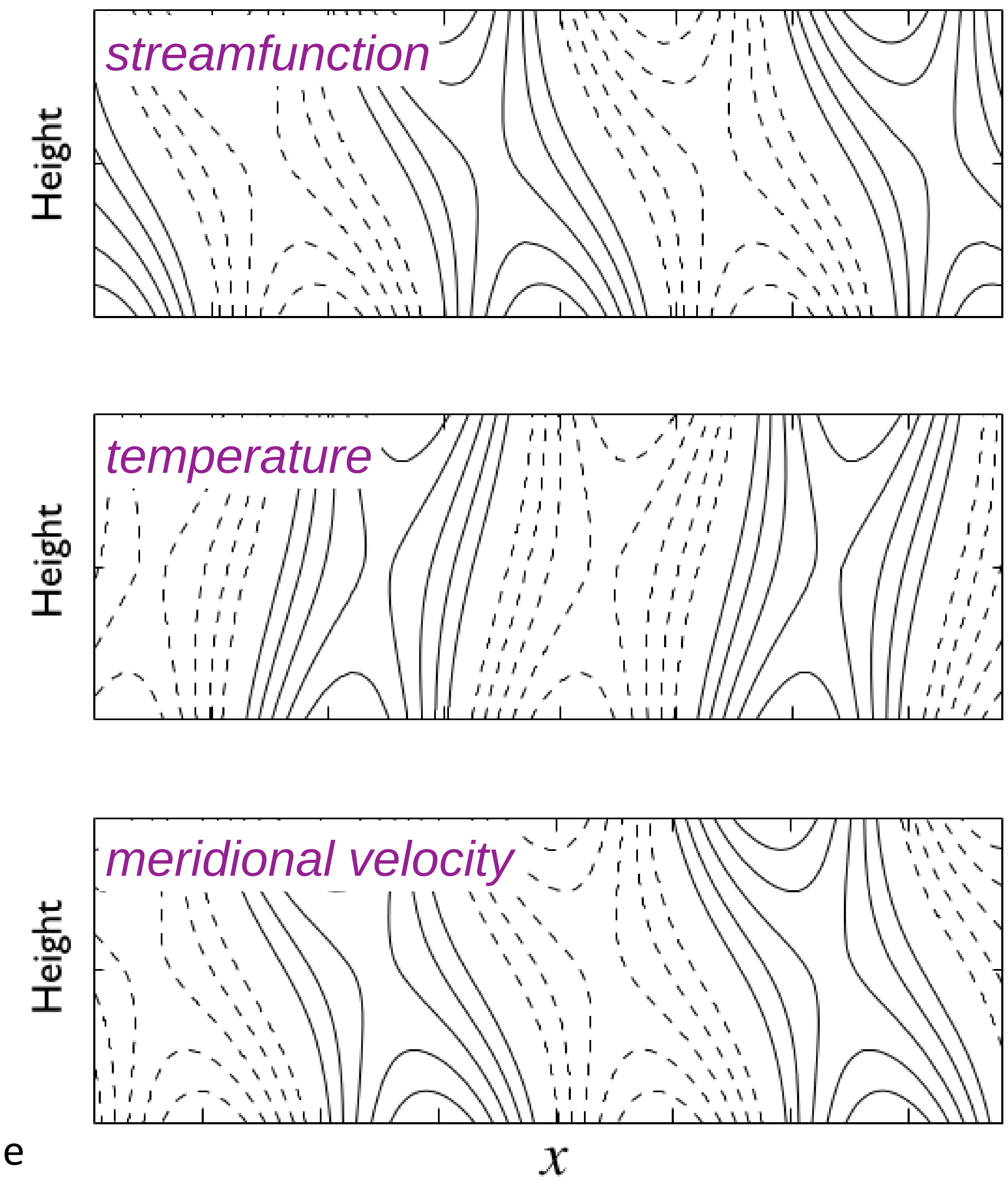


Max. growth rate

$$\sim 0.31 \frac{U}{L_R}$$

Wavenumber and wavelength at which the instability is the greatest are:

$$k_m = \frac{1.61}{L_R} \qquad \lambda_m = \frac{2\pi}{k_m} = \frac{2\pi}{1.6} L_R$$



What we learn from the Eady problem

- length scale of maximum instability characterised by the deformation scale (factor of about four times)
- the most unstable growth rate is 0.3 U/L_R = 0.3 f_0/N du/dz
- there is a short wave cutoff - short waves are not unstable
- the circulation (meridional current, streamfunction) must slope westwards with height in westerly shear to extract energy from the basic state.

Some results of the Eady calculation applied in an oceanic context:

H ~ 1 km, U ~ 0.1 m/s, N ~ 10^{-2} s^{-1} leads to

deformation radius L_R = NH/f = 10^{-2} x 1000 / 10^{-4} = 100 km
scale of maximum instability = 3.9 L_R ~ 400 km
growth rate = 0.3 U/L_R ~ 0.3 x 0.1 / 10^5 ~ 0.026 $days^{-1}$ (period ~ 40 days)

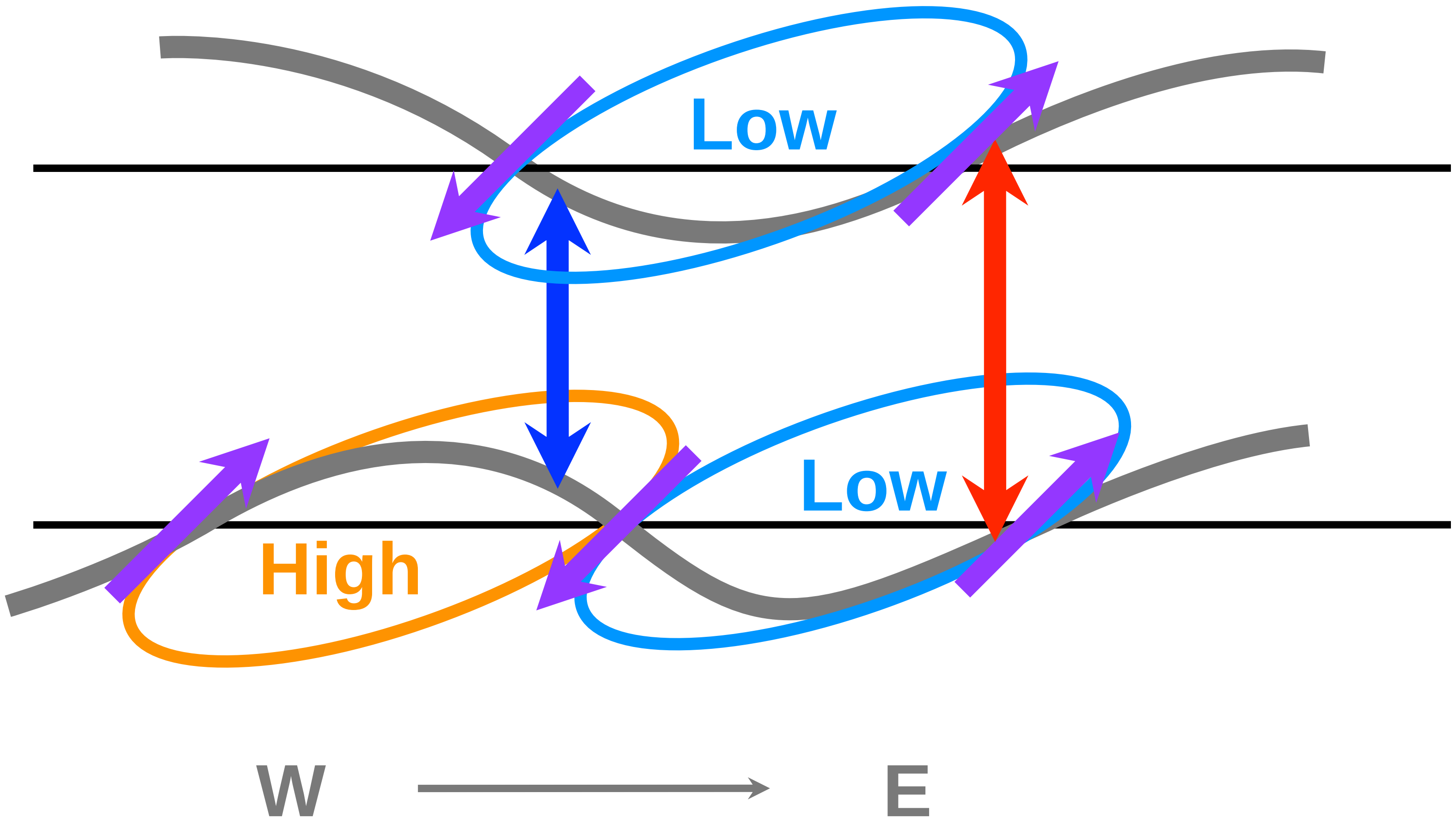
Compare with the atmosphere

H ~ 10 km, U ~ 10 m/s, N ~ 10^{-2} s^{-1} leads to
L_R ~ 1000 km, instability scale ~ 4000 km, growth rate ~ 0.26 $days^{-1}$ (period 4 days)

In the Eady problem is theoretical, the instability relies on an interaction between waves at the upper and lower boundaries. If either boundary is removed, the instability dies.

Heat transport in a baroclinic system

Growing structures tilt westwards with height.
Consistent with thermal wind balance this structure also transports heat polewards.



Baroclinic instability: summary

- There is clear evidence of a preferred scale for turbulent motions in the ocean

- Simple scaling arguments and more sophisticated stability analyses show that there is a preferred scale on which growth can occur.

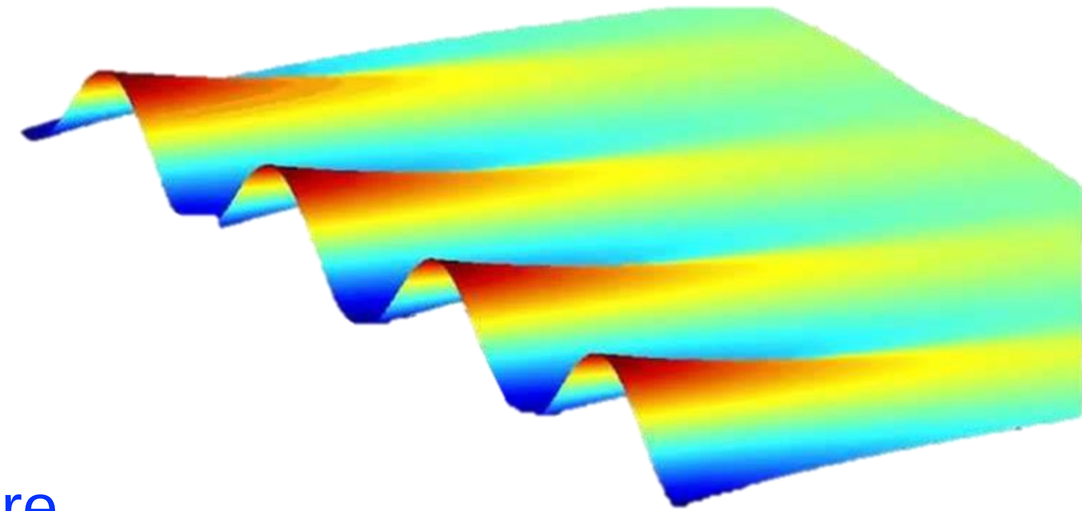
- If this growth depends on extracting energy from sloping density surfaces (or equivalently, vertical wind shear, or horizontal temperature gradients), then there must be an interplay between vortex stretching and relative vorticity terms in the conservation of PV.

- This naturally selects structures around the Rossby deformation scale.

- These structures can grow exponentially provided certain criteria are met: notably if extrema exist in the potential vorticity of the basic state.

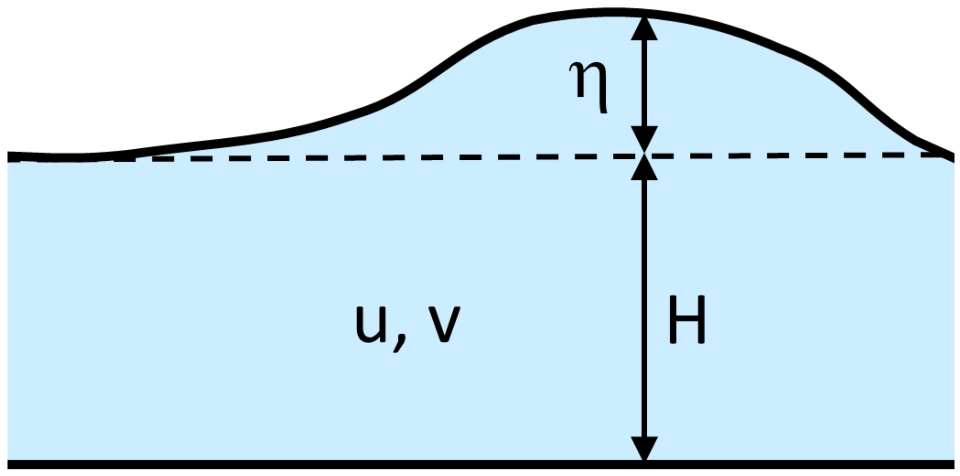
Chapter 4: Gravity waves and tropical dynamics

- ⇨ Gravity waves and rotation
- ⇨ Coastal Kelvin waves
- ⇨ Scales and approximations near the equator
- ⇨ Equatorial Kelvin waves
- ⇨ Equatorial wave solutions: dispersion and structure



Gravity waves in shallow water

Let's start with something simple: a one-dimensional non-rotating linear system.
Which terms shall we cross out ?



$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$ x-momentum

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$ y-momentum

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -\overset{H}{h}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$ continuity $(h = H + \eta)$

This leaves us with

$$\frac{du}{dt} = -g\frac{dh}{dx}, \quad \frac{dh}{dt} = -H\frac{du}{dx} \quad \Rightarrow \quad \frac{d^2u}{dt^2} = gH\frac{d^2u}{dx^2}$$

and the solution of this equation is $u = Re\,\tilde{u}\,e^{i(lx-\omega t)}$

a gravity wave with a simple dispersion relation $\omega^2 = gHl^2, \quad c = \frac{\omega}{l} = \pm\sqrt{gH}\left(= \frac{d\omega}{dl}\right)$

Adding rotation

Put the rotation back into the linear system, we need two dimensions and three equations again:

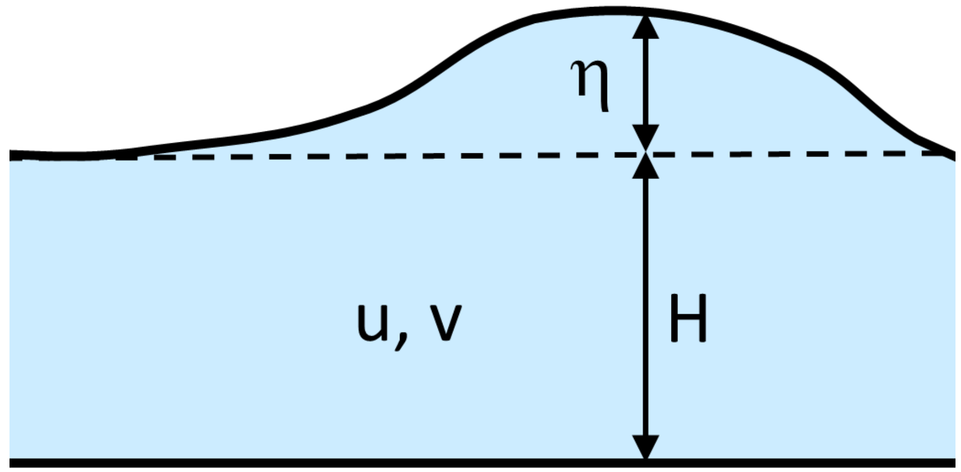
$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

substitute solution $(u, v, \eta) = (\tilde{u}, \tilde{v}, \tilde{\eta})e^{i(lx+my-\omega t)}$

remember $\frac{\partial}{\partial x} \rightarrow il\times \quad \frac{\partial}{\partial y} \rightarrow im\times \quad \frac{\partial}{\partial t} \rightarrow -i\omega\times$



Differential equations become linear algebraic equations

$$-i\omega\tilde{u} - f\tilde{v} = -igl\tilde{\eta}$$

$$-i\omega\tilde{v} + f\tilde{u} = -igm\tilde{\eta}$$

$$-i\omega\tilde{\eta} + H(il\tilde{u} + im\tilde{v}) = 0$$

The unknowns are the wave amplitudes $\tilde{u}, \tilde{v}, \tilde{\eta}$

The parameters are the wave properties l, m, ω *and the geophysical constants* f, g, H

Inertia-gravity (Poincaré) waves

We need to solve the algebraic system

$$\begin{pmatrix} -i\omega & -f & igl \\ f & -i\omega & igm \\ ilH & imH & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

⇨ Trivial solution $\tilde{u} = \tilde{v} = \tilde{\eta} = 0$ (no flow)

⇨ The condition for having non-trivial solutions is that the determinant of the matrix is zero.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

This leads to: $\omega \left[\omega^2 - f^2 - gH(l^2 + m^2)\right] = 0$

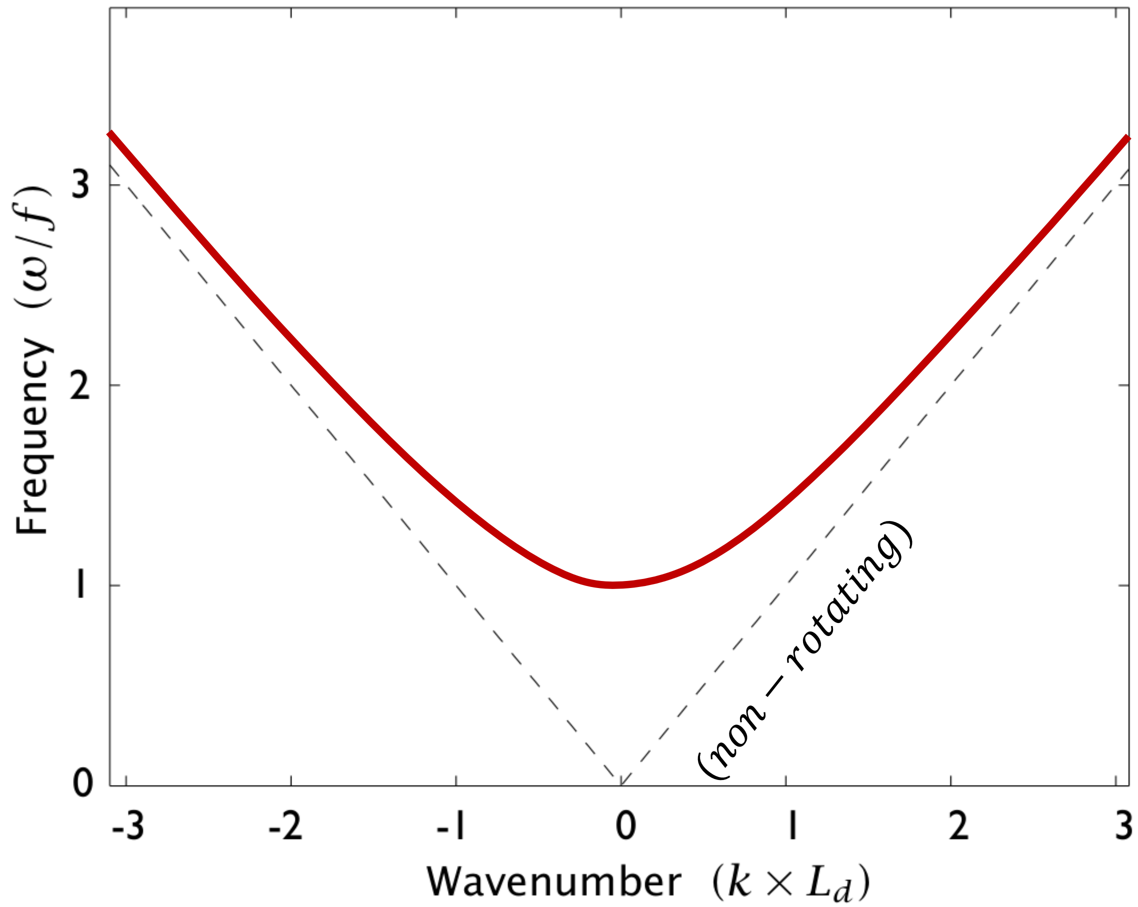
So the solutions are either $\omega = 0$ (steady geostrophic flow)

Or "Inertia-gravity" waves or Poincaré waves:

$$\omega = \pm\sqrt{f^2 + gHk^2}$$

- For short waves (large l) these waves behave like ordinary, non-dispersive gravity waves.
- For long waves (small l) the frequency has a lower limit of f, and the waves become very dispersive, to the point where they break down into coherent but unconnected free motion in inertial circles.

Is there any way to have a large-scale gravity wave that propagates normally on a rotating planet ? Yes !



Boundary Kelvin waves

Add a lateral boundary to the problem, cross out terms involving flow perpendicular to the boundary

$$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x}$$

Geostrophic balance

$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y}$$

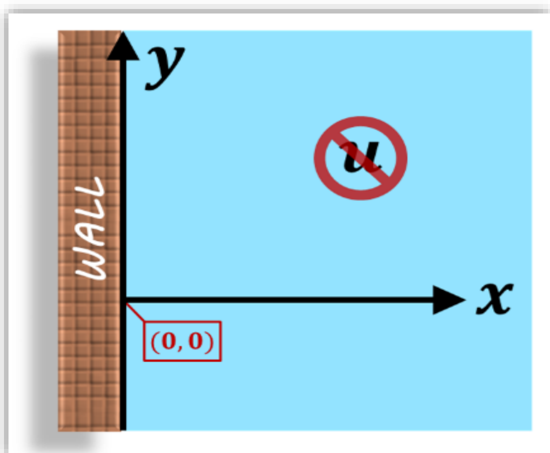
$$\frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0$$

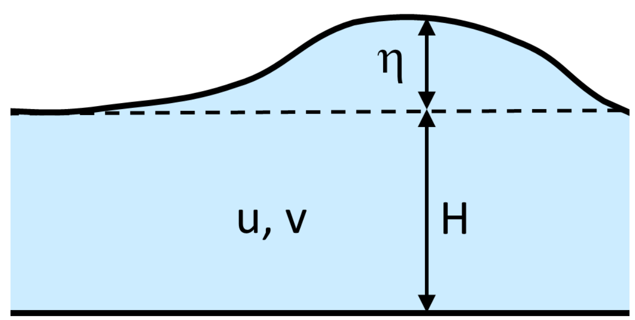
non-dispersive waves

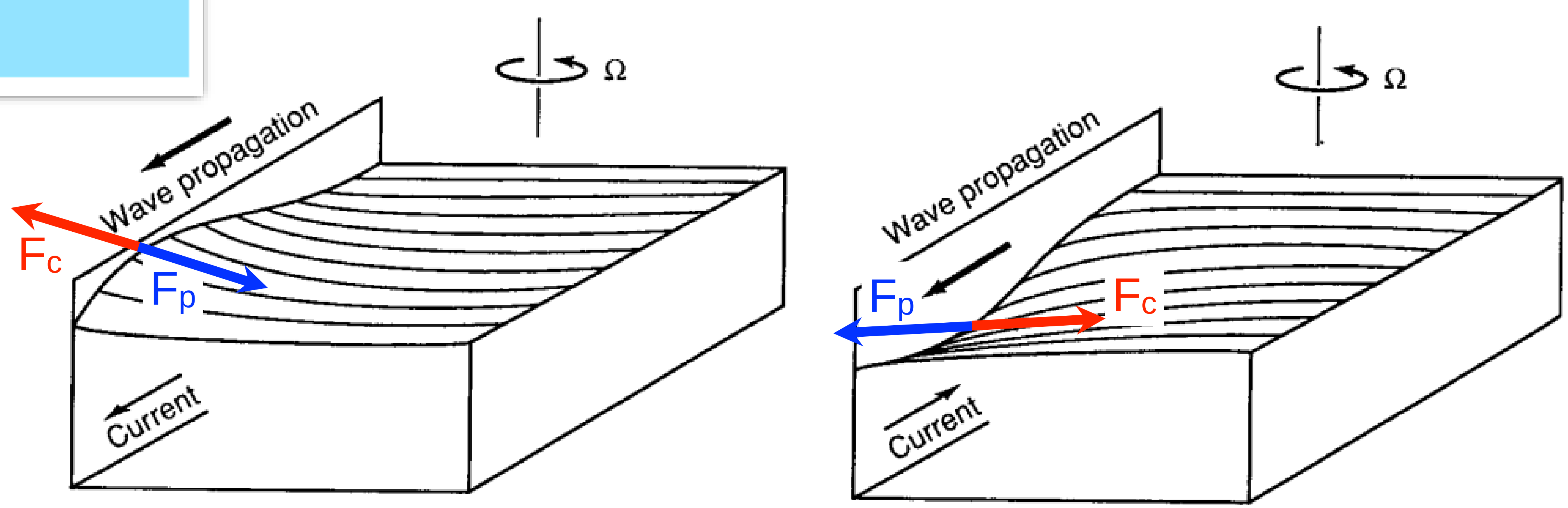
In the x direction we have geostrophic balance, with pressure and Coriolis forces alternating in direction as crests and troughs propagate meridionally.

In the y direction we have non-dispersive gravity waves with a fixed phase speed independent of horizontal scale

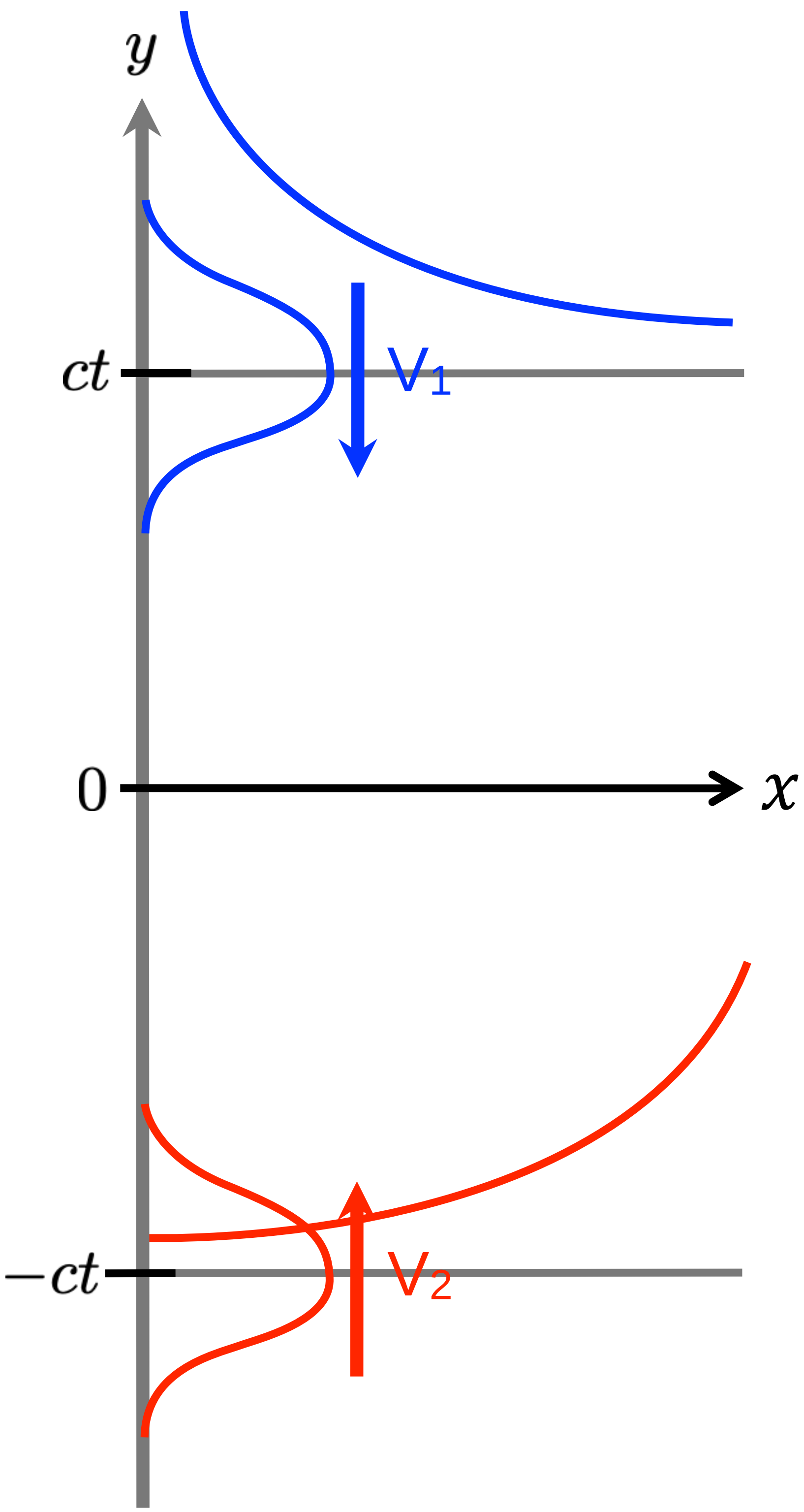
$$|c| = \sqrt{gH}$$







Boundary Kelvin waves



⇨ *Since the wave is non-dispersive, all signals must travel at speed c. The solution for v at y=0 and time t consists of two waves traveling in opposite directions:*

$$v = V_1(x, y + ct) + V_2(x, y - ct)$$

⇨ *The corresponding solution for η is* $\eta = \sqrt{H/g}\,(-V_1 + V_2)$

(check this by substitution:

$$\rightarrow \frac{\partial}{\partial t}(V_1 + V_2) = -\sqrt{gH}\frac{\partial}{\partial y}(-V_1 + V_2) \qquad \rightarrow \frac{\partial V_1}{\partial t} = c\frac{\partial V_1}{\partial y}$$

$$\frac{\partial}{\partial t}(-V_1 + V_2) = -\sqrt{gH}\frac{\partial}{\partial y}(V_1 + V_2) \qquad \frac{\partial V_2}{\partial t} = -c\frac{\partial V_2}{\partial y}$$

)

↳ *Substituting this solution into the geostrophic balance equation we can derive the x-dependence*

$$-fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial V_1}{\partial x} = -\frac{f}{\sqrt{gH}}V_1 \qquad \frac{\partial V_2}{\partial x} = \frac{f}{\sqrt{gH}}V_2$$

⇨ *These relations have exponential solutions in x with a scale distance of the Rossby radius of deformation* $L_R = c/f$.

$$V_1(x, ct)e^{-\frac{x}{L_R}} \qquad V_2(x, ct)e^{\frac{x}{L_R}}$$

$x \geq 0$

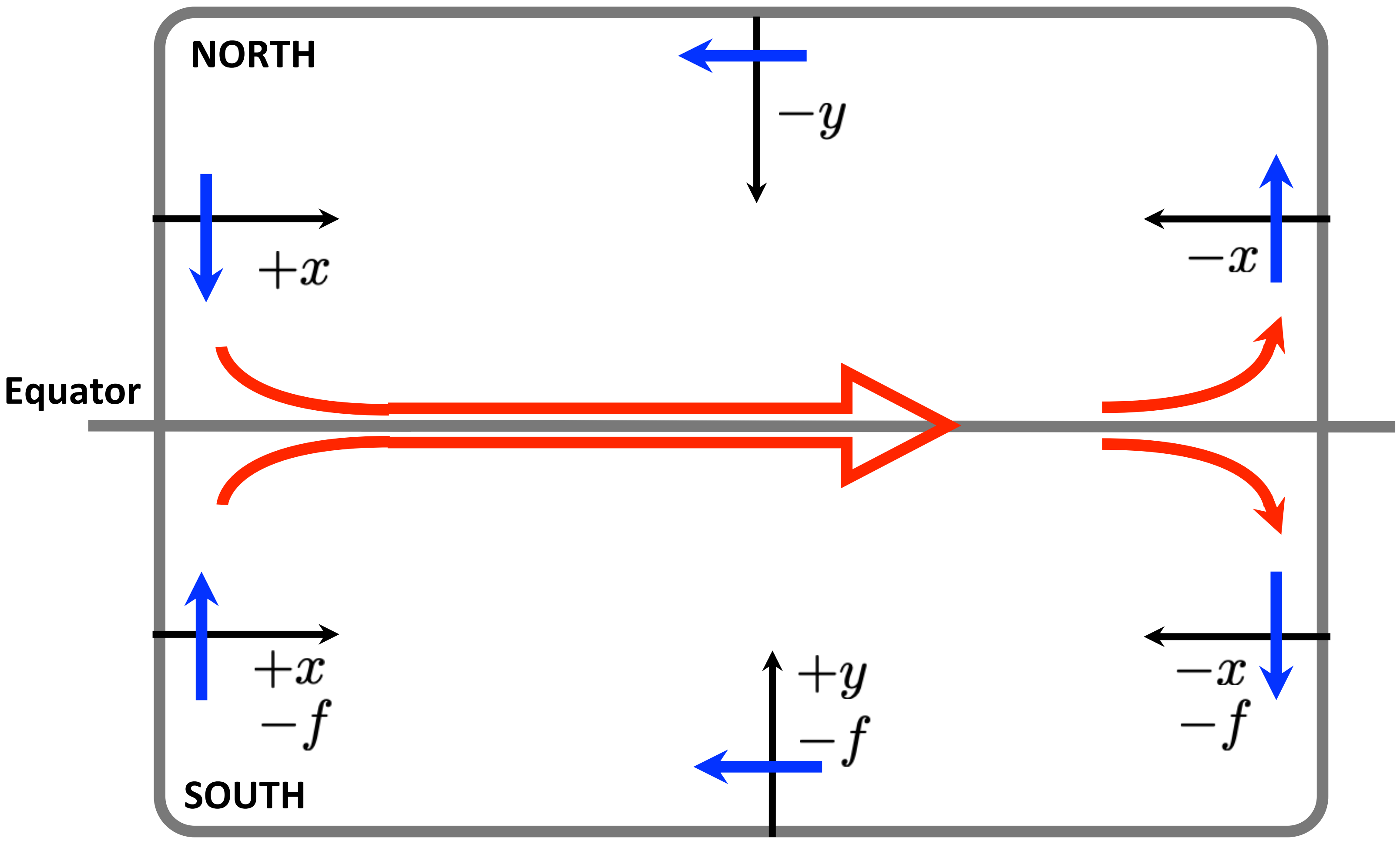
$f > 0$

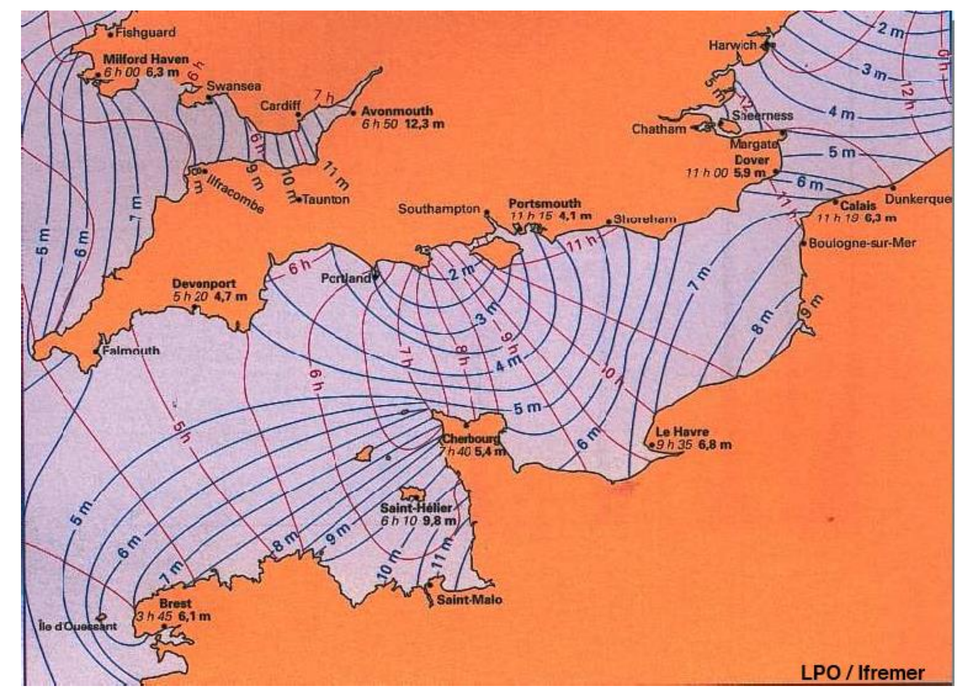
V_2 is a growing solution so we reject it as unphysical $(x > 0)$.

V_1 decays away from the coast with boundary layer width L_R

Properties of Kelvin waves

Since the only admissible solution is V_1, we conclude that for a system bounded on the west (x positive) the wave propagates in the negative y direction, i.e. to the south. If x is negative this reverses so on the eastern side of the basin the Kelvin wave goes northwards. So in the northern hemisphere a Kelvin wave will keep the coast to its right as it is pushed against it by the Coriolis force.





Tides are higher on the French side because the signal propagates in from the west

In the southern hemisphere f changes sign so all these considerations are reversed, and Kelvin waves propagate with the coast to the left.

What happens at the Equator ?
Can northern and southern Kelvin waves get pushed against each other for mutual support ?

Scales of motion near the Equator

At the equator

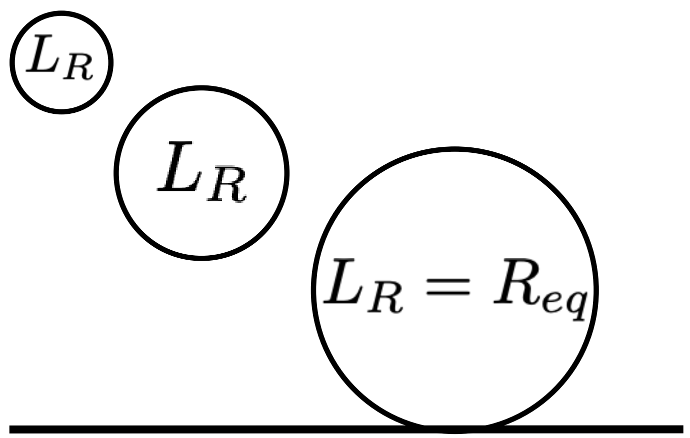
$$\phi = 0,\ \ f = 0,\ \ \beta = \frac{2\Omega}{a} = 2.28 \times 10^{-11}\, m^{-1} s^{-1}$$

β approximation: $f = \beta y$

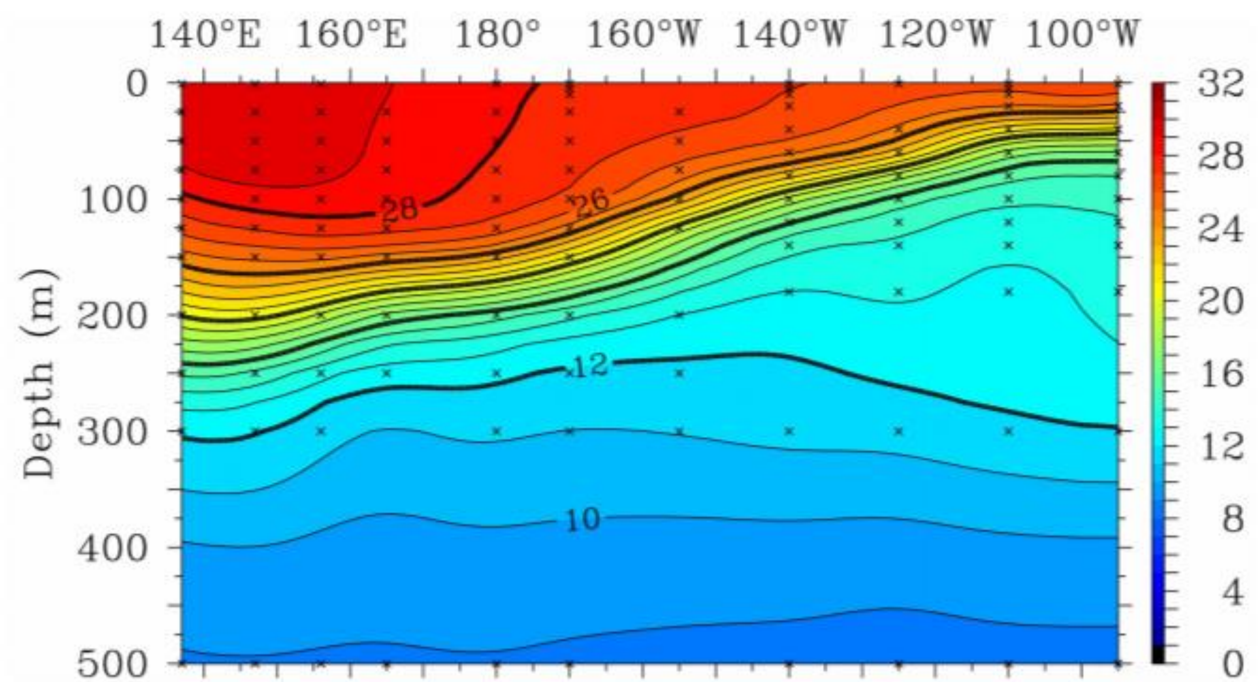
⇨ *Consider a single layer overlying the abyss.*
The internal Rossby radius is

$$L_R = \frac{\sqrt{g'H}}{f} = \frac{c}{f}$$

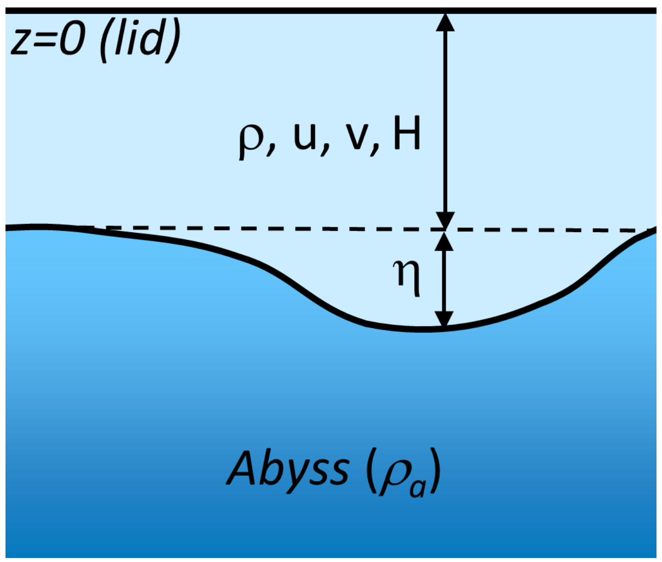
⇨ *How does this work at the equator where f=0 ?*



$$L_R = \frac{c}{\beta R_{eq}}$$



Δρ/ρ = 0.002 (*g'* = *g Δρ/ρ*), *H* = 100m
Gives gravity wave speed *c* = 1.4 m/s



⇨ *Define equatorial radius of deformation* $R_{eq} = \sqrt{\frac{c}{\beta}} \sim 250\text{km} \ \ \sim 2.2^0$

⇨ *The time* T_{eq} *for a wave to travel distance* R_{eq} $\ T_{eq} = \frac{1}{\sqrt{\beta c}} \ \sim 2\text{ days}$

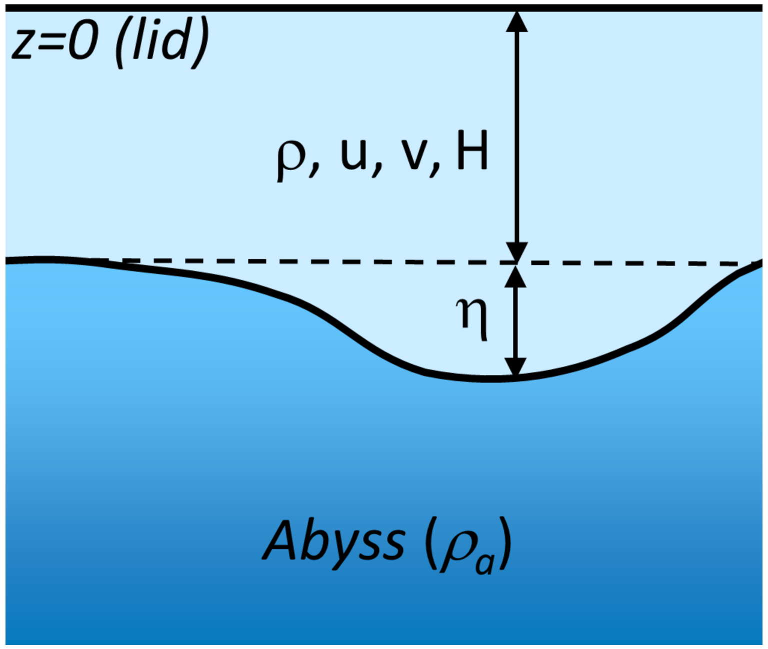
Linear Equatorial shallow water model

⇨ *Consider linear perturbations on a resting basic state*

$$\frac{\partial u}{\partial t} - \beta y v = -g' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \beta y u = -g' \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$



⇨ *First we'll look for a special case - with v = 0*

The equatorial Kelvin wave solution

Assume no meridional flow

$$v = 0$$

$$\frac{\partial u}{\partial t} - \beta y v = -g' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

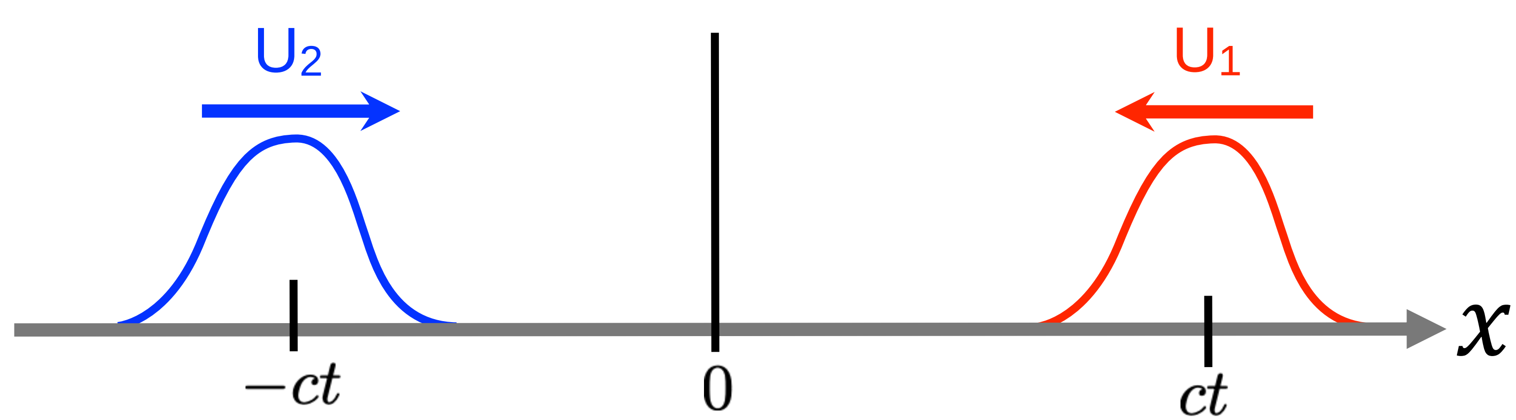
non-dispersive Waves

$c = \sqrt{g'H}$

$$\frac{\partial v}{\partial t} + \beta y u = -g' \frac{\partial \eta}{\partial y}$$

Cross-equatorial geostrophic balance

As before, this is a wave equation that has non-dispersive solutions with wave speed c for all wavenumbers. So any function of x will translate at speed c. The solution at x can be any function of (x ± ct).



The Kelvin wave solution

Assume no meridional flow

$$v = 0$$

$$\frac{\partial u}{\partial t} - \cancel{\beta y v} = -g'\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}\right) = 0$$

non-dispersive Waves
$c = \sqrt{g'H}$

$$\cancel{\frac{\partial v}{\partial t}} + \beta y u = -g'\frac{\partial \eta}{\partial y}$$

Cross-equatorial geostrophic balance

As before, this is a wave equation that has non-dispersive solutions with wave speed c for all wavenumbers. So any function of x will translate at speed c. The solution at x can be any function of (x ± ct).

As for coastal Kelvin waves, we can postulate solutions of the form the superposition of 2 independant waves

$$u = U_1(x+ct) + U_2(x-ct)$$

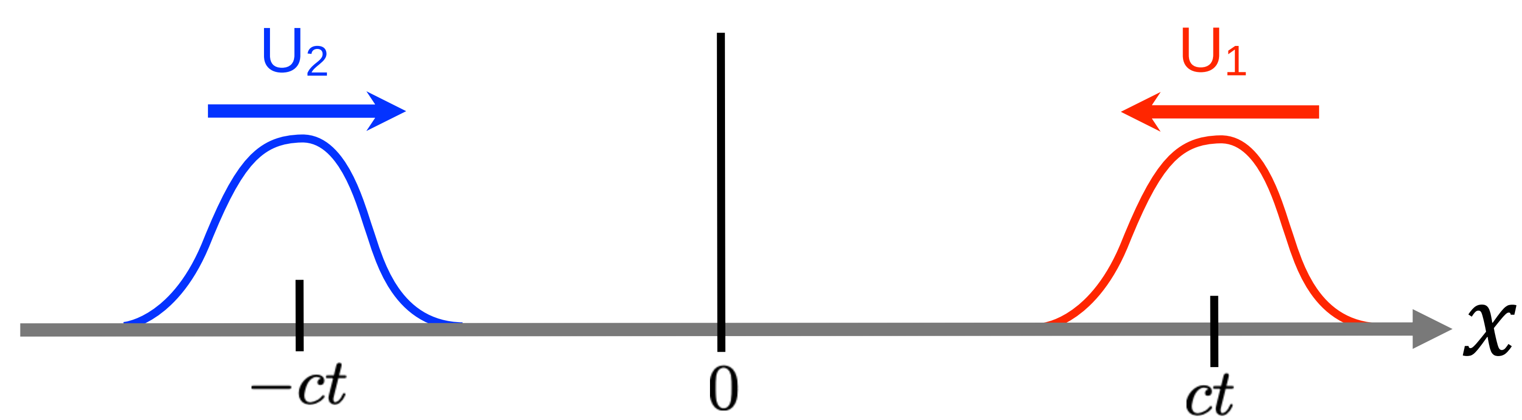
*where **U_1** propagates westwards and **U_2** propagates eastwards*

As before, the solution for η can be written in terms of U_1 and U_2

$$\eta = \sqrt{\frac{H}{g'}}(-U_1 + U_2)$$

(which can be verified by substitution, to give

$$\frac{\partial U_1}{\partial t} = c\frac{\partial U_1}{\partial x}$$
$$\frac{\partial U_2}{\partial t} = -c\frac{\partial U_2}{\partial x}$$
)



The EKW wave properties

⇨ The meridional structure is given by the remaining equation which expresses cross-equatorial geostrophic balance !

$$\beta y u = -g' \frac{\partial \eta}{\partial y}$$

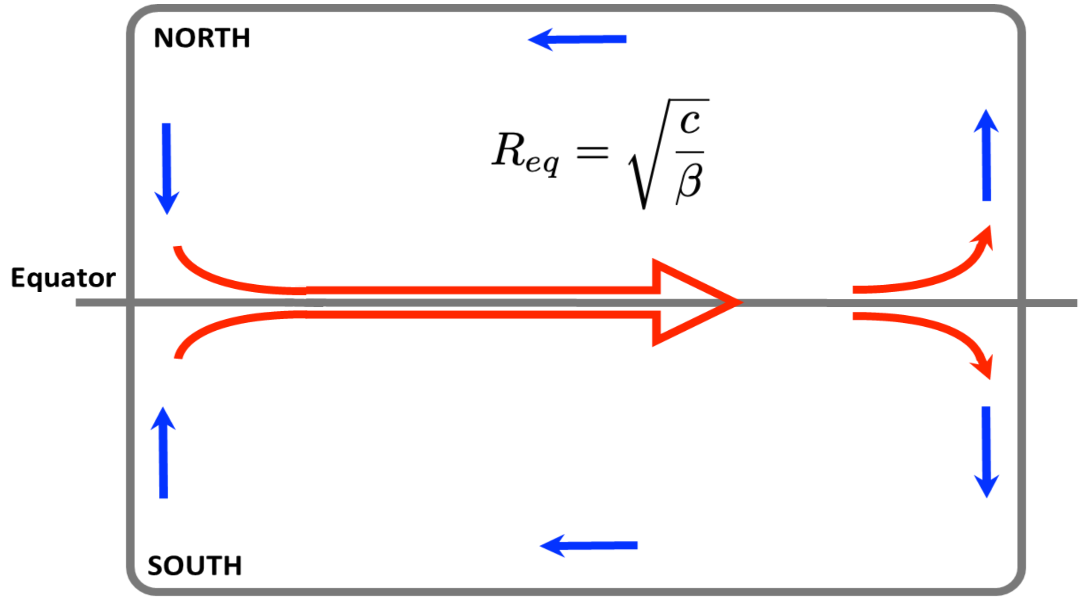
⇨ Substituting our solutions gives:

$$\beta y(U_1 + U_2) = -c\frac{\partial}{\partial y}(-U_1 + U_2)$$

Meridional structures are: $U_1 \sim e^{\frac{\beta}{2c}y^2}$ $\quad U_2 \sim e^{-\frac{\beta}{2c}y^2}$

↳ Only the **eastward propagating** solution U_2 is exponentially decaying in y^2. Note the difference with coastal waves that depended on nonzero f, and thus, y. Now we have a y^2 dependence that works both to the north and south with the same propagation direction.

⇨ If we write $U_2(x - ct) = c\psi(x - ct)$, where ψ is a dimensionless wave form in the x-direction, **equatorial Kelvin wave solution** can be written:



$$u = c\,\psi(x - ct)\,e^{-y^2/2R_{eq}^2}$$
$$v = 0$$
$$\eta = H\,\psi(x - ct)\,e^{-y^2/2R_{eq}^2}$$

EKW have the following properties:

- propagates eastwards
- non-dispersive $c = \sqrt{g'H}$
- maximum on equator

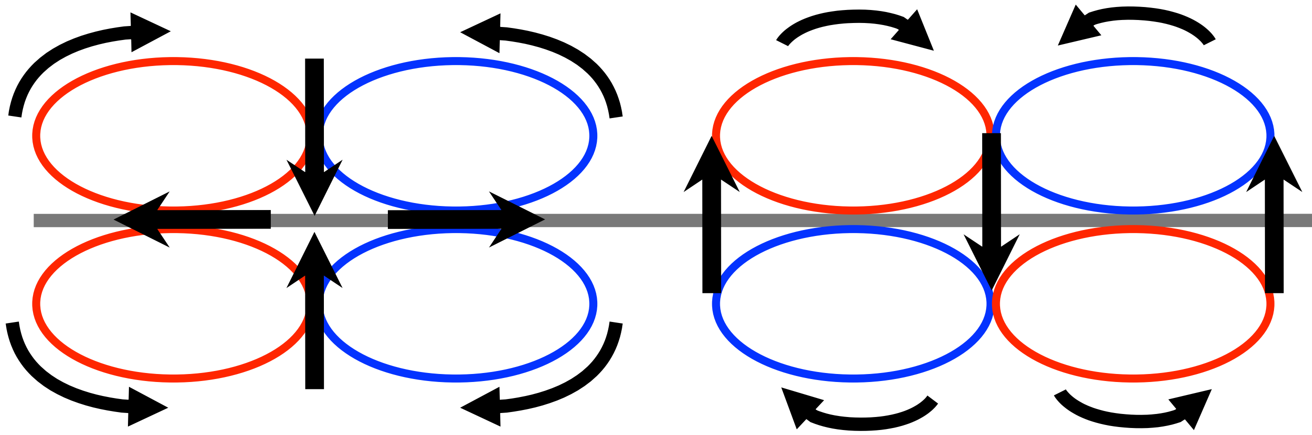
Cf. coastal Kelvin wave, propped up against the coast.
An EKW is "propped up" against another equatorial Kelvin wave.

The general solution

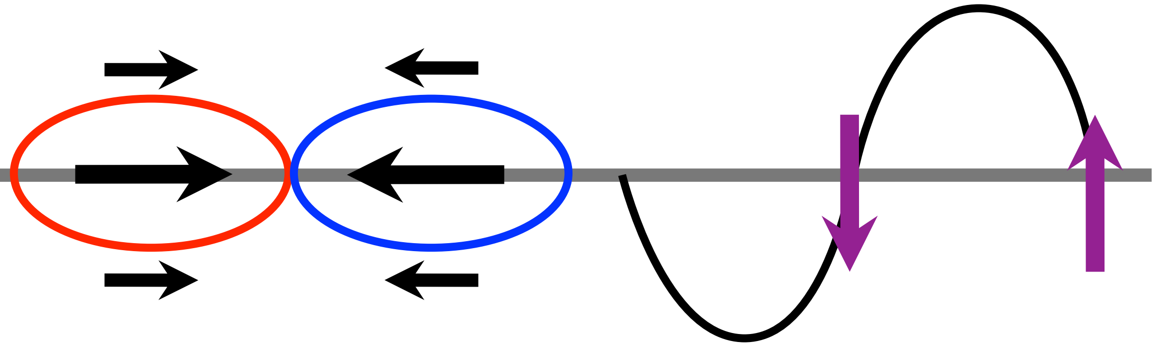
Now we allow wavelike variations in the zonal direction including for v

$$u = \tilde{u}(y)e^{i(lx-wt)} \qquad v = \tilde{v}(y)e^{i\left(lx-wt+\frac{\pi}{2}\right)} \qquad \eta = \tilde{\eta}(y)e^{i(lx-wt)}$$

Note that we specify u and η in phase with one another, but v is in quadrature with them.



rotational flow



divergent flow

Substitution into equatorial shallow water equations …

$$\frac{\partial u}{\partial t} - \beta y v = -g'\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \beta y u = -g'\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

details

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta y v = -g' \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial v}{\partial t} + \beta y u = -g' \dfrac{\partial \eta}{\partial y} \\ \dfrac{\partial \eta}{\partial t} + H \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \right) = 0 \end{cases}$$

$$u = \tilde{u}(y) e^{i(kx - \omega t)} \qquad v = \tilde{v}(y) e^{i(kx - \omega t \pm \pi/2)}$$

$$\eta = \tilde{\eta}(y) e^{i(kx - \omega t)} \qquad = \tilde{v}(y) e^{i(kx - \omega t)} e^{\pm i\pi/2}$$

$$= \pm i \tilde{v}(y) e^{i(kx - \omega t)}$$

v in quadrature with u,
+ or - makes no difference, we choose +

$$\begin{cases} -i\omega\tilde{u} - i\beta y \tilde{v} + i g' k \tilde{\eta} = 0 \\ \omega \tilde{v} + \beta y \tilde{u} + g' \dfrac{\partial \tilde{\eta}}{\partial y} = 0 \\ -i\omega\tilde{\eta} + H \left(ik\tilde{u} + i \dfrac{\partial \tilde{v}}{\partial y} \right) = 0 \end{cases}$$

We want to eliminate u and η to get an equation for v.

We drop tildes and prime on g, and we use subscript notation for derivatives. The linear system can be written:

$$\begin{cases} \omega u + \beta y v - g k \eta = 0 & (1) \\ \omega v + \beta y u + g \dfrac{\partial \eta}{\partial y} = 0 & (2) \\ -\omega \eta + H k u + H \dfrac{\partial v}{\partial y} = 0 & (3) \end{cases}$$

details

$$\begin{cases} \partial/\partial y(1) + k \times (2) \rightarrow \omega u_y + \beta v + \beta y v_y + \omega k v + \beta y k u = 0 & (A) \\ \omega \times (2) + g \times \partial/\partial y(3) \rightarrow \omega^2 v + \beta y \omega u + gHku_y + gHv_{yy} = 0 & (B) \\ \omega \times (1) - gk \times (3) \rightarrow \omega^2 u + \beta y \omega v - gk^2 Hu - gkHv_y = 0 & (C) \end{cases}$$

⇨ $gHk \times (A) - \omega \times (B) \rightarrow$

$$gHk(\beta v + \beta y v_y + \omega k v) + gHk^2 \beta y u - \omega^3 v - \beta y \omega^2 u - gH\omega v_{yy} = 0$$

$$-gH\omega v_{yy} + gHk\beta y v_y + (gHk\beta + gH\omega k^2 - \omega^3)v + (gHk^2\beta y - \beta y \omega^2)u = 0 \qquad (D)$$

⇨ $(D) + \beta y \times (C) \rightarrow$

$$-gH\omega v_{yy} + gHk\beta y v_y + (gHk\beta + gH\omega k^2 - \omega^3)v + \beta^2 y^2 \omega v - \beta y gHkv_y = 0$$

⇨ $\div - gH\omega \rightarrow \dfrac{d^2\tilde{v}}{dy^2} + \left[\dfrac{\omega^2}{g'H} - k^2 - \dfrac{k\beta}{\omega} - \dfrac{\beta^2}{g'H}y^2\right]\tilde{v} = 0$

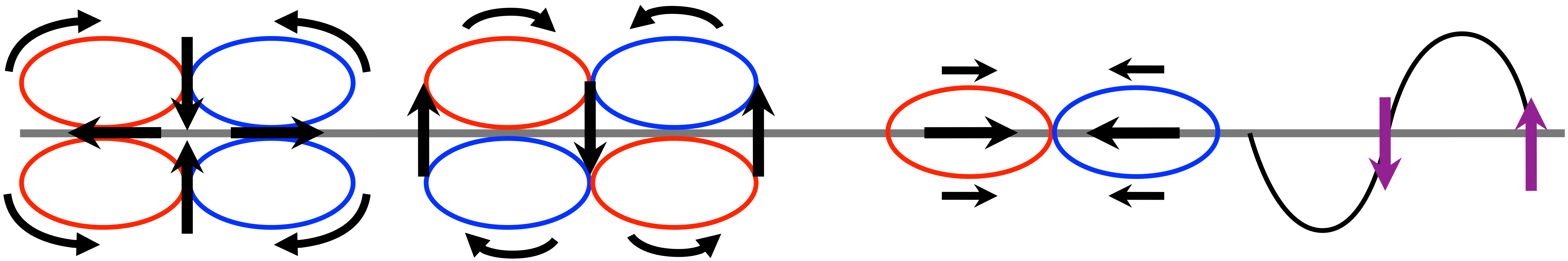
or $\dfrac{d^2\tilde{v}}{dy^2} + \dfrac{\beta^2}{c^2}(Y^2 - y^2)\tilde{v} = 0$ where $\begin{cases} c \text{ is the gravity wave speed)} \\ Y^2 = \dfrac{g'H}{\beta^2}\left[\dfrac{\omega^2}{g'H} - k^2 - \dfrac{k\beta}{\omega}\right] \end{cases}$

The general solution

Now we allow wavelike variations in the zonal direction including for v

$$u = \tilde{u}(y)e^{i(lx-wt)} \qquad v = \tilde{v}(y)e^{i\left(lx-wt+\frac{\pi}{2}\right)} \qquad \eta = \tilde{\eta}(y)e^{i(lx-wt)}$$

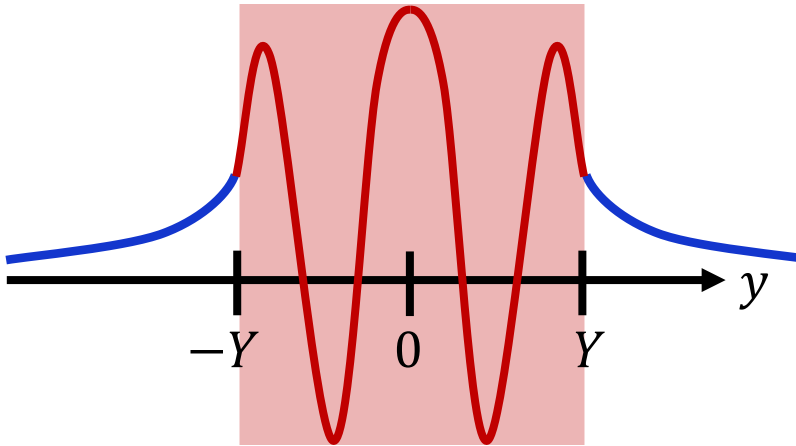
Note that we specify u and η in phase with one another, but v is in quadrature with them.



Substitution into equatorial shallow water equations gives

$$\frac{d^2\tilde{v}}{dy^2} + \frac{\beta^2}{c^2}(Y^2 - y^2)\tilde{v} = 0 \quad \text{where} \quad Y^2 = \left(\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega}\right)\frac{c^2}{\beta^2}$$

y < Y : oscillating solutions in y
y > Y : decaying solutions in y



Y is the width of the "equatorial waveguide".

Y depends on wavelength and frequency but scales similar to R_{eq}.
It represents the zone in which we have some meridional wave structure.
Outside this zone the amplitude decays exponentially with latitude.

Meridional structure

It can be shown that the general solution is of the form

$$\tilde{v} \propto H_n(y')\ e^{-y'^2/2} \qquad (y' = y/R_{eq})$$

Remember that u and η have opposite symmetry to v

and substitution of this form into the differential equation for v leads to the dispersion relation

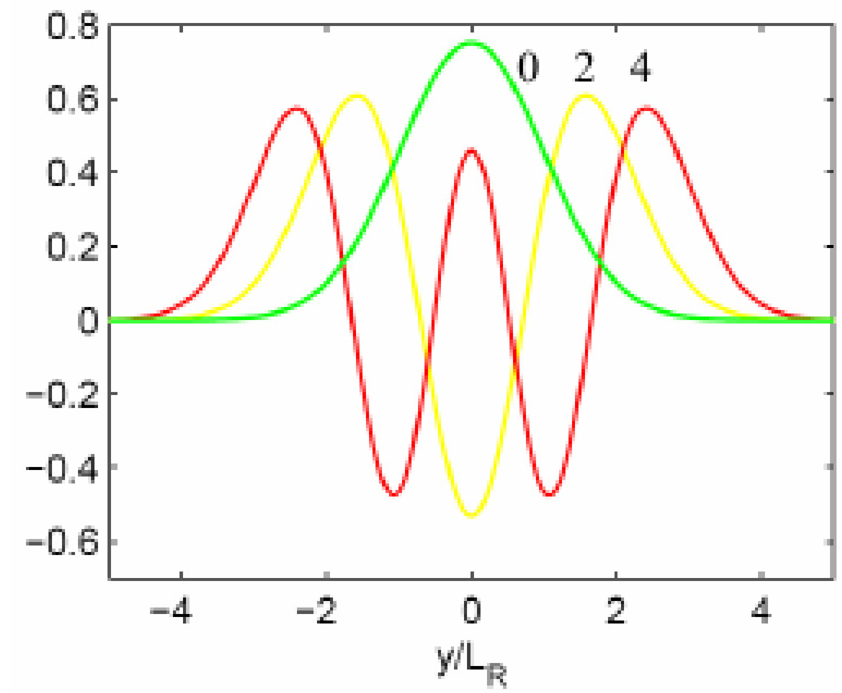
$$\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega} = (2n+1)\frac{\beta}{c} = \frac{(2n+1)}{R_{eq}^2}$$

In fact this is a set of dispersion relations corresponding to a discrete set of meridional structures $H_n(y')$, the "Hermite polynomials".

$$H_0(y') = 1$$
$$H_1(y') = 2y'$$
$$H_2(y') = 4y'^2 - 2$$
$$H_3(y') = 8y'^3 - 12y'$$
$$H_4(y') = 16y'^4 - 48y'^2 - 12 \quad \ldots$$

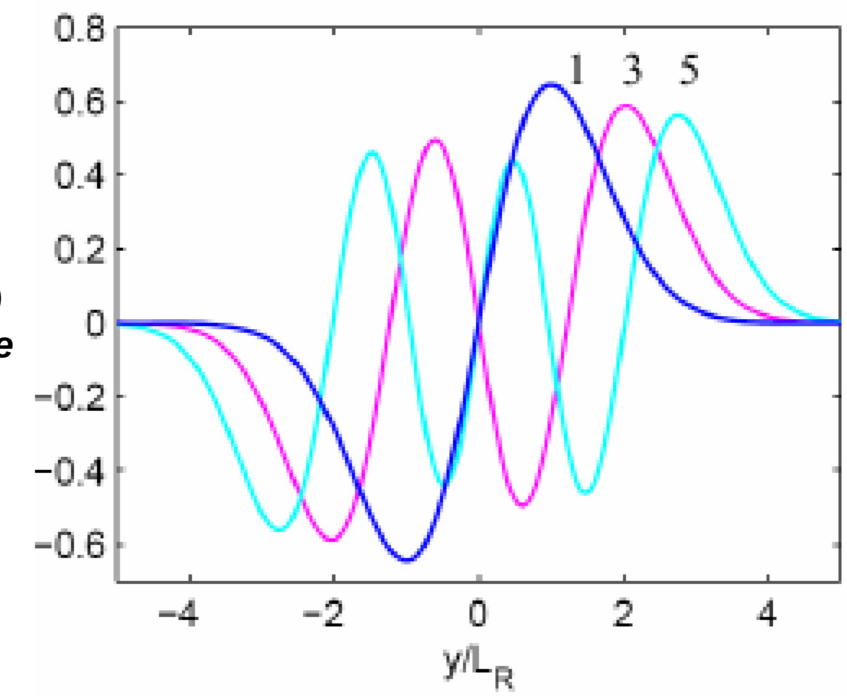
$$y' H_n = nH_{n-1} + \frac{1}{2}H_{n+1} \qquad \frac{dH_n}{dy'} = 2nH_{n-1}$$

Symmetric structures for v: n=0,2,4...



Cross-equatorial flow and anti-symmetric η

Anti-symmetric structures for v: n=1,3,5...



No cross-equatorial flow (convergence/divergence) and symmetric thermocline displacements

details

$$\frac{d^2v}{dy^2} + \frac{1}{R_{eq}^4}(Y^2 - y^2)v = 0$$

$$R_{eq} = \sqrt{\frac{c}{\beta}}, \qquad Y^2 = \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right) R_{eq}^4 \quad \{= (2n+1)R_{eq}^2\}$$

$$y' = y/R_{eq}, \quad Y' = Y/R_{eq} \rightarrow \quad \frac{1}{R_{eq}^2}\frac{d^2v}{dy'^2} + \frac{1}{R_{eq}^4}(Y'^2 - y'^2)vR_{eq}^2 = 0$$

dropping primes $\dfrac{d^2v}{dy^2} + (Y^2 - y^2)v = 0$ solution $v = H_n e^{-y^2/2}$

should lead to non-dimensional dispersion relation $Y^2 = 2n + 1$

using $\dfrac{dH_n}{dy} = 2nH_{n-1}$ and $yH_n = nH_{n-1} + \dfrac{H_{n+1}}{2}$

details

$$\frac{dv}{dy} = \frac{dH_n}{dy} e^{-y^2/2} - yH_n e^{-y^2/2} = \left[\frac{dH_n}{dy} - yH_n\right] e^{-y^2/2}$$

$$\frac{dv}{dy} = \left[2nH_{n-1} - \left(nH_{n-1} + \frac{H_{n+1}}{2}\right)\right] e^{-y^2/2} = \left[nH_{n-1} - \frac{H_{n+1}}{2}\right] e^{-y^2/2} = \left[yH_n - H_{n+1}\right] e^{-y^2/2}$$

$$\frac{d^2v}{dy^2} = \left[H_n + y\frac{dH_n}{dy} - \frac{dH_{n+1}}{dy} - y\left(yH_n - H_{n+1}\right)\right] e^{-y^2/2}$$

$$= \left[H_n + 2ynH_{n-1} - 2(n+1)H_n - y^2 H_n + yH_{n+1}\right] e^{-y^2/2}$$

$$= \left[H_n + y\left(2nH_{n-1} - yH_n + H_{n+1}\right) - 2(n+1)H_n\right] e^{-y^2/2}$$

$$= \left[-H_n - 2nH_n + y^2 H_n\right] e^{-y^2/2}$$

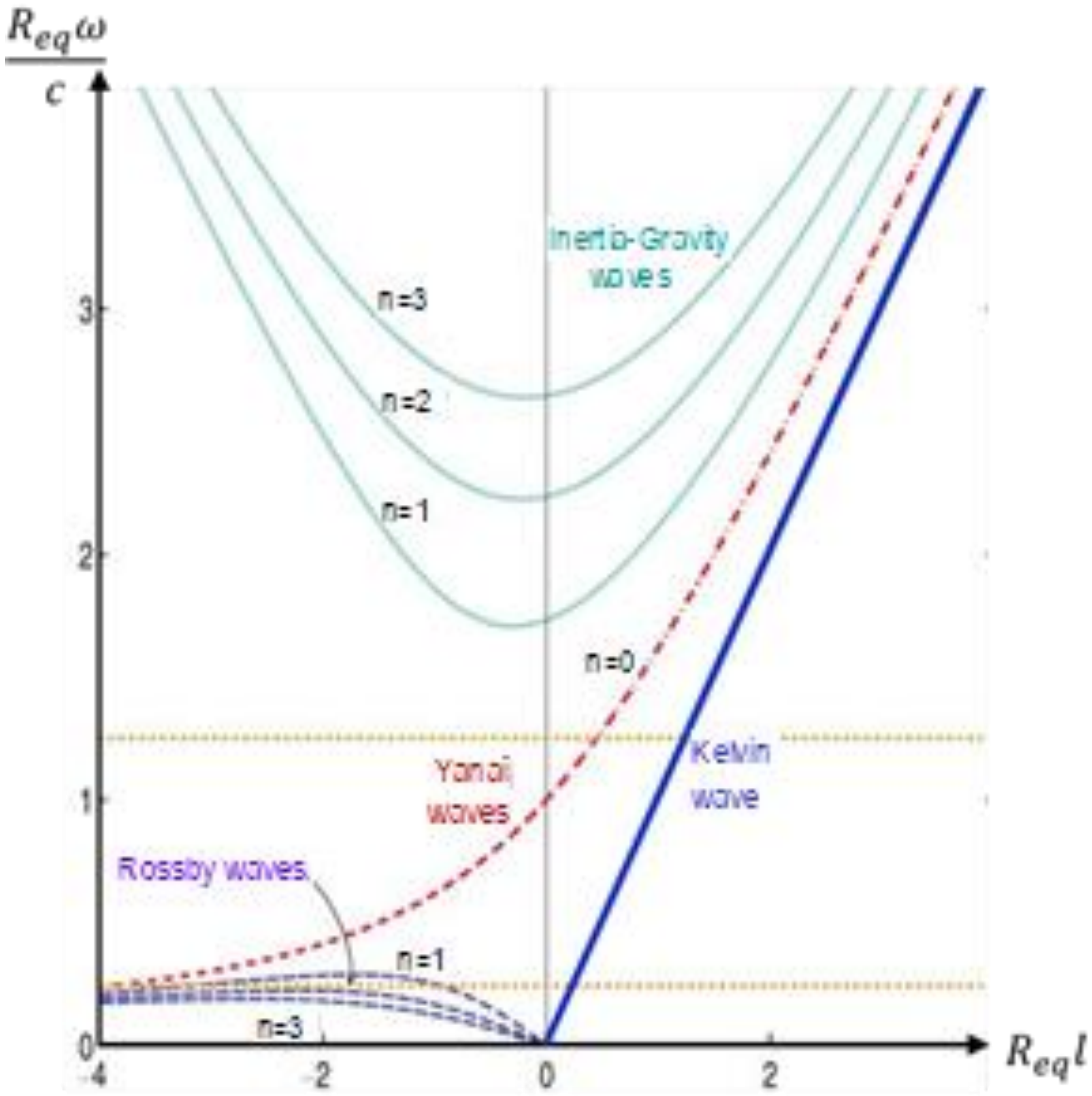
so $\left[y^2 - (2n+1)\right] H_n e^{-y^2/2} + (Y^2 - y^2) H_n e^{-y^2/2} = 0$

thus $Y^2 = 2n + 1$

The dispersion relations

⇨ Substitution of the general solutions into the differential equation for v leads to a set of dispersion relations: $\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega} = (2n+1)\frac{\beta}{c}$ ⇨ *There are 3 roots for each value of n ≥ 1.*

⇨ The entire family of equatorially trapped waves: $\omega = \omega(l, n)$



- *The largest roots are for high frequencies ($T < T_{eq}$). ⇨ They are **inertia-gravity wa**ves slightly modified by the beta effect.*

- *The smaller root for ω are equatorial **Rossby waves***

- *A mixed Rossby / Inertia Gravity wave (sometimes called “Yanai wave”) exists for n = 0.*

- *The special case of v = 0 corresponds to n = –1, this is the Kelvin wave.*

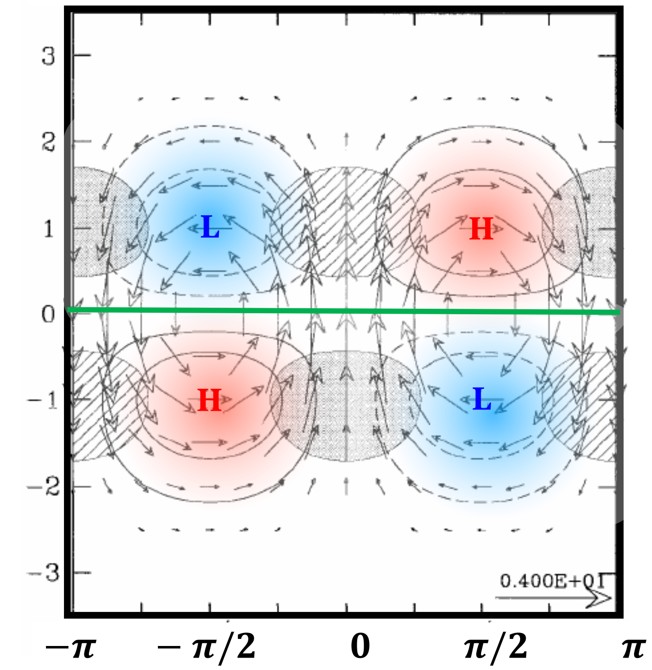
Wave properties

⇨ *Odd order waves (n = –1,1,3..) are symmetric in η : Kelvin, Rossby and Inertia Gravity waves.*

⇨ *Even order waves (n = 0,2...) are antisymmetric in η : mixed Rossby-Gravity waves*

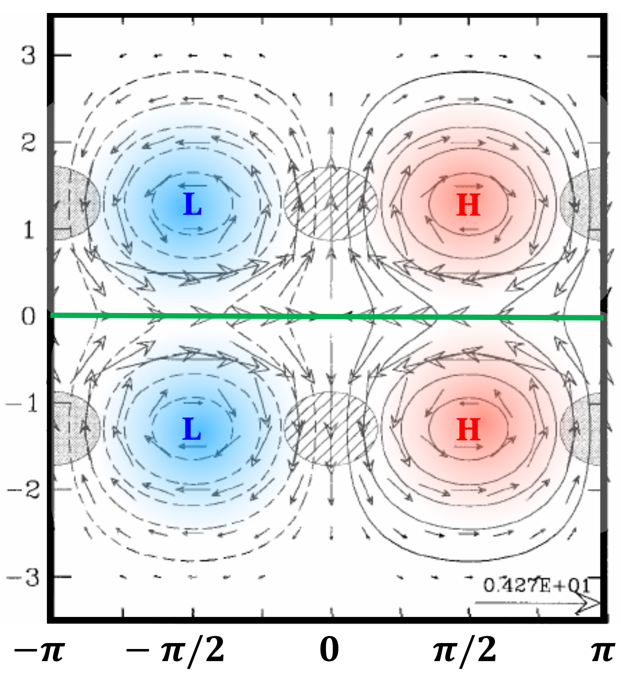
$n = 0, l^* = 1$

Mixed Rossby-gravity wave



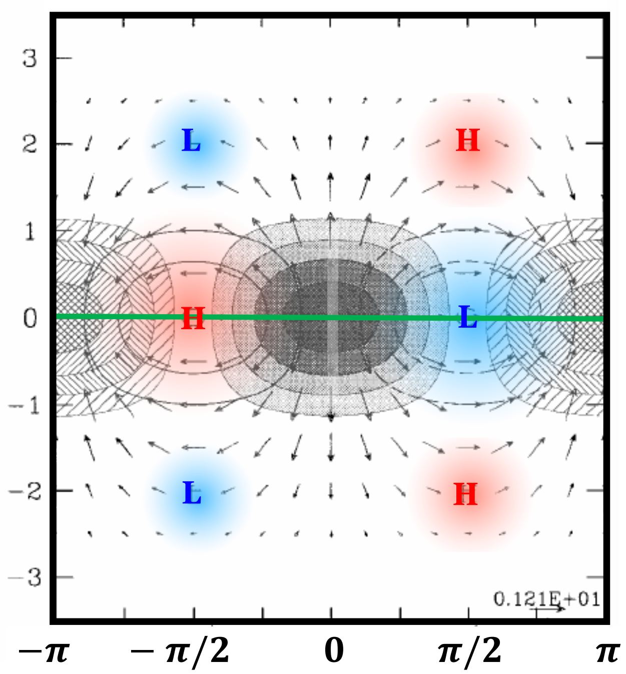
$n = 1,\ l^* = 1$

Equatorial Rossby wave



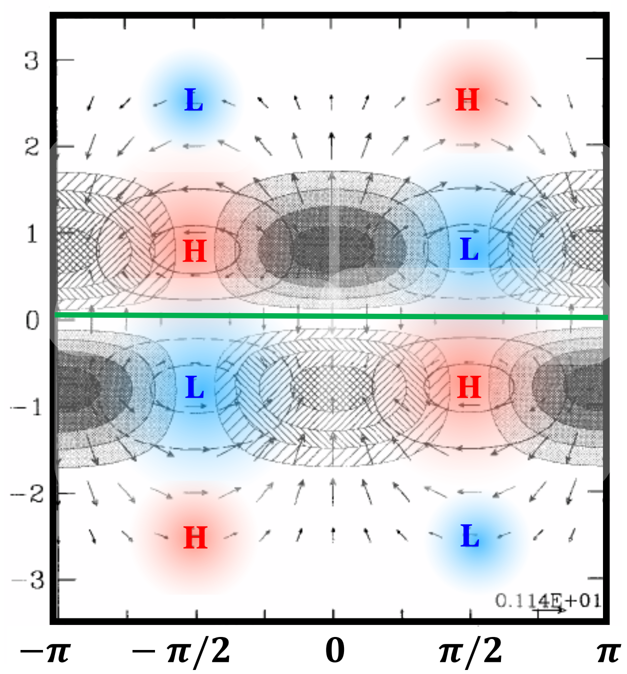
$n = 1, l^* = 1$

Westward IG wave



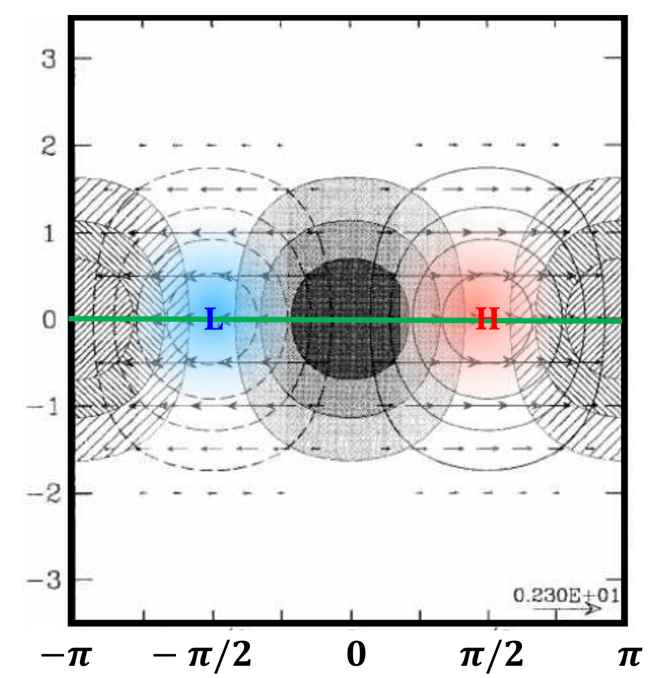
$n = 2, l^* = 1$

Eastward IG wave



$n = -1, l^* = 1$

Equatorial Kelvin wave



n≥1 Equatorial wave structures

Fig. 4. Pressure and velocity distributions of eigensolutions for $n=1$
a: Eastward propagating inertio-gravity wave
b: Westward propagating inertio-gravity wave
c: Rossby wave.

from Matsuno (1966)

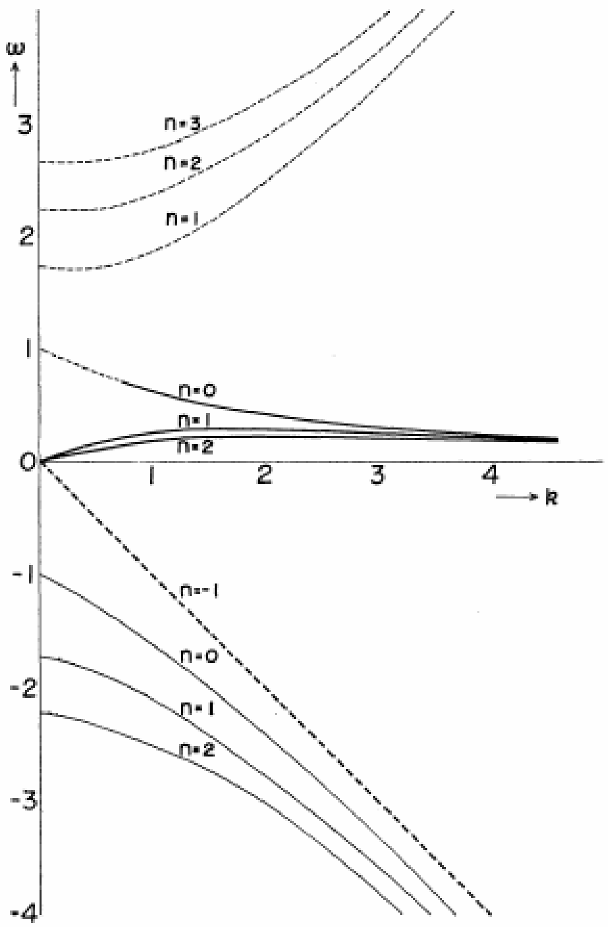
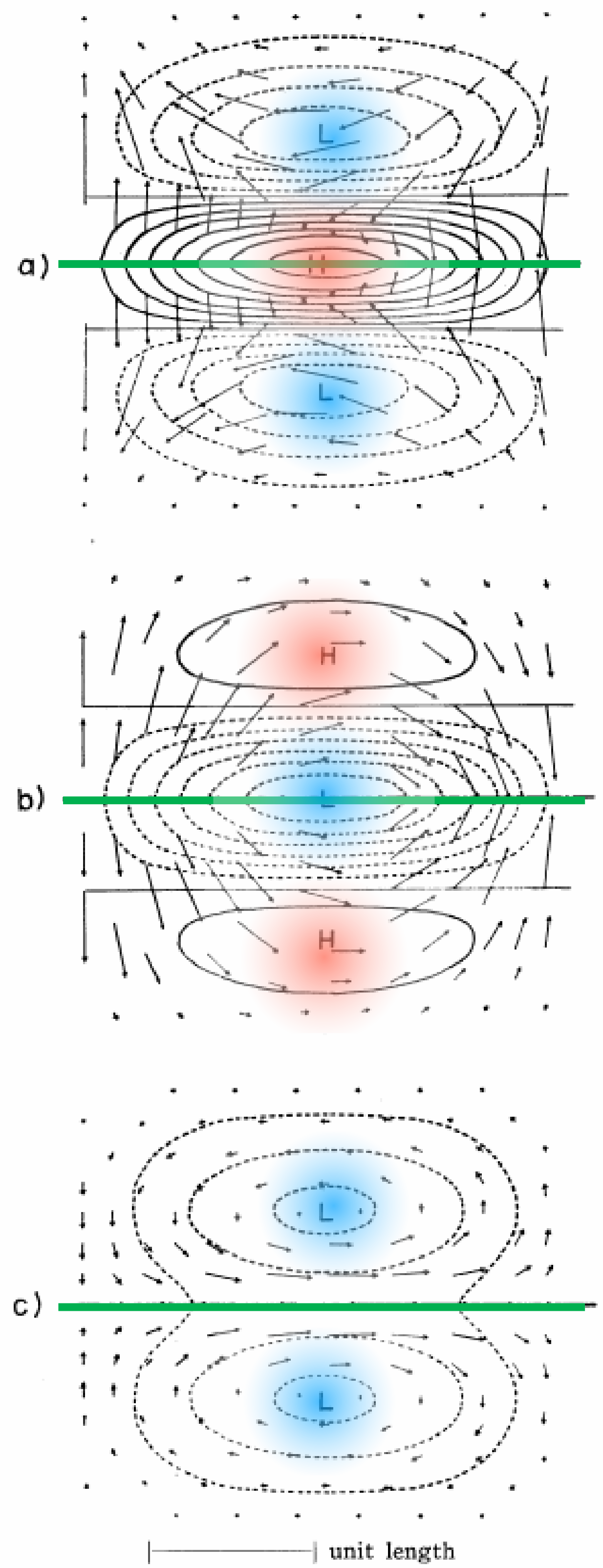


Fig. 3a. Frequencies as functions of wave number.
Thin solid line: eastward propating inertio-gravity waves.
Thin dashed line: westward propagating inertio-gravity waves.
Thick solid line: Rossby (quasi-geostrophic) waves.
Thick dashed line: The Kelvine wave like wave.



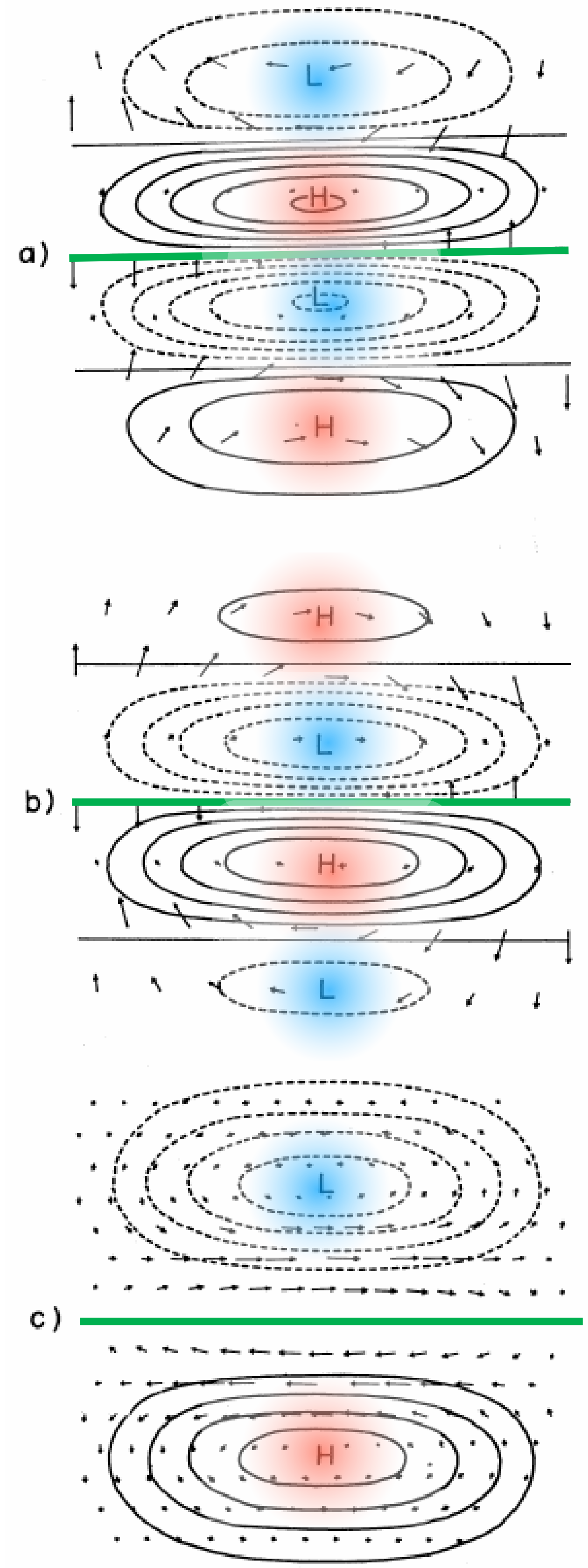


Fig. 5. Same as Fig. 4 but for $n=2$.

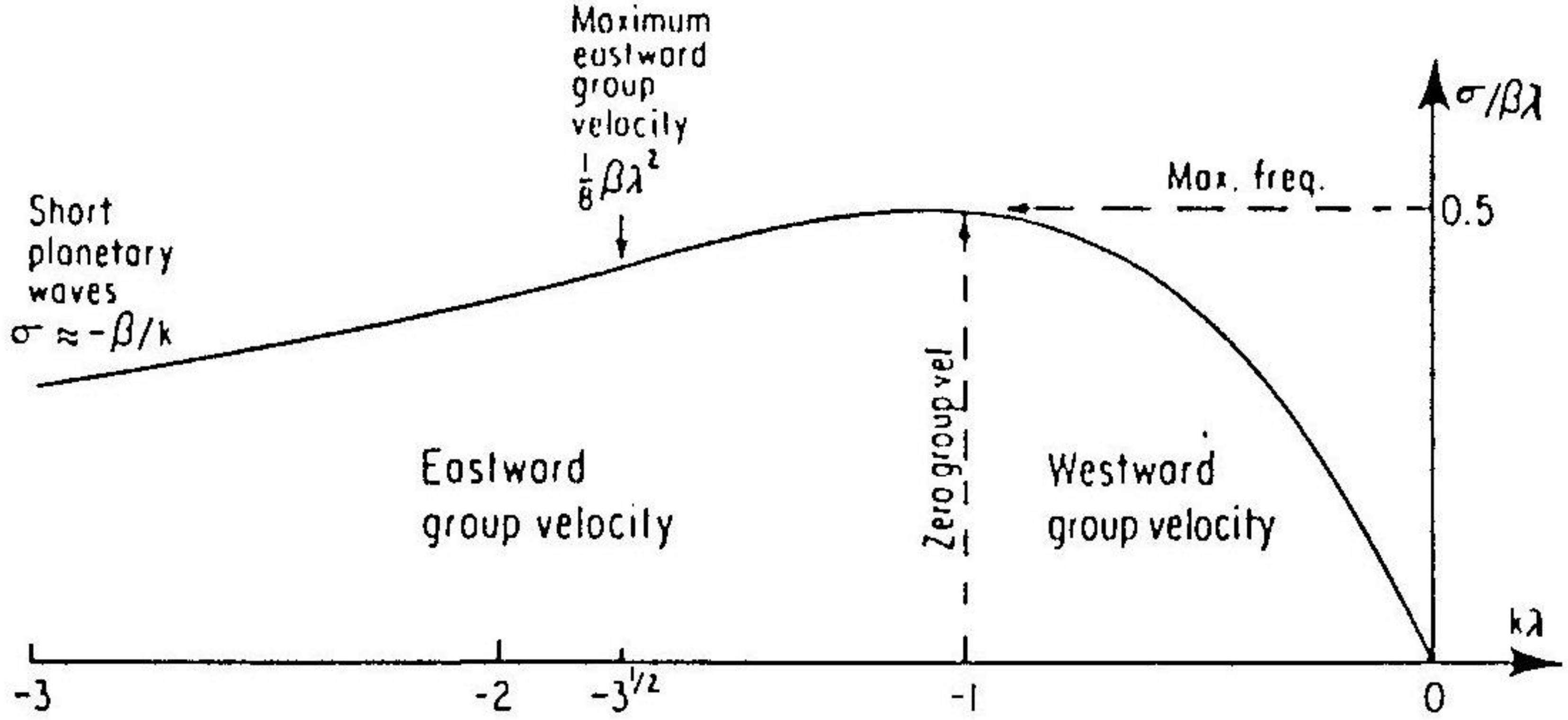
from Matsuno (1966)

Equatorial Rossby waves

$$\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega} = \frac{2n+1}{R_{eq}^2}$$

At low frequencies ω << f

$$\omega = -\frac{\beta l}{l^2 + (2n+1)R_{eq}^{-2}}$$



=> k negative, Rossby waves have westward phase propagation.
But the group velocity depends on the wavelength.

In practice the shorter Rossby waves with eastward group propagation are of little importance because they are dispersive, slow, and tend to dissipate.

Equatorial Rossby rays

Generally as a wave propagates its dispersion relation changes.

This is because it may change latitude, and f enters into the dispersion relation.
We will consider that f is "slowly varying". The direction of the group velocity is given by

$$\frac{dx}{dy} = \frac{\partial\omega/\partial n}{\partial\omega/\partial k} = -\frac{2\omega}{\beta}\left(-\frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right)^{\frac{1}{2}} \qquad \Rightarrow y = -\left(\frac{c^2 k}{\omega\beta}\right)^{\frac{1}{2}} \cos\left(\frac{2\omega}{c}x + \theta_0\right)$$

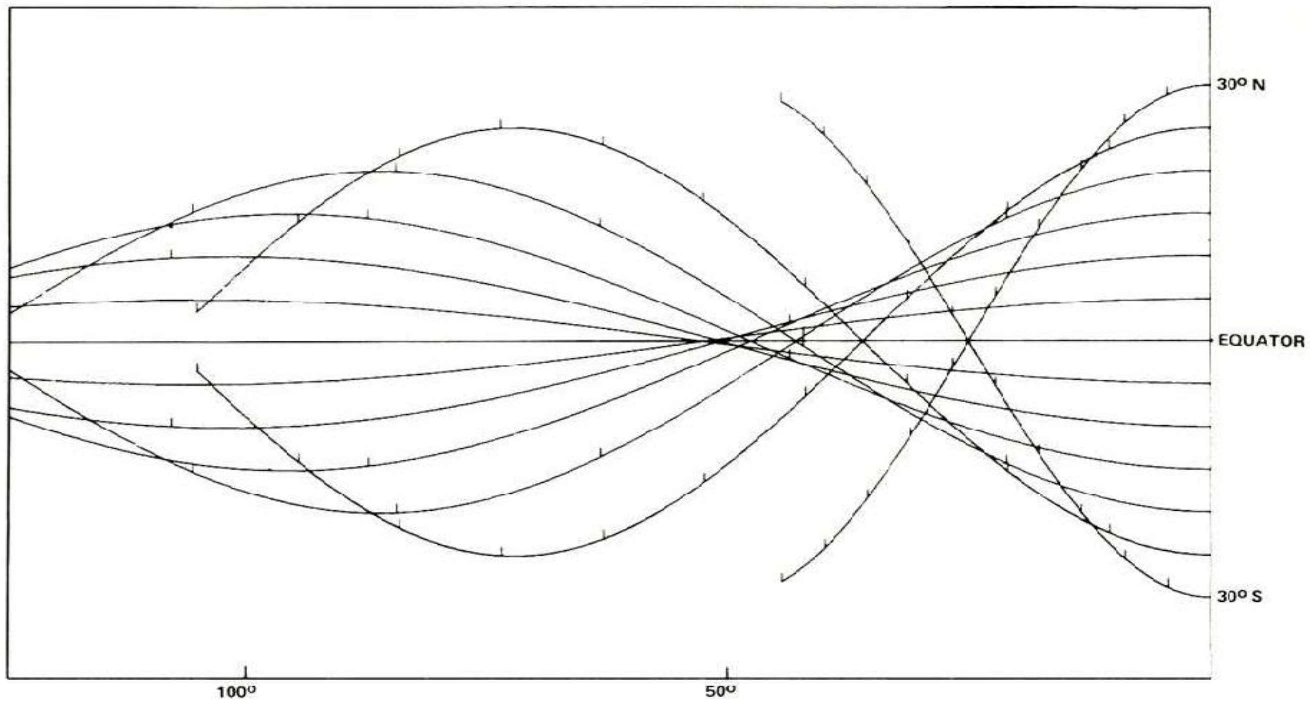
(for long R-waves)

Waves of constant frequency and zonal wavenumber will change their meridional wavenumber and thus their direction of propagation.

- they end up oscillating about the equator by refraction

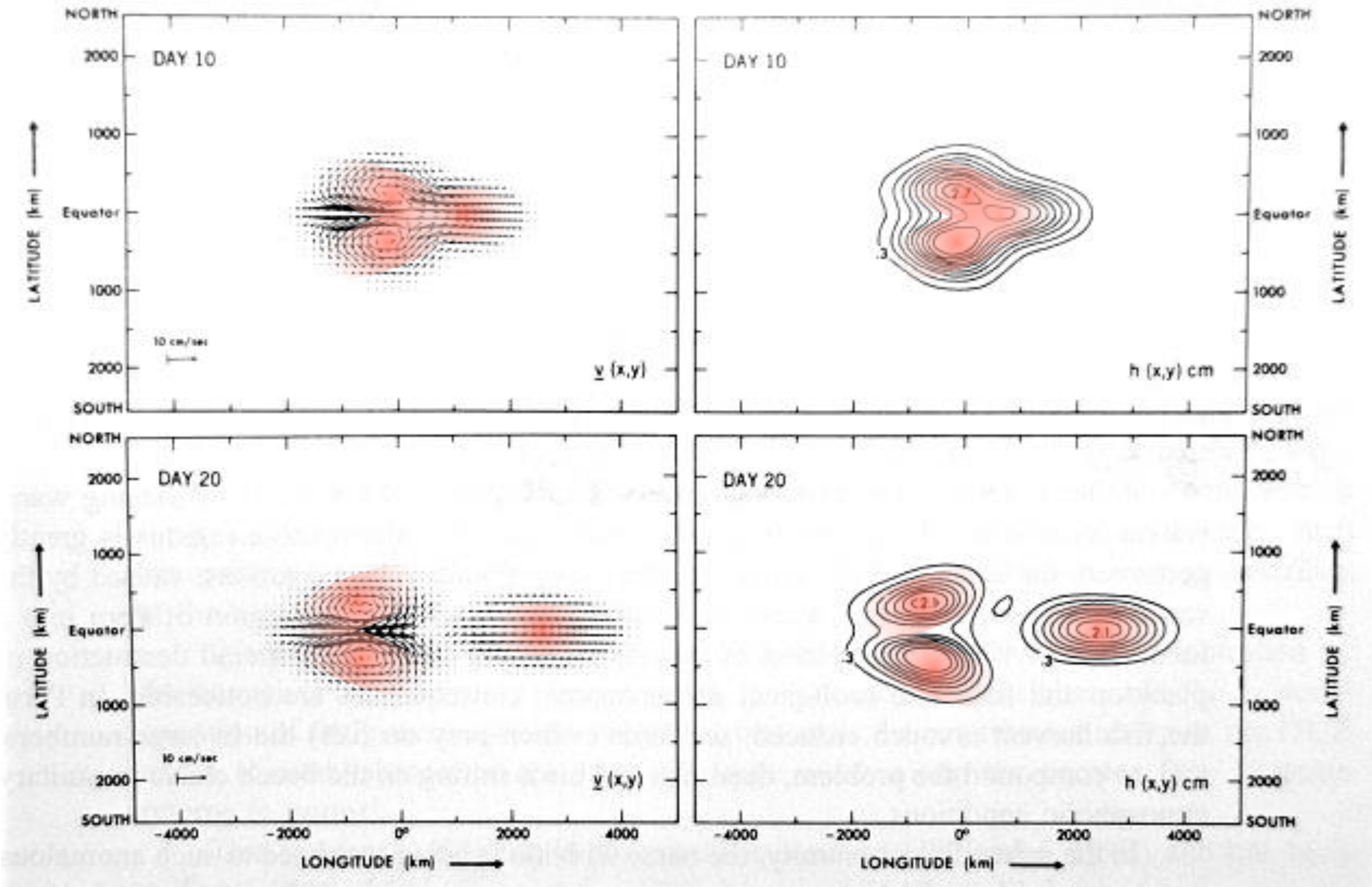
- its another way to show that they are "equatorially trapped"

- this behaviour is modified by the presence of mean currents



Oceanic adjustment

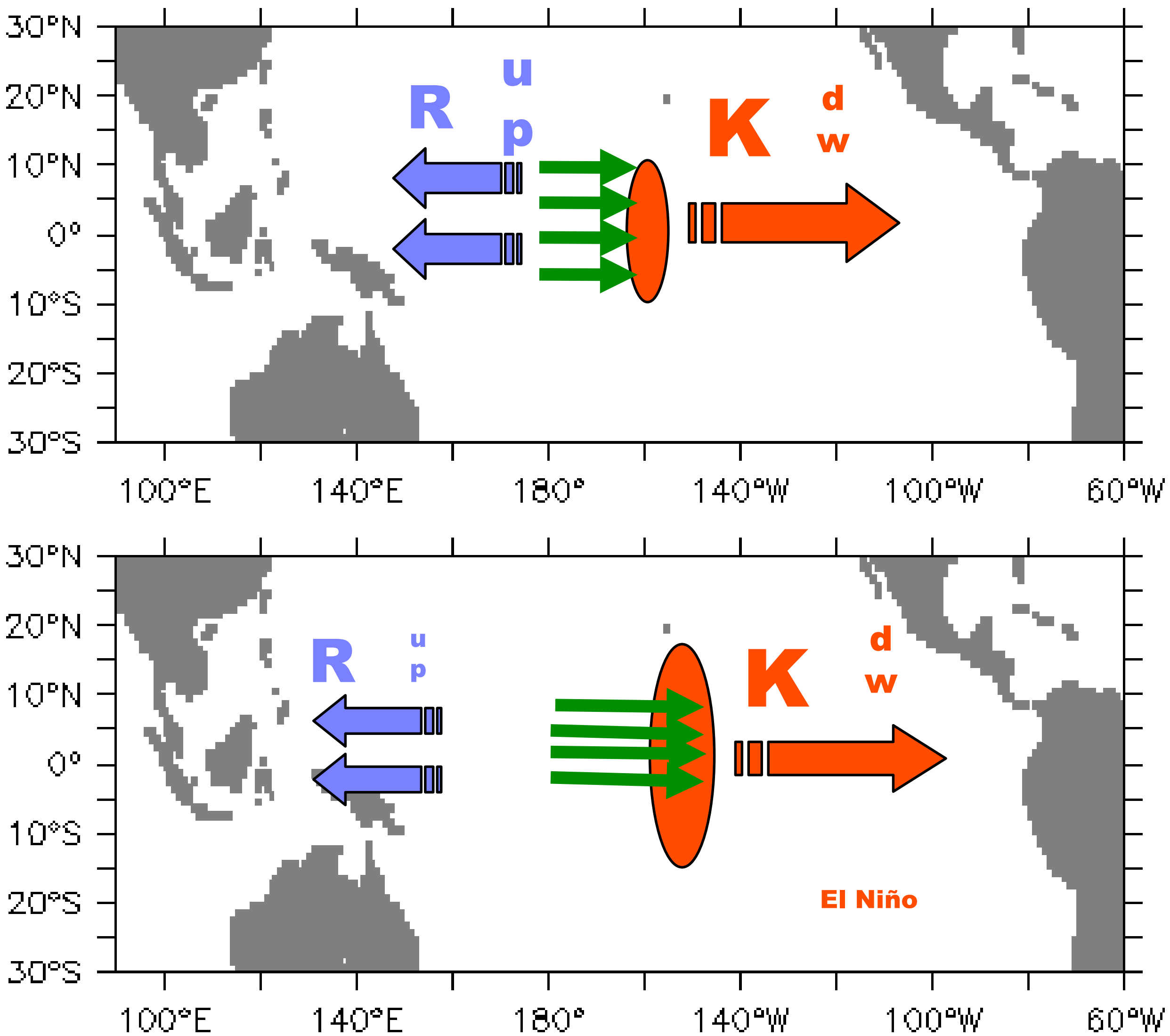
An abrupt change in the wind forcing can generate waves.
In this experiment an initial bell shaped perturbation to the thermocline is allowed to dissipate in a shallow water model. We see the single bulge (n = –1) Kelvin wave propagating eastwards and the double bulge (n = 1) Rossby wave propagating westwards.



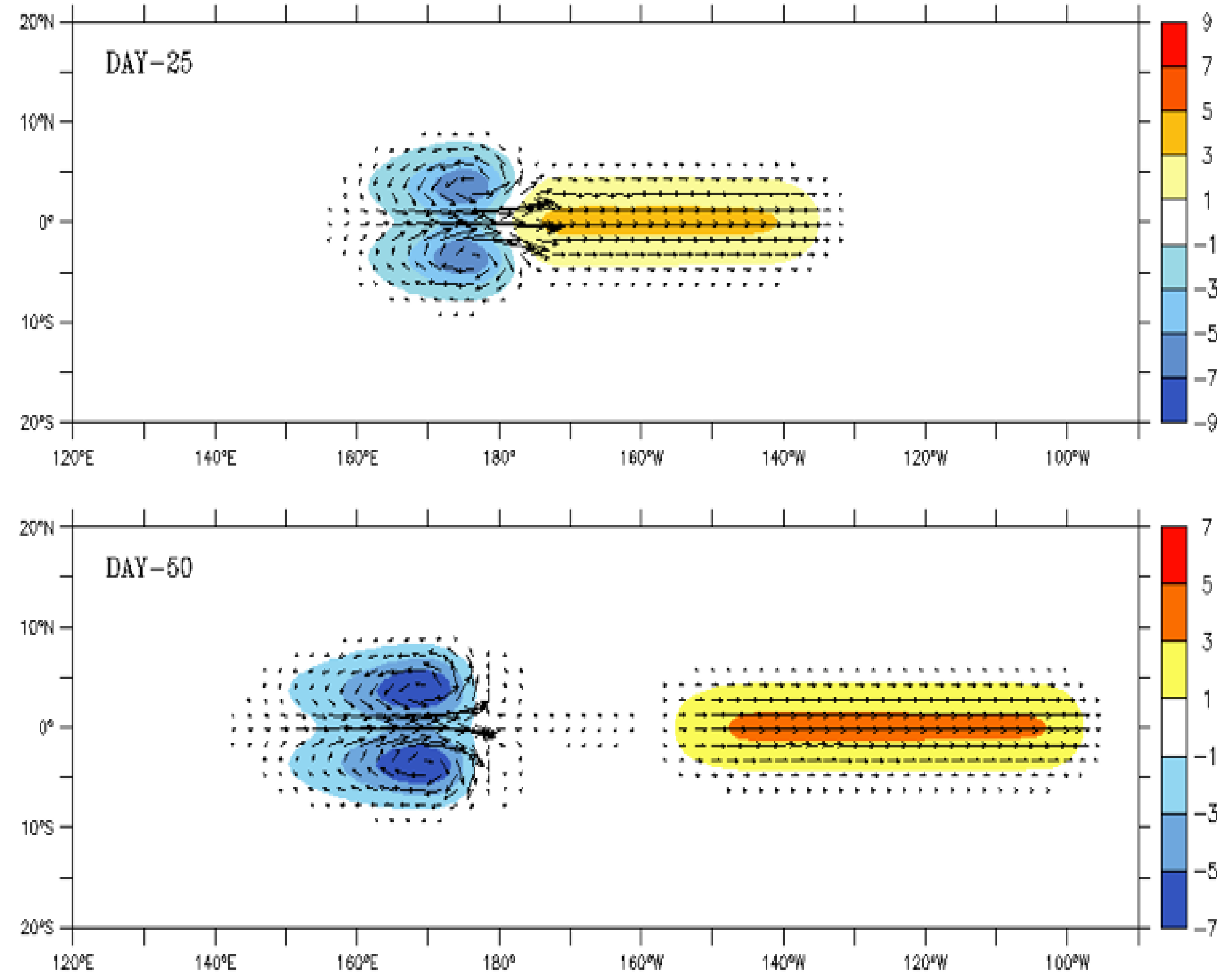
ENSO theories: the delayed oscillator

A mechanism proposed to explain how El Niño can cancel itself out the following season. Depends on wave reflection at boundaries.

SCEMATIC

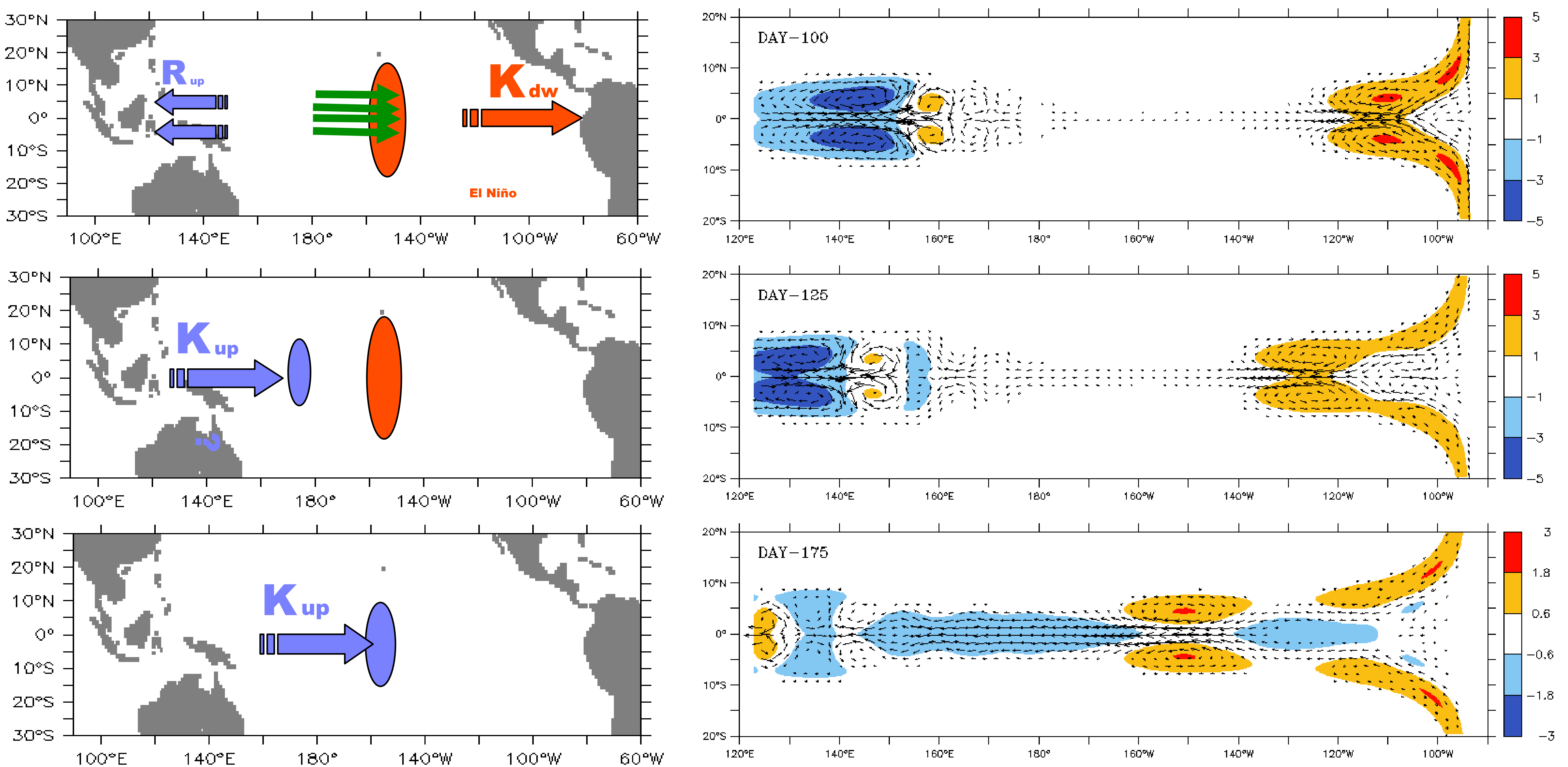


SURFACE CURRENTS and THERMOCLINE DEPTH



ENSO theories: the delayed oscillator

The upwelling Rossby wave at the base of the thermocline becomes an upwelling Kelvin wave traveling the other way. In the meantime, the original Kelvin wave leaks energy away at the eastern side of the basin through coastal Kelvin waves



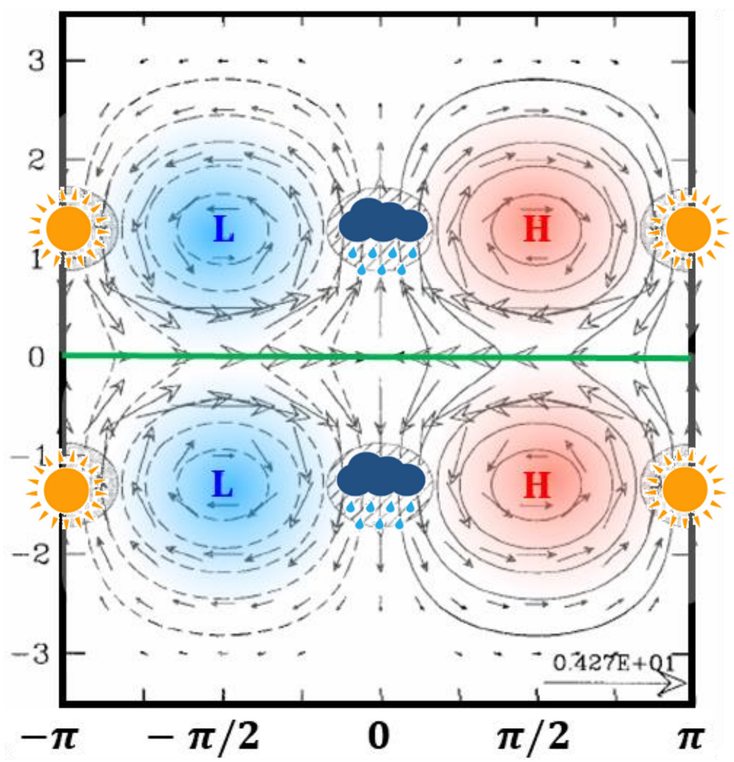
Tropical convection in the atmosphere

⇨ Divergence ➤ downward motion ➤ evaporation ➤ clear sky

⇨ Convergence ➤ upward motion ➤ condensation ➤ clouds

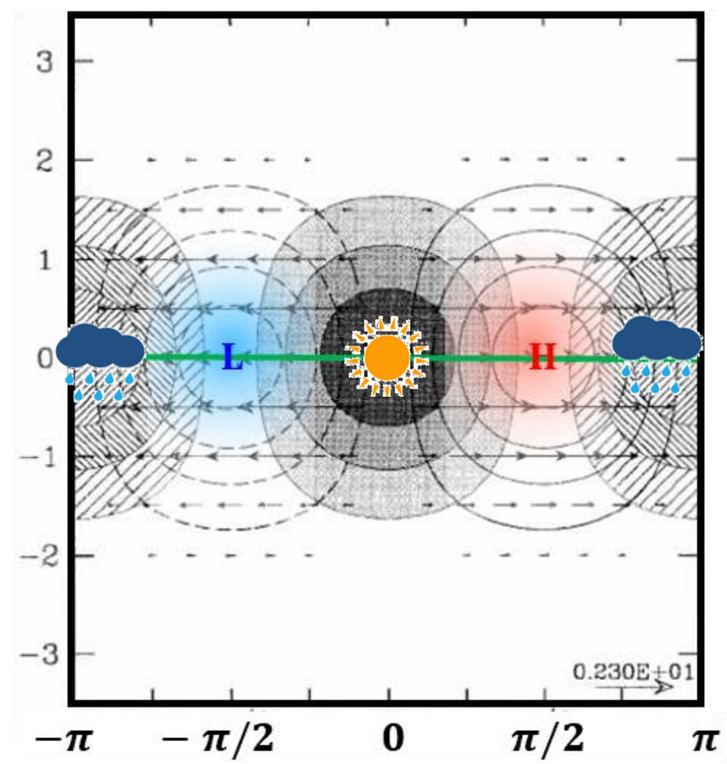
$n = 1,\ l^* = 1$

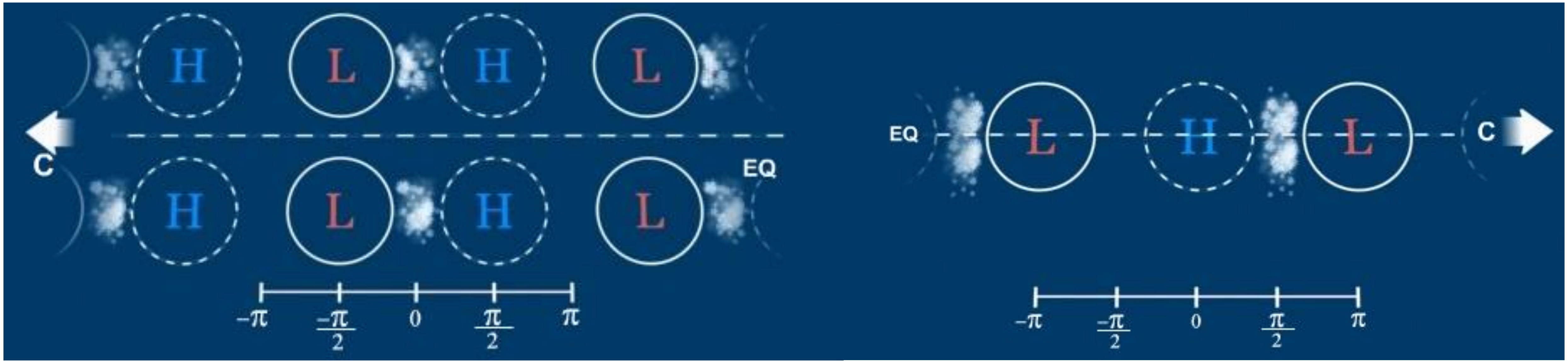
Equatorial Rossby wave



$n = -1, l^* = 1$

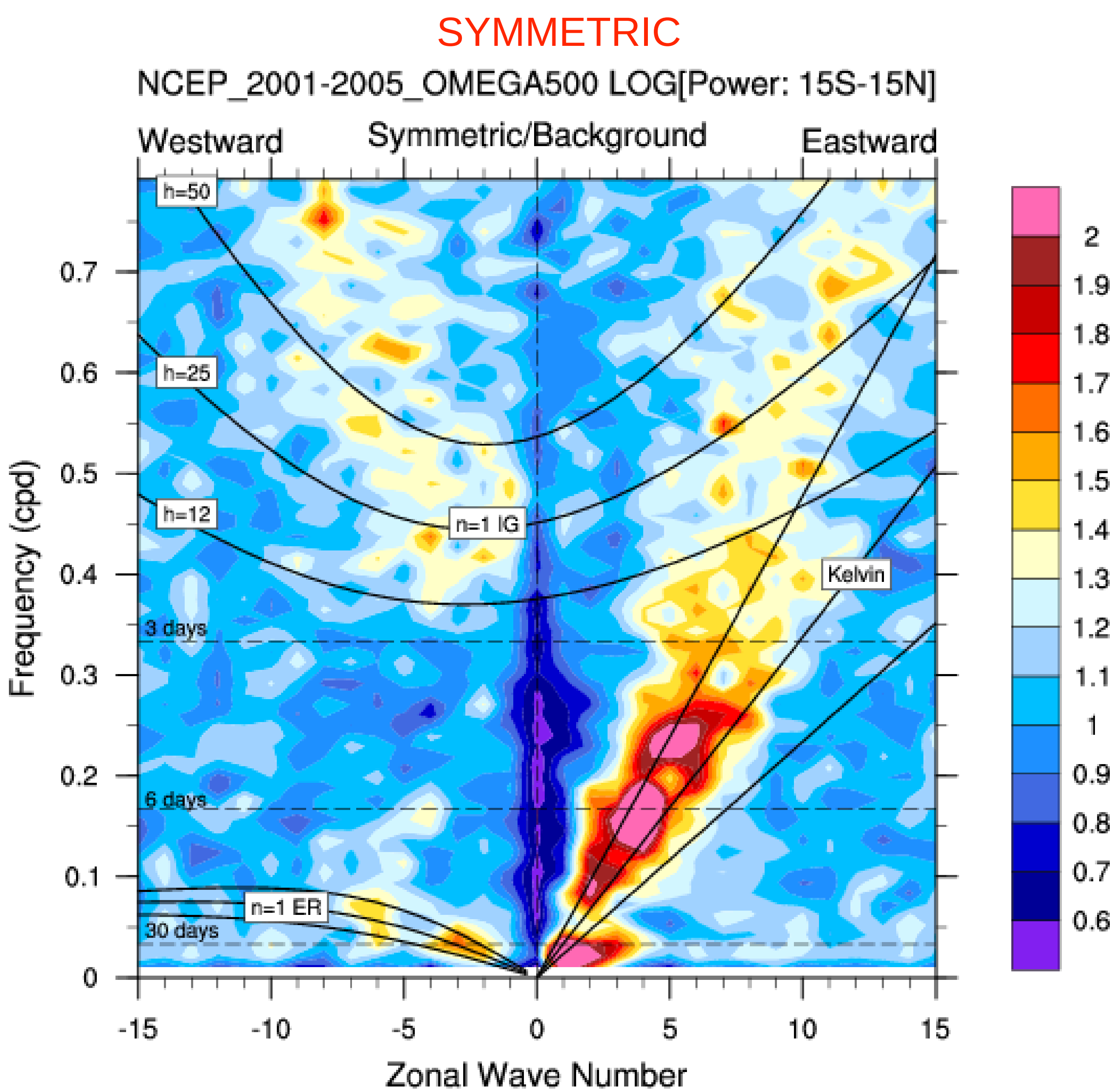
Equatorial Kelvin wave

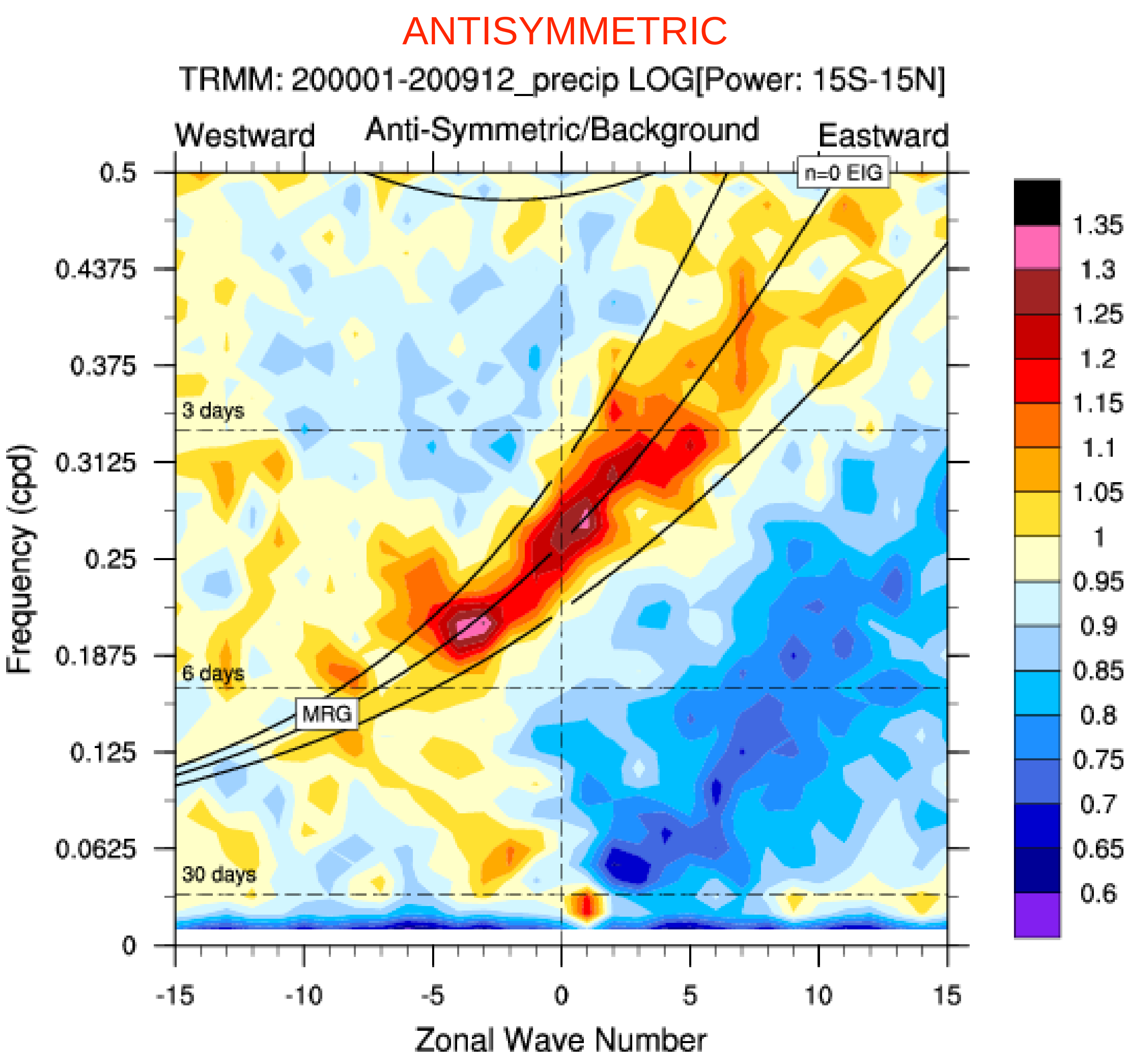




Tropical convection in the atmosphere

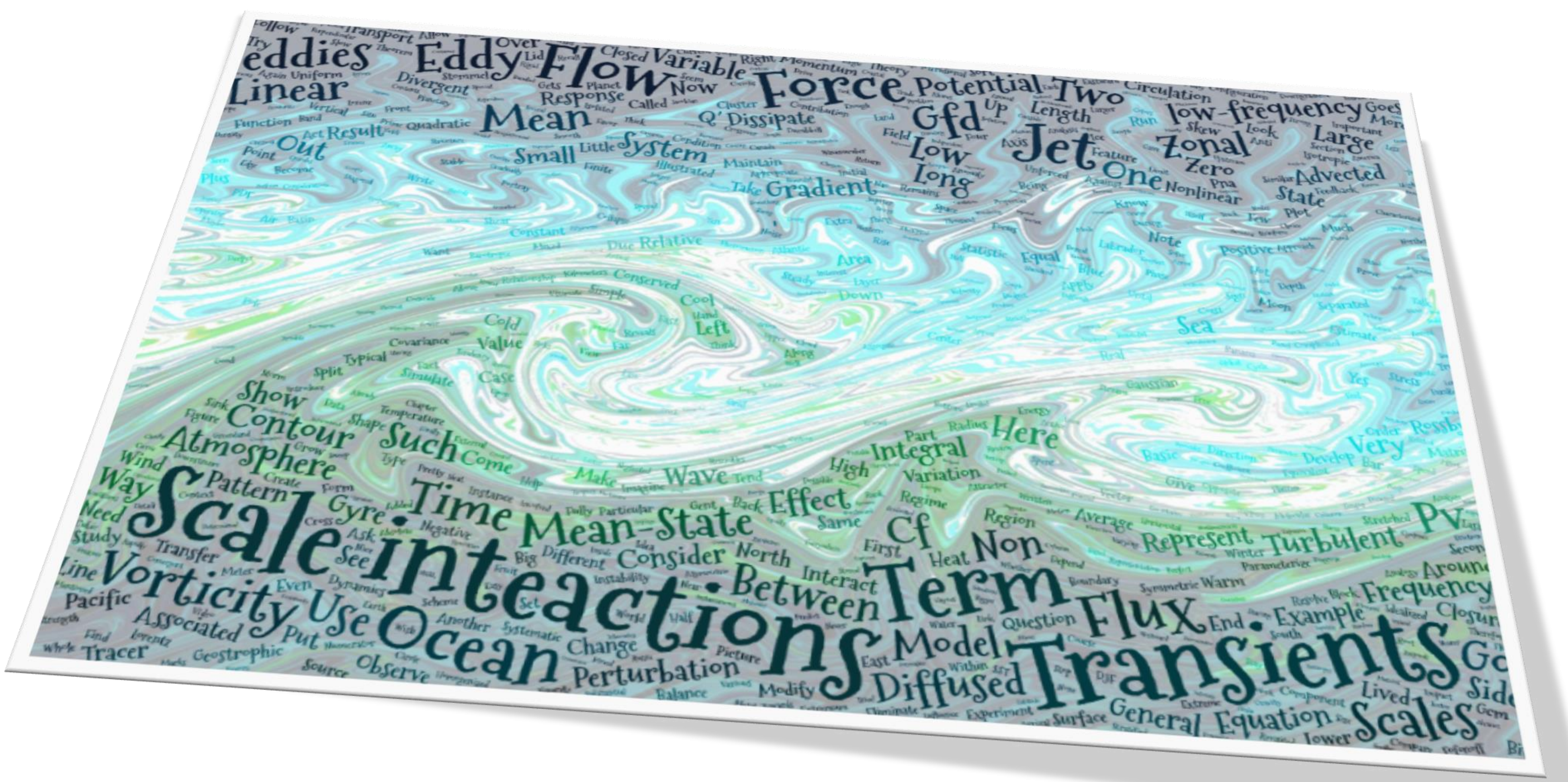
Wheeler-Kiladis space-time variance spectra



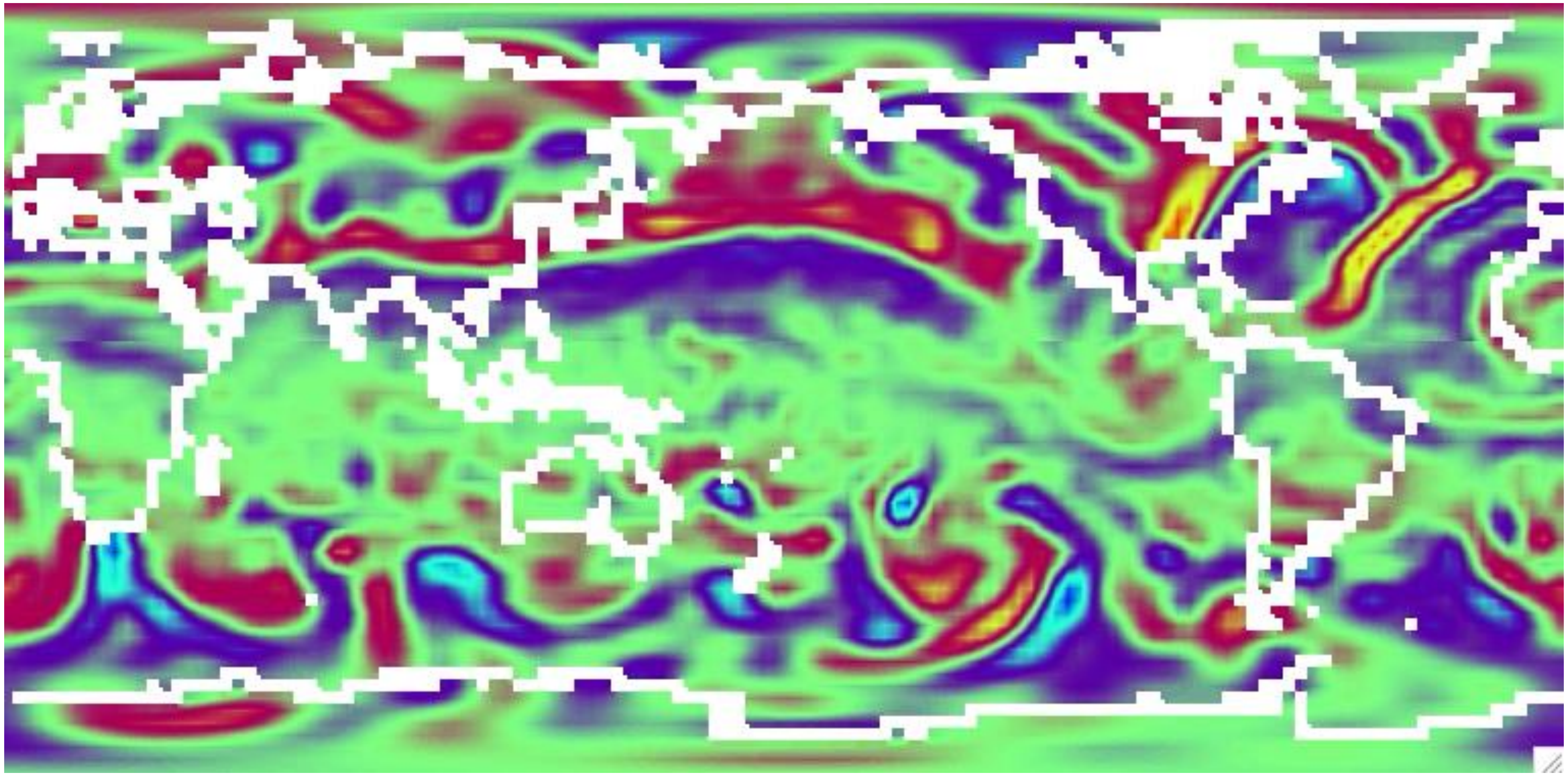


Scale interactions in the atmosphere and ocean

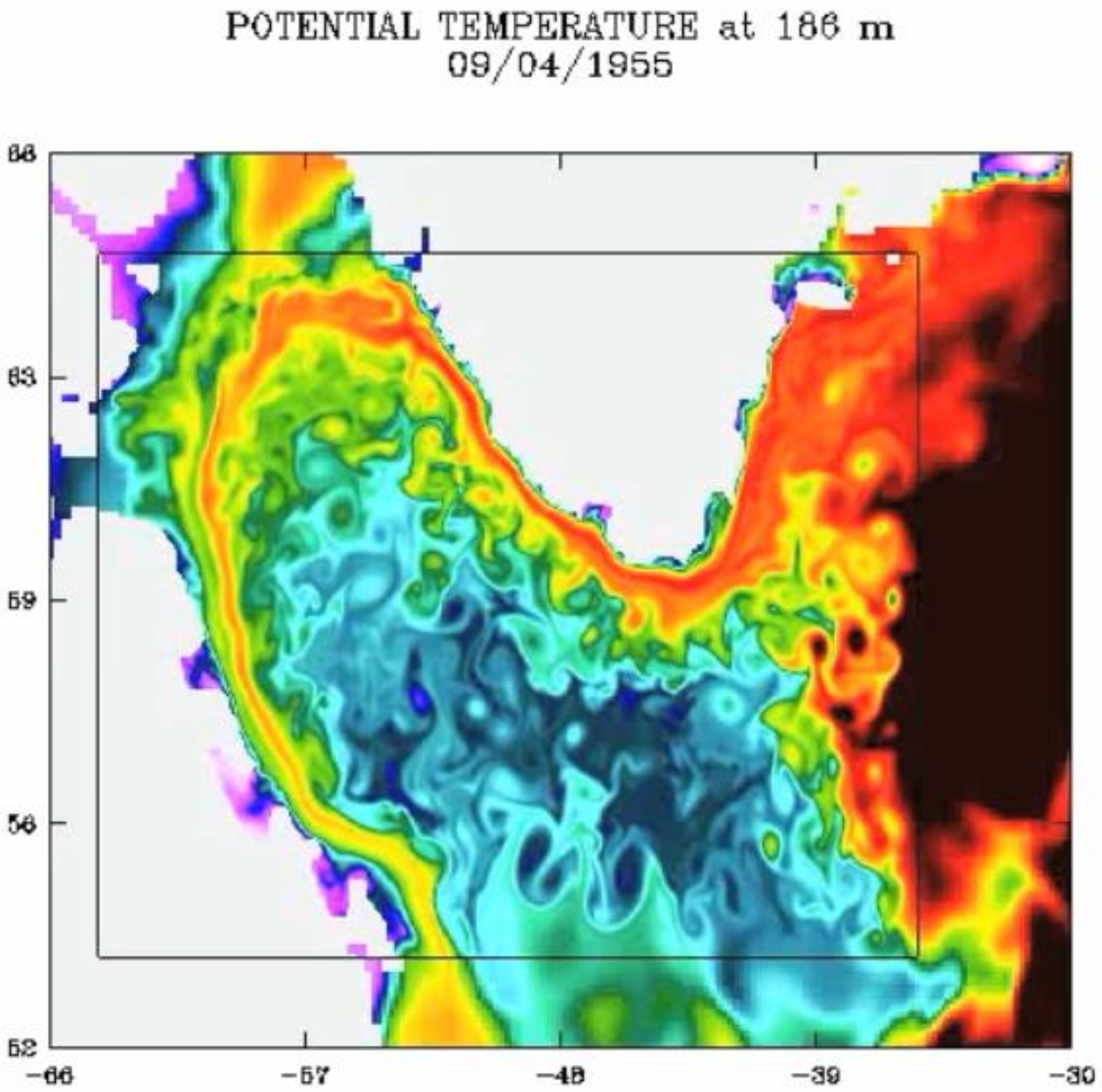
- Large-scale forcing and transport due to transient systems
- Closure and diffusion
- Modification of large-scale potential vorticity: Impact on ocean circulation.
- Long-lived features of the atmosphere and low frequency variability.
- Atmospheric response to forcing anomalies.
- The wave-turbulence crossover and zonal jets



250 mb relative vorticity (ERAi DJF)



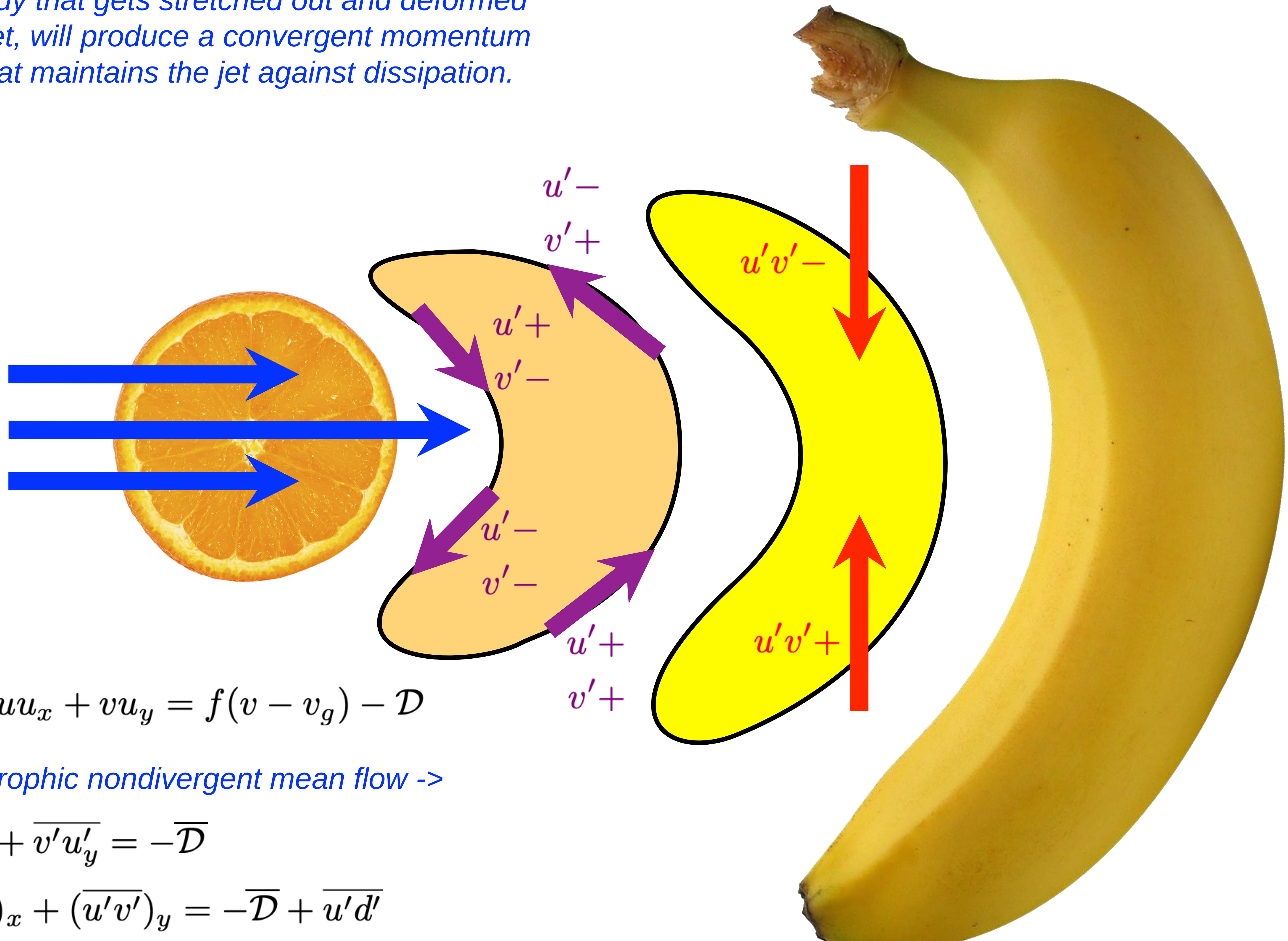
Restratification of the Labrador Sea (MEOM)



(Chanut et al 2008)

Example: Momentum transport in zonal jets

An eddy that gets stretched out and deformed by a jet, will produce a convergent momentum flux that maintains the jet against dissipation.



$$u_t + uu_x + vu_y = f(v - v_g) - \mathcal{D}$$

Geostrophic nondivergent mean flow ->

$$\overline{u'u'_x} + \overline{v'u'_y} = -\overline{\mathcal{D}}$$

$$(\overline{u'u'})_x + (\overline{u'v'})_y = -\overline{\mathcal{D}} + \overline{u'd'}$$

General considerations for tracer transport

⇨ *So far we have looked at small perturbations, linear systems, oscillating or exponentially growing / decaying solutions.*

↳ *Consider nonlinear system* $\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{F} - \mathcal{D}$ *or* $\frac{\partial q}{\partial t} + J(\psi, q) = \mathcal{F} - \mathcal{D}$

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

⇨ *Tendency equation for q* $q = \overline{q} + q' \qquad \psi = \overline{\psi} + \psi'$

$$\frac{\partial q'}{\partial t} + J(\overline{\psi}, \overline{q}) + J(\overline{\psi}, q') + J(\psi', \overline{q}) + J(\psi', q') = \mathcal{F} - \mathcal{D}$$

mean flow advection — linear waves — turbulence

⇨ *Budget equation for* $\overline{q}$ $J(\overline{\psi}, \overline{q}) = -\overline{J(\psi', q')} + \overline{\mathcal{F}} - \overline{\mathcal{D}}$

transient "forcing"

⇨ *If we have steady unforced flow* $J(\psi, q) = 0 \;\Rightarrow\; q = q(\psi)$

↳ *This describes a closed circulation - q contours coincide with ψ contours. So nonlinearity is associated with closed circulations*

If the closed circulation is large scale, this describes a gyre.
If the closed circulation is small scale, we need to relate it to the large-scale circulation.

Forcing due to transients: Closure

⇨ Imagine we wish to simulate or predict the slow, **large-scale flow**. Because the system is nonlinear the fast, small-scale component (maybe unresolved) will affect the slow, large scale.

↳ Consider the nonlinear system $\dfrac{du}{dt} + uu + ru = 0$

⇨ The average is $\dfrac{d\overline{u}}{dt} + \overline{uu} + r\overline{u} = 0$

↳ But the problem is that $\overline{uu} \neq \overline{u}\,\overline{u}$ it's $\overline{uu} = \overline{u}\,\overline{u} + \overline{u'u'}$

⇨ Multiply the equation by u and take time mean $\dfrac{1}{2}\dfrac{d}{dt}\overline{uu} + \overline{uuu} + r\overline{uu} = 0$

↳ This gives us an equation for $\overline{uu}$, but now we have a cubic term ! ☹

In general we need to represent the $(n + 1)^{th}$ order term in terms of the n^{th} order term.
We must make additional physical assumptions to do this.

⇨ What is the relation between the transport of transients and the mean flow?

Diffusion and diffusivity

⇨ *Let's go back to our tracer equation and consider a diffusive representation for the flux of the tracer q. For the moment we ignore other forms of forcing and dissipation. Consider advection by a* ***nondivergent flow****:*

$$\frac{\partial q}{\partial t} + \nabla.\mathbf{v}q = 0$$

take an ensemble average

$$\frac{\partial \overline{q}}{\partial t} + \nabla.\overline{\mathbf{v}}\,\overline{q} = -\nabla.\overline{\mathbf{v}'q'}$$

⇨ *Let's represent this eddy covariance through* ***analogy with molecular diffusion****, i.e. transport down the mean gradient:*

$$\overline{\mathbf{v}'q'} = -K\nabla\overline{q}$$

↳ *So* $\frac{D\overline{q}}{Dt} = \nabla.(K\nabla\overline{q}) \quad (= \nabla.F)$ *where* $\boldsymbol{F}$ *is the diffusive flux of q.*

⇨ *In general, K is a second rank tensor. It is usually* ***not isotropic*** *for large-scale flows.*

↳ *For example* $\overline{\mathbf{v}'q'} = -\kappa^{vy}\frac{\partial \overline{q}}{\partial y} - \kappa^{vz}\frac{\partial \overline{q}}{\partial z}$

↳ *We can estimate* $\kappa^{vy} \sim v'l'$ *where* v' *is a typical eddy velocity and* l' *is a"mixing length" (various scalings can be used).*

But it can be more complicated…

Symmetric and asymmetric diffusion

⇨ *If we decompose K into symmetric and antisymmetric parts* $\boldsymbol{K} = \boldsymbol{S} + \boldsymbol{A}$.

⇨ ***Isotropic diffusion corresponds to***

$$K = S = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$ *and the diffusive flux* $F = -\kappa \nabla q$ *is downgradient.*

⇨ *But* ***if K is antisymmetric*** $F = -A \nabla q$ *and the flux is* ***parallel to contours of q****.*

(if A has zero diagonal and opposite sign off-diagonal elements then $A\boldsymbol{x} \perp \boldsymbol{x}$ *)*

$$F.\nabla q = -(A \nabla q).\nabla q = 0$$

↳ *The flux is neither upgradient nor downgradient. It's called a "**skew flux**".*

↳ *A skew flux is equivalent to* ***advection*** *by a nondivergent flow with velocity* $\tilde{\mathbf{v}} = \nabla_{\wedge} \psi$
The elements of A can be expressed in terms of ψ

⇨ *Whether or not it is appropriate to use a downgradient diffusion depends on the quantity being diffused. We might expect conserved quantities to behave like tracers, with diffusion eroding their gradients. For non-conserved quantities more elaborate schemes need to be considered.*

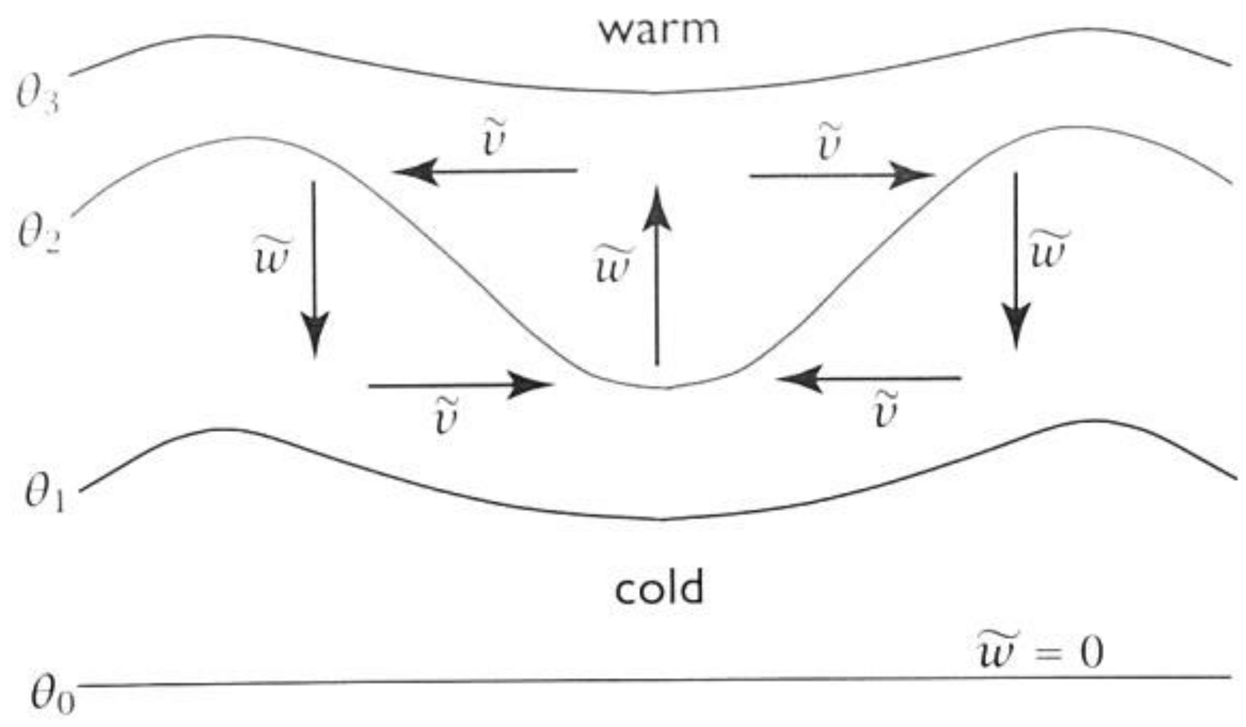
Parameterization

Sophisticated parameterization schemes use physical assumptions to determine the elements of the tensor K.

This often involves a symmetric part and an antisymmetric part so the scheme will be equivalent to a diffusion along the gradient of the tracer plus an advection by a residual flow.

*An example is the **Gent and McWilliams scheme**, which hypothesizes energy conversion by baroclinic instability and formulates the eddy closure in terms of asymmetric diffusion of thickness. Such a scheme will tilt density contours to liberate available potential energy, rather than just erode gradients.*

Other schemes have been formulated in terms of potential vorticity diffusion, or in terms of flow dependent coefficients of K.

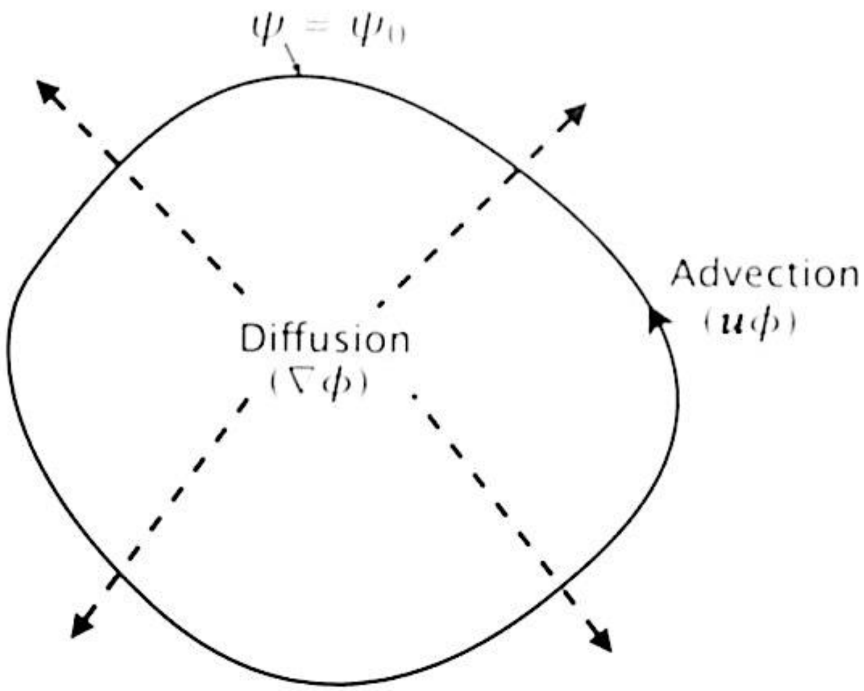


⇨ *Whether or not it is appropriate to use a downgradient diffusion depends on the quantity being diffused. We might expect conserved quantities to behave like tracers, with diffusion eroding their gradients. For non-conserved quantities more elaborate schemes need to be considered.*

Potential vorticity homogenization

⇨ *Let's look again at our tracer equation for q, and add in some downgradient diffusion*

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nabla.(\kappa \nabla q) + \mathcal{S}$$



⇨ *For **steady nondivergent** flow in a region **isolated** from the source S*

$$\nabla.(\mathbf{v}q) = \nabla.(\kappa \nabla q)$$

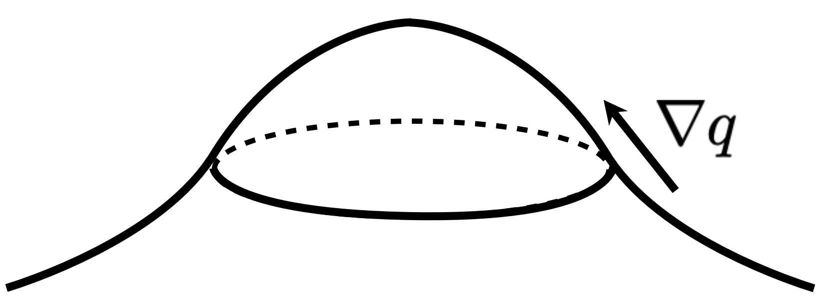
⇨ ***Integrate** over a region enclosed by a contour of q.*

- *The left hand side integrates to zero*

$$\iint_A \nabla.(\mathbf{v}q)\, dA = \oint (\mathbf{v}q).\hat{\mathbf{n}}\, dl = q \oint \mathbf{v}.\hat{\mathbf{n}}\, dl = q \iint_A \nabla.\mathbf{v}\, dA = 0$$

- *So the right hand side must also be zero*

$$\iint_A \nabla.(\kappa \nabla q)\, dA = \oint \kappa \nabla q.\hat{\mathbf{n}}\, dl = 0$$



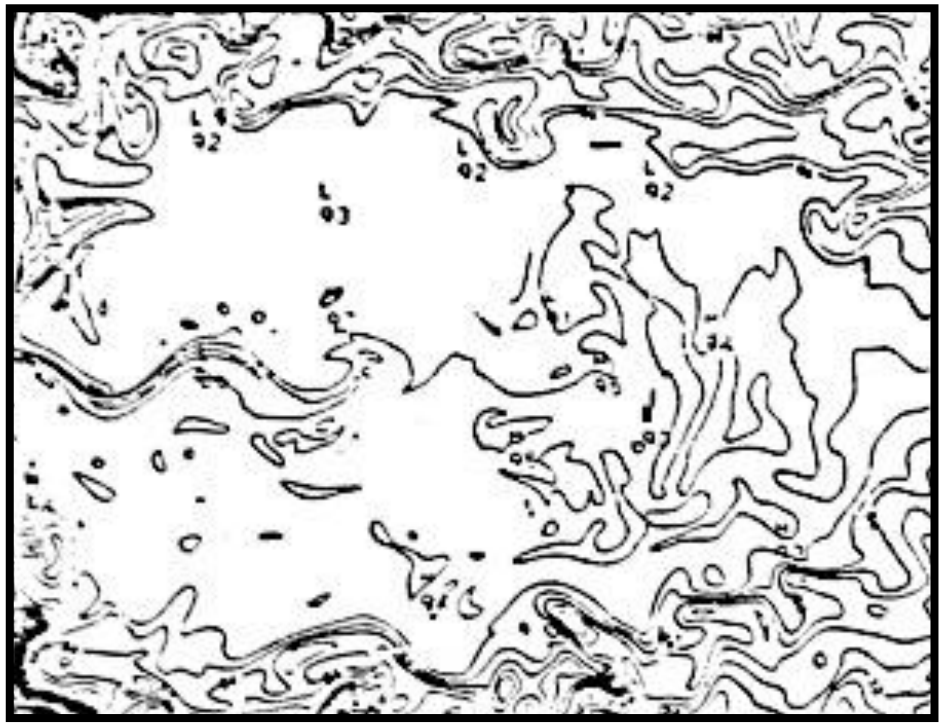
Away from forcing
(in depth)

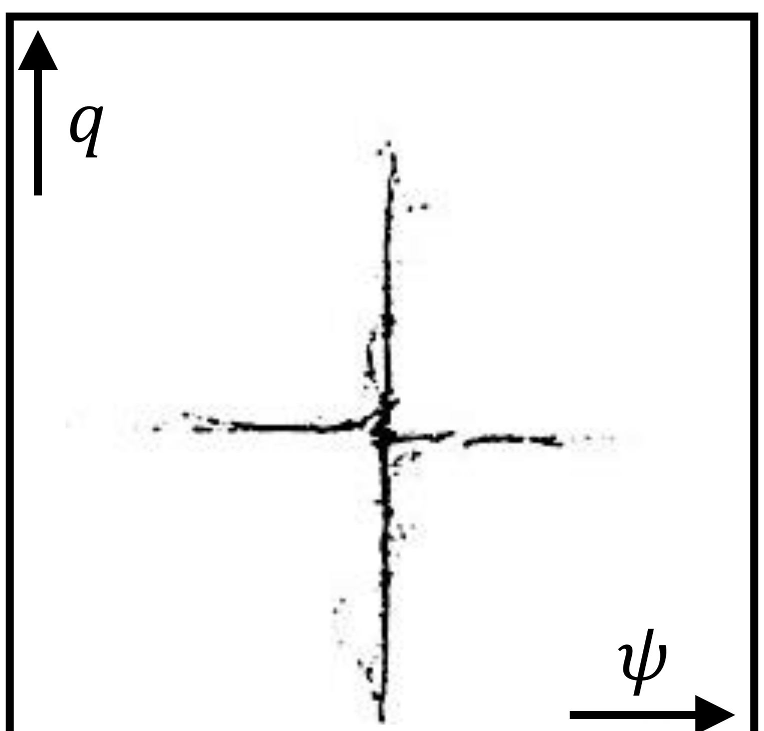
⇨ *This **cannot be true if the contour encloses an extremum of q**.*
The gradient of q must integrate to zero around this contour.
So there can be no extremum of q within the contour.

⇨ *Gradients of q are eliminated, resulting in **Homogenization** to a uniform value in regions remote from sources of q.*

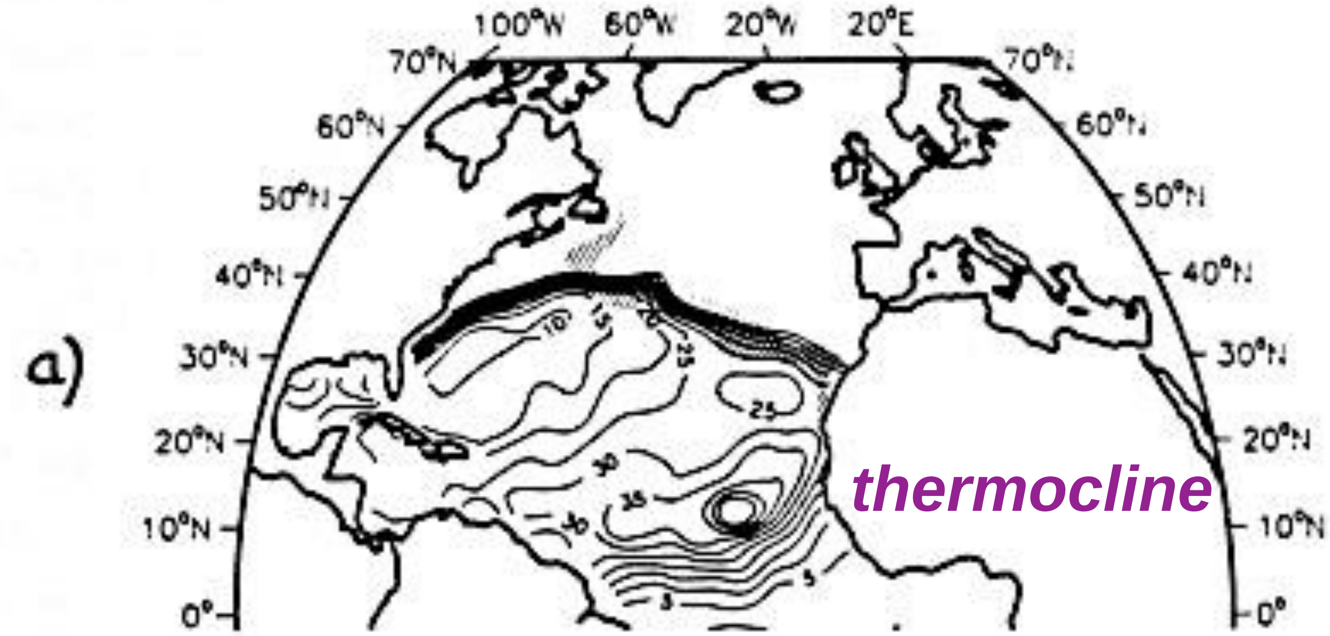
Examples in models and observations

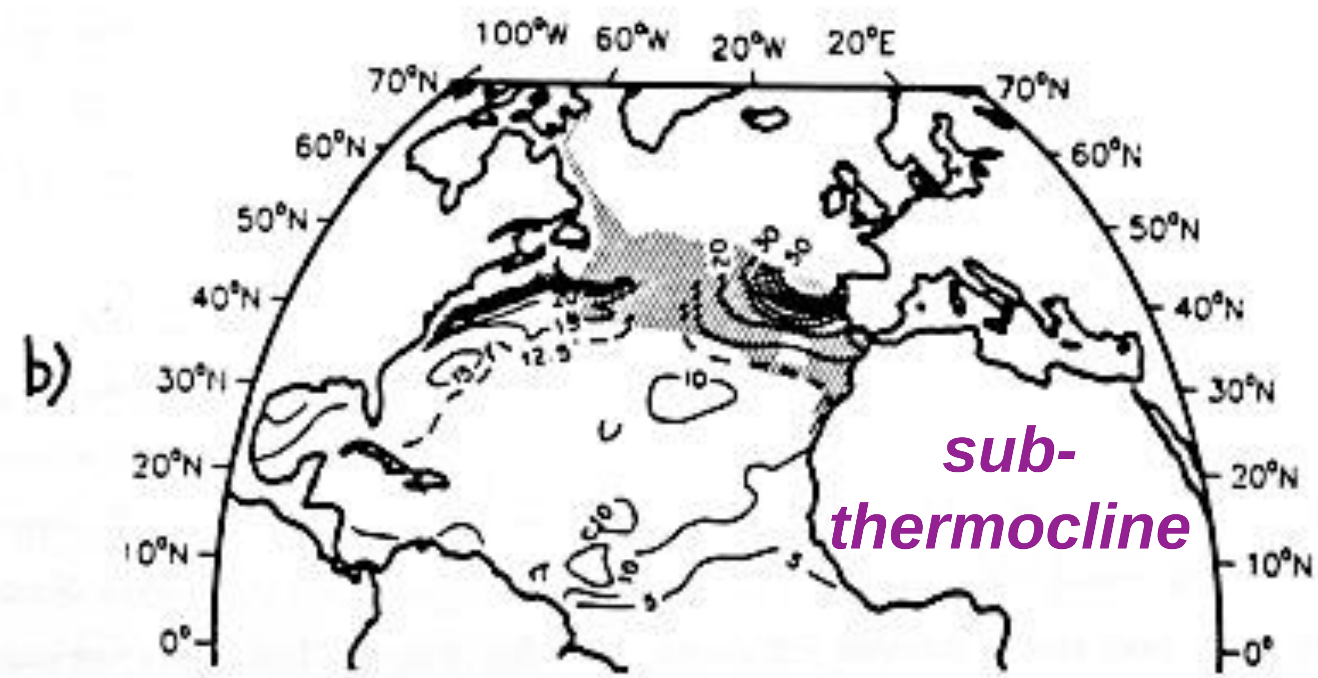
QG model - mid-level PV





Observed PV on isopycnal surfaces





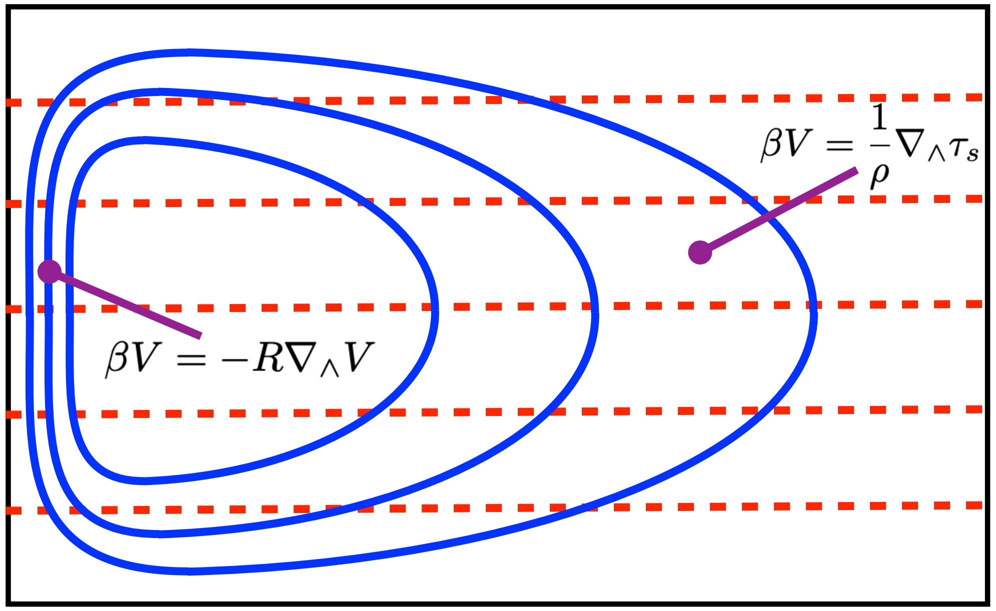
(Keffer 1985)

Stommel vs Fofonoff

Two extreme paradigms of gyre-scale flow:

A Stommel gyre has different vorticity balance in different regions

$$\beta V = \frac{1}{\rho}\nabla_\wedge \tau_s - R\nabla_\wedge V$$

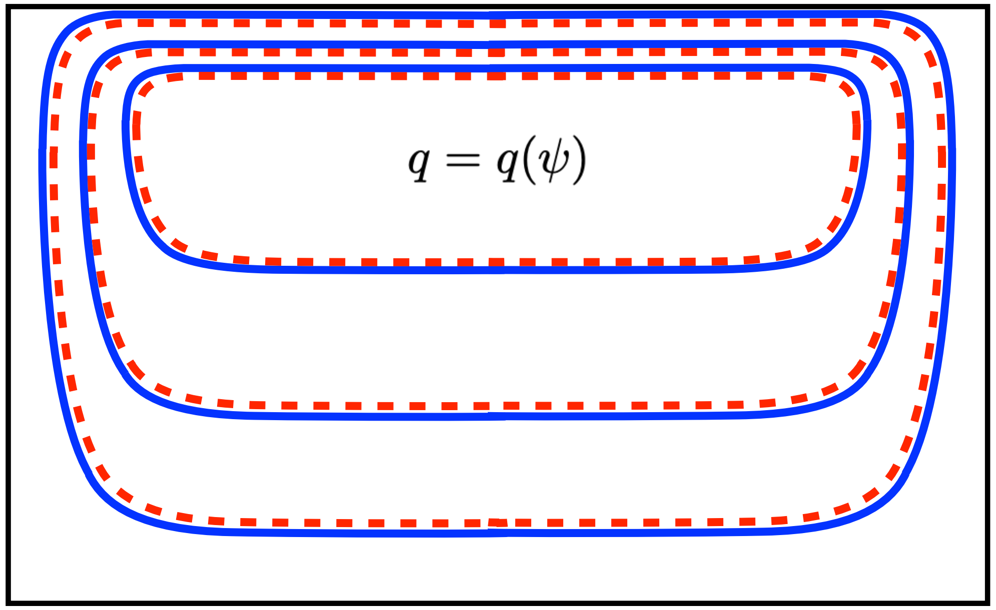


Flow is forced south across contours of planetary vorticity. Its changing vorticity is supplied by forcing and dissipation.

A Fofonoff gyre corresponds to unforced flow, so the balance is between advection of planetary vorticity and relative vorticity.

$$v.\nabla\xi + \beta v = v.\nabla q = 0 \quad \textit{or} \quad J(\psi, q) = 0, \quad q = q(\psi)$$

so $\nabla^2\psi + \beta y = fn(\psi)$



Flow conserves its absolute vorticity, so contours of absolute vorticity are parallel to contours of the streamfunction.

Diffusion and the strength of the gyre

In a Fofonoff gyre we don't know the relathionship between q and ψ, so the strength of the flow is not constrained. Let's assume that the relation is linear, and reintroduce some forcing and downgradient diffusion of q.

$$\nabla q \approx \frac{dq}{d\psi}\nabla\psi, \quad J(\psi, q) = \nabla.(\kappa\nabla q) + \mathcal{S}$$

Integrating within a streamline of ψ: $$0 = \iint_A \nabla.(\kappa\nabla q)\, dA + \iint_A \mathcal{S}\, dA$$

$$\Rightarrow \quad \iint_A \mathcal{S}\, dA = -\oint_\psi \kappa\nabla q.\hat{\mathbf{n}}\, dl = -\oint_\psi \kappa\frac{dq}{d\psi}\nabla\psi.\hat{\mathbf{n}}\, dl$$

so $$\frac{dq}{d\psi} = -\frac{\iint_A \mathcal{S}\, dA}{\oint_\psi \kappa\, \mathbf{v}.dl}$$

- The relationship between q and ψ is determined by integrals of forcing and dissipation around the closed gyre circulation.
- Integrated eddy diffusion provides the link between the q / ψ relationship and the strength of the circulation.
- In regions isolated from forcing, the numerator is zero but the denominator is non-zero, so the field of q must be uniform. q is homogenized.

Long-lived atmospheric flow anomalies

Can we imagine similar mechanisms at work within closed atmospheric circulations ?

How are low-frequency patterns in the atmosphere maintained against dissipation ?

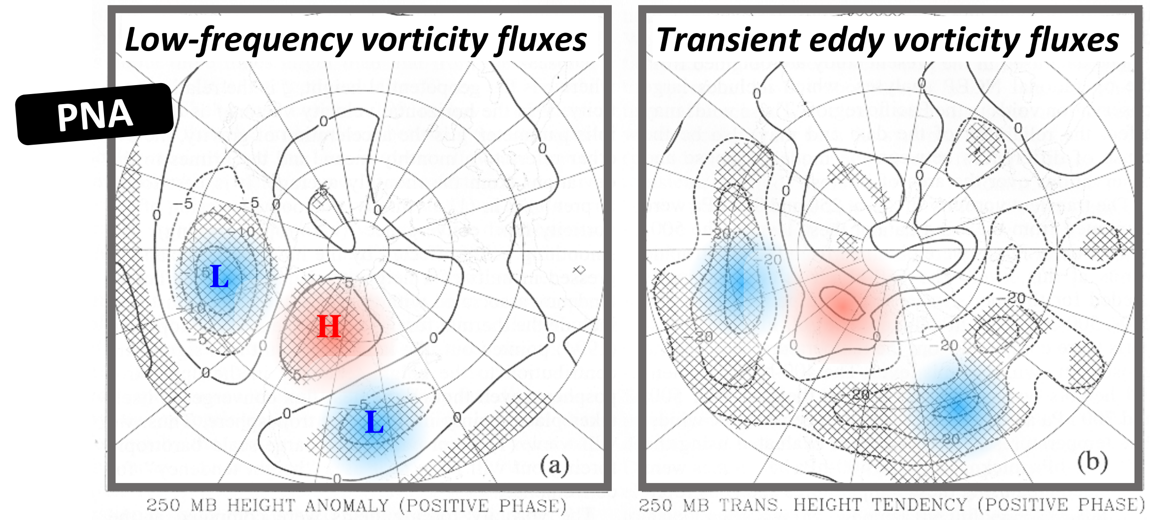
Transient fluxes of heat and vorticity have rotational and divergent components.

The divergent components are associated with development or maintenance of long-lived structures.

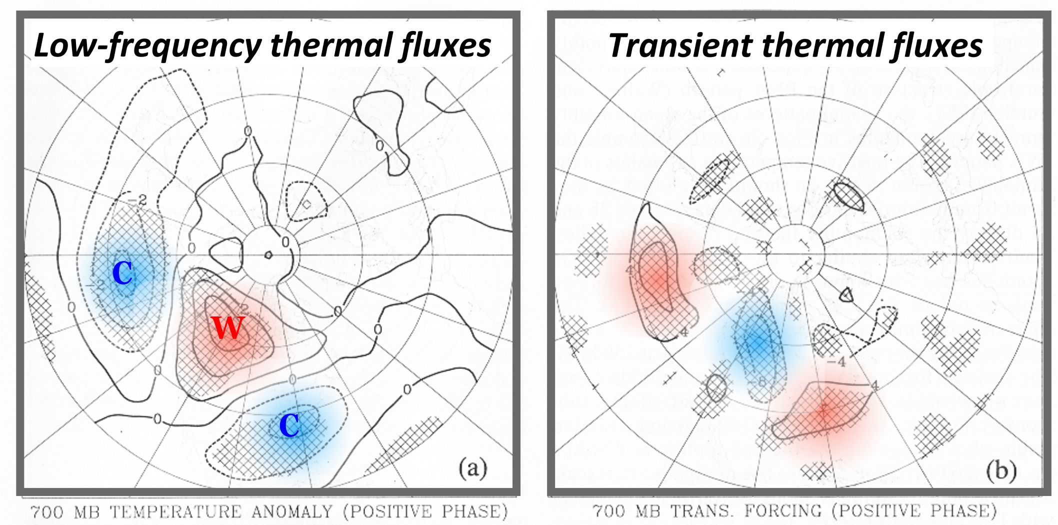
Observational analyses consistently show that high-frequency transient eddy vorticity fluxes reinforce the low-frequency patterns, while transient thermal fluxes dissipate them.

Some anomaly structures may be well configured for maintenance by transients

PNA height field and tendency due to vorticity fluxes

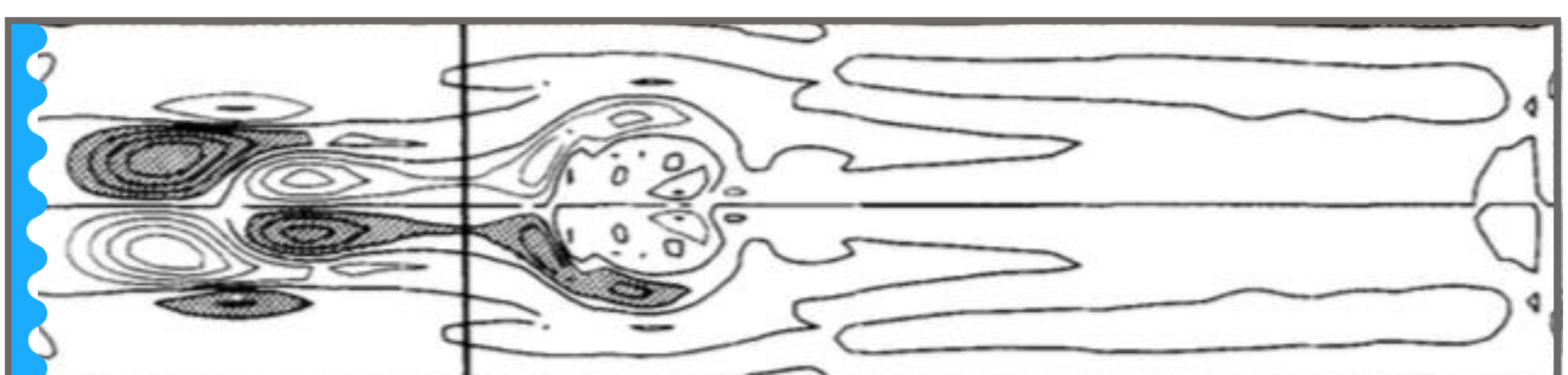


PNA temperature field and tendency due to heat fluxes



(Sheng et al 1998)

Wave maker



Transient potential vorticity flux divergence in an idealised model of atmospheric blocking

(Haynes and Marshall 1986)

Transient feedback on a forced response

Imagine the generic development

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{F} - \mathcal{D}$$

Time average of this

$$\overline{\mathbf{v}.\nabla q} = \overline{\mathcal{F}} - \overline{\mathcal{D}} = \mathcal{G}$$

"Forcing" for mean flow can be written

$$\overline{\mathbf{v}}.\nabla \overline{q} = \overline{\mathcal{F}} - \overline{\mathcal{D}} - \overline{\mathbf{v}'.\nabla q'} = \mathcal{H}$$

Use G to force an empirical GCM

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{G}$$

Add a perturbation to the forcing (say an SSTA)

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{G} + f'$$

SET1

The difference between runs gives the average response Δq.

We can also diagnose the difference in transient forcing Δ(v'.∇q').

Now run the same model but force with H, and initialise with $\overline{q}$

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{H}$$

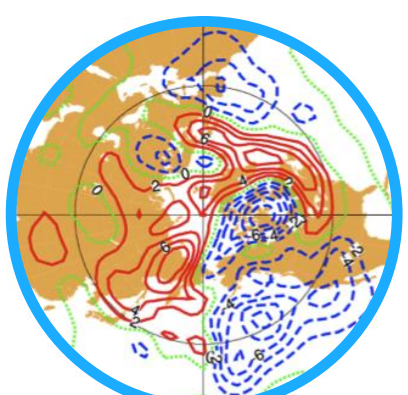
No development !

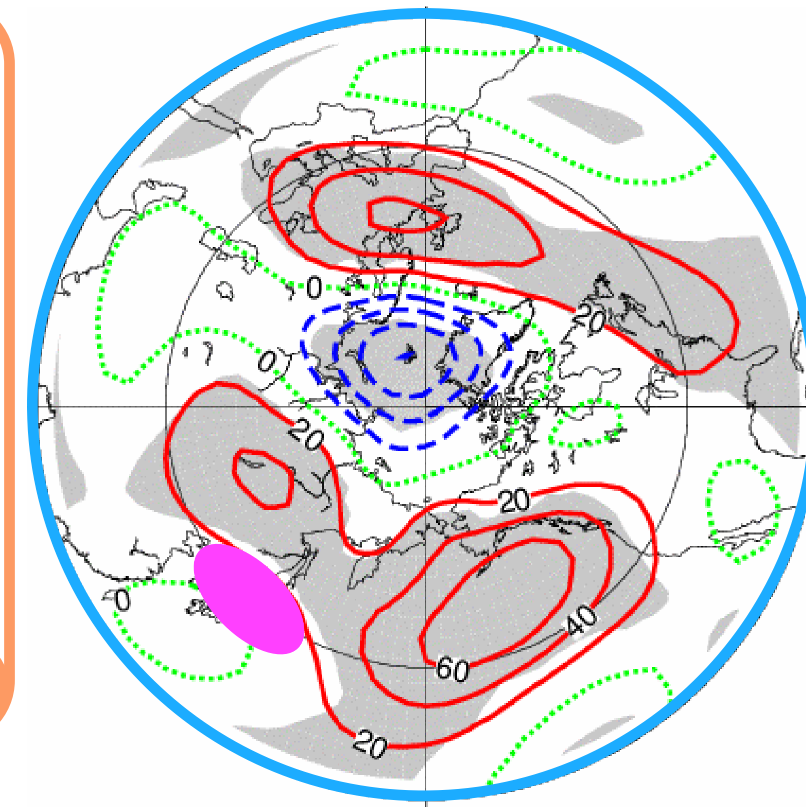
Now add the perturbation.

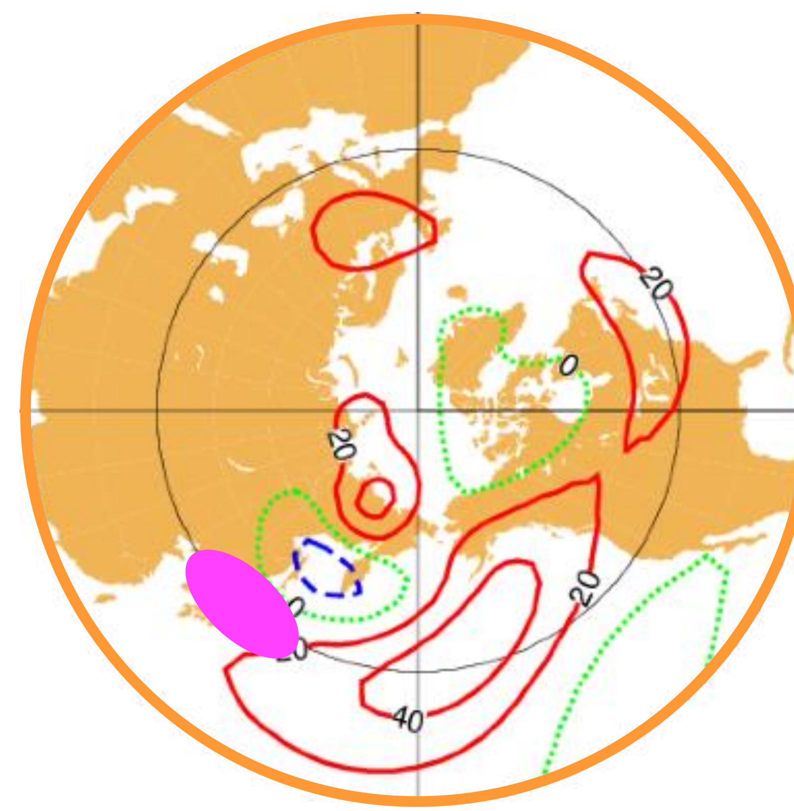
$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{H} + f'$$

SET2

The response is not the same as before.







(Hall et al 2001)

Transient feedback on a forced response

Imagine the generic development

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Time average of this

$$\overline{\mathbf{v}.\nabla q} = \overline{\mathcal{F}} - \overline{\mathcal{D}} = \mathcal{G}$$

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$$\overline{\mathbf{v}}.\nabla\overline{q} = \overline{\mathcal{F}} - \overline{\mathcal{D}} - \overline{\mathbf{v}'.\nabla q'} = \mathcal{H}$$

Use G to force an empirical GCM

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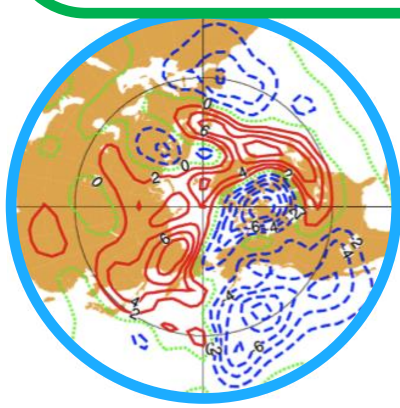
We can also diagnose the difference in transient forcing Δ(v'.∇q').



Now run the same model but force with H, and initialise with $\overline{q}$

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{H}$$

No development !

Now add the perturbation.

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{H} + f'$$

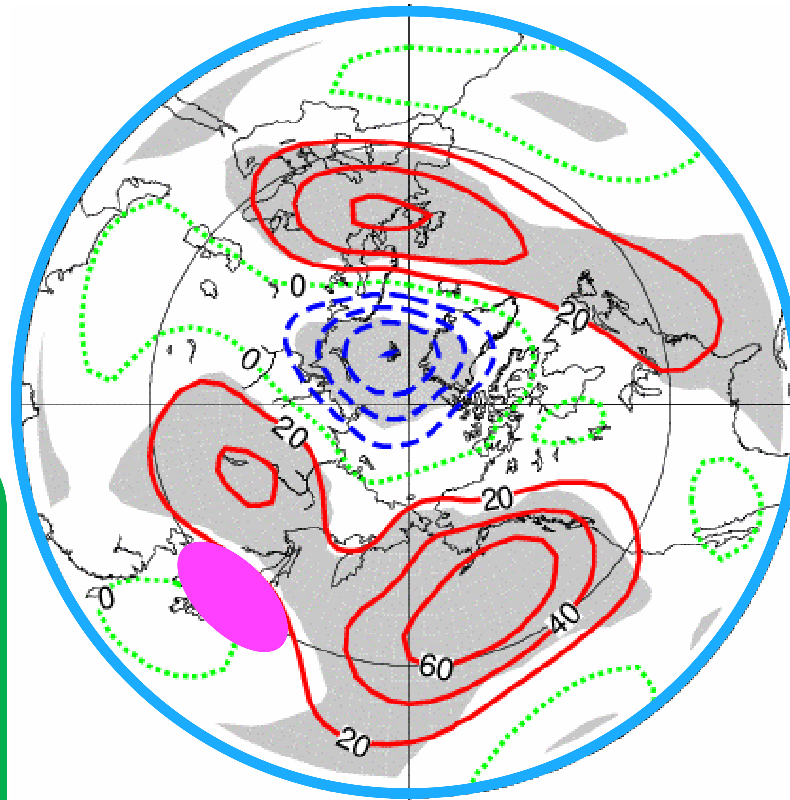
SET2

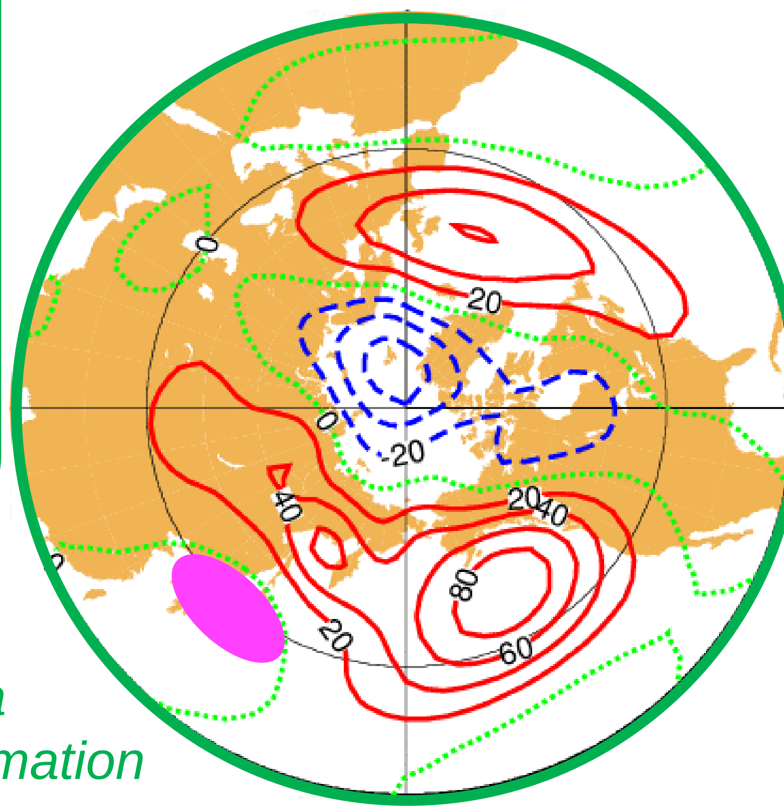
The response is not the same as before.

But if we add the extra transient forcing

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{H} + f' - \Delta(\overline{v'.\nabla q'})$$

SET3





The linear model gives a good approximation to the full response

(Hall et al 2001)

The importance of nonlinearity

There is no doubt that atmospheric dynamics is nonlinear. One need only look at the difference between cyclones and anticyclones.

Does this mean we need a nonlinear framework to analyse lower frequency variability ?

Non-Gaussian and even multi-modal statistics are features of nonlinear systems.

But synoptic timescale nonlinearity can be represented as stochastic noise plus linear damping.

The response of a linear system to external forcing can be written:

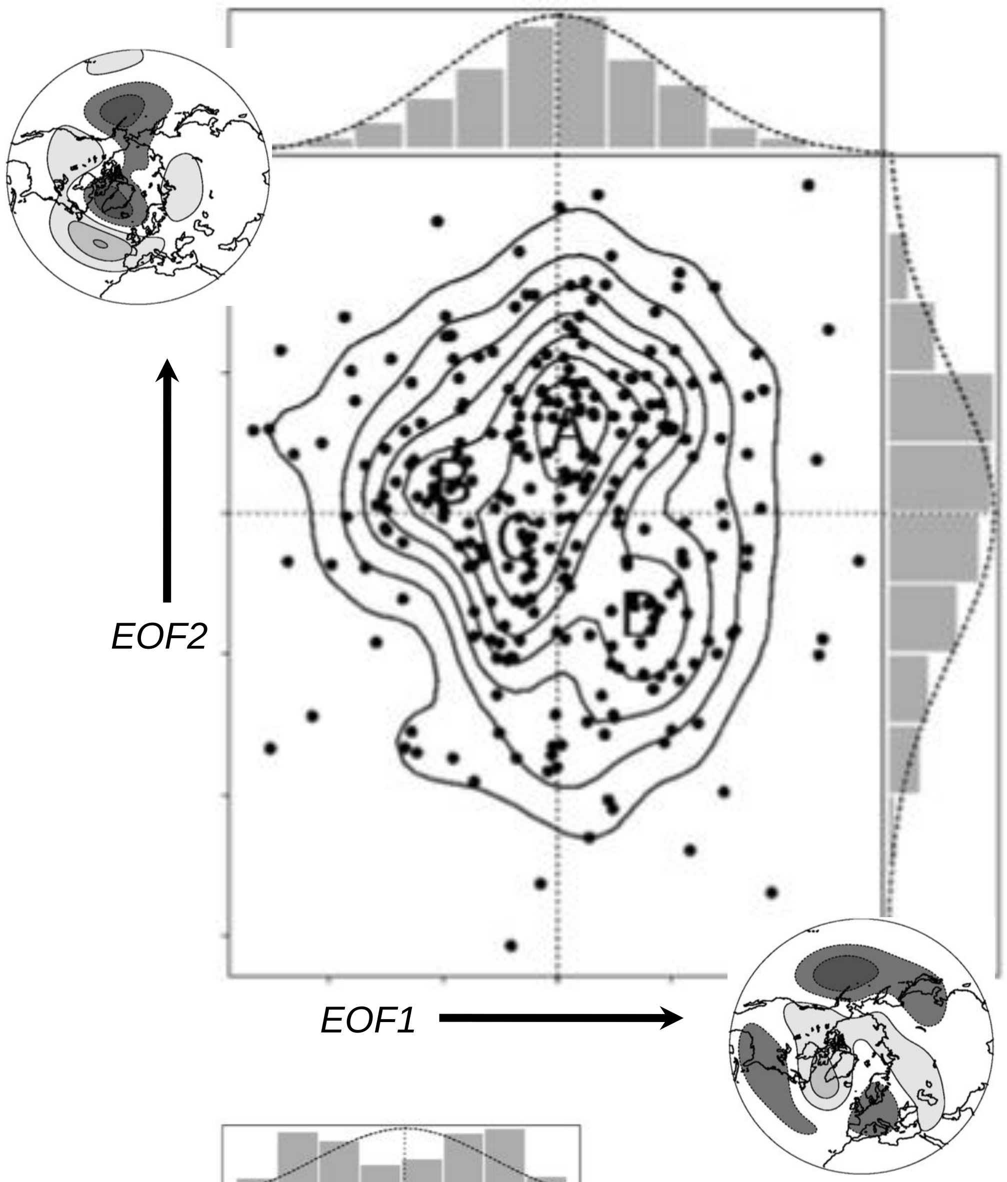
$$\frac{dx}{dt} = Lx + f + B\eta$$

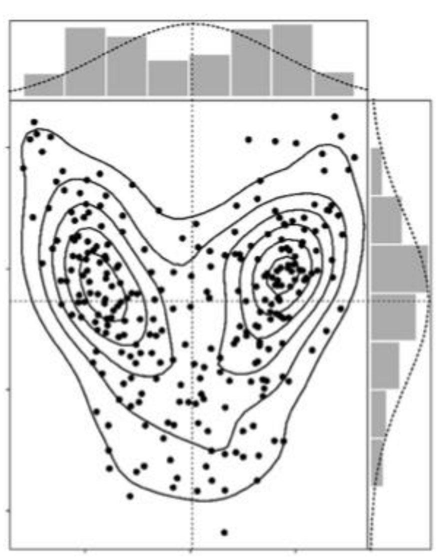
linear operators (L, B); *state vector* (x); *external forcing* (f); *Gaussian noise* (η)

This linear system yields Gaussian statistics if B is constant, but can deliver non-Gaussian unimodal statistics if B=B(x).

(Sardeshmukh and Sura, 2009)

Monthly mean 500mb height





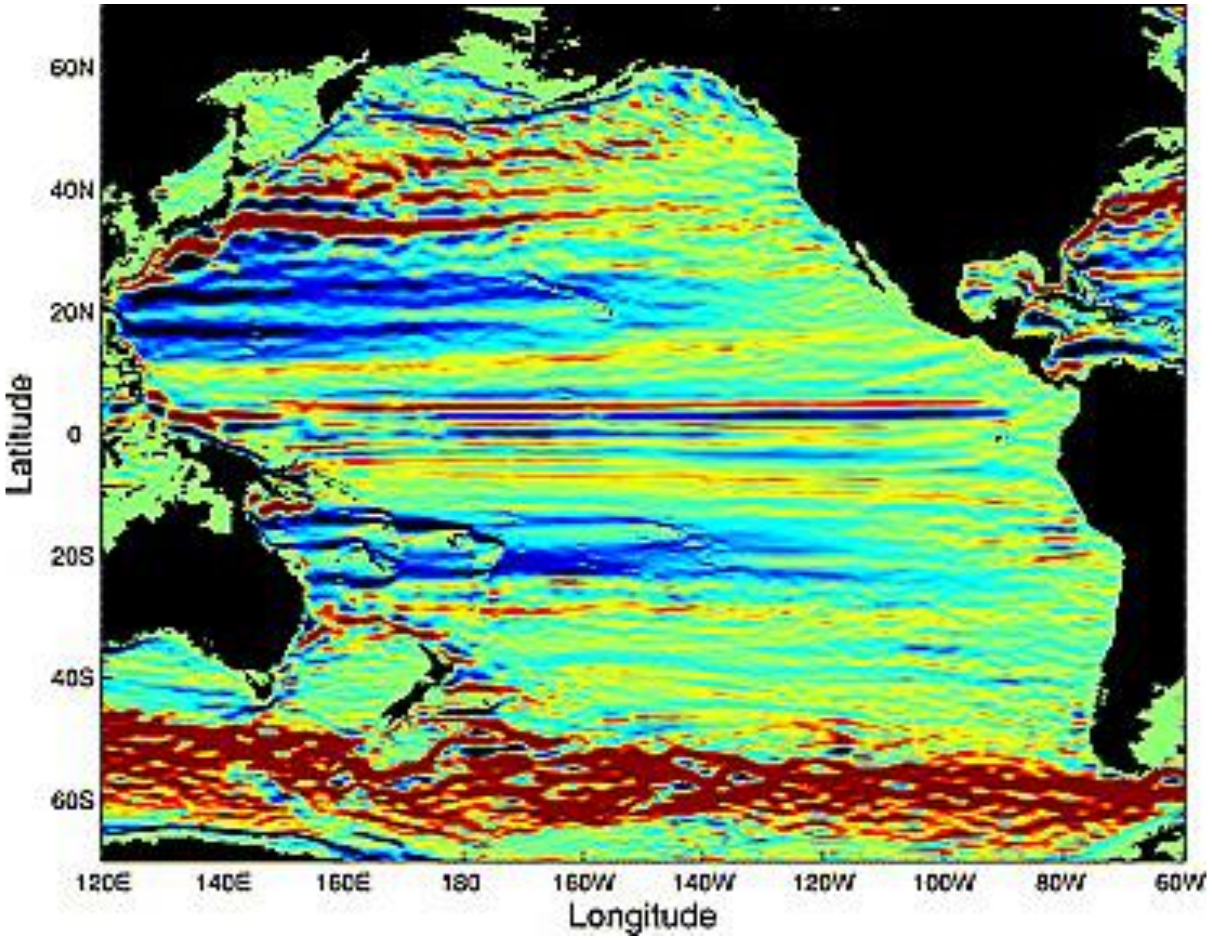
Lorenz system

(Stephenson et al 2004)

Zonal jets revisited: Ocean currents

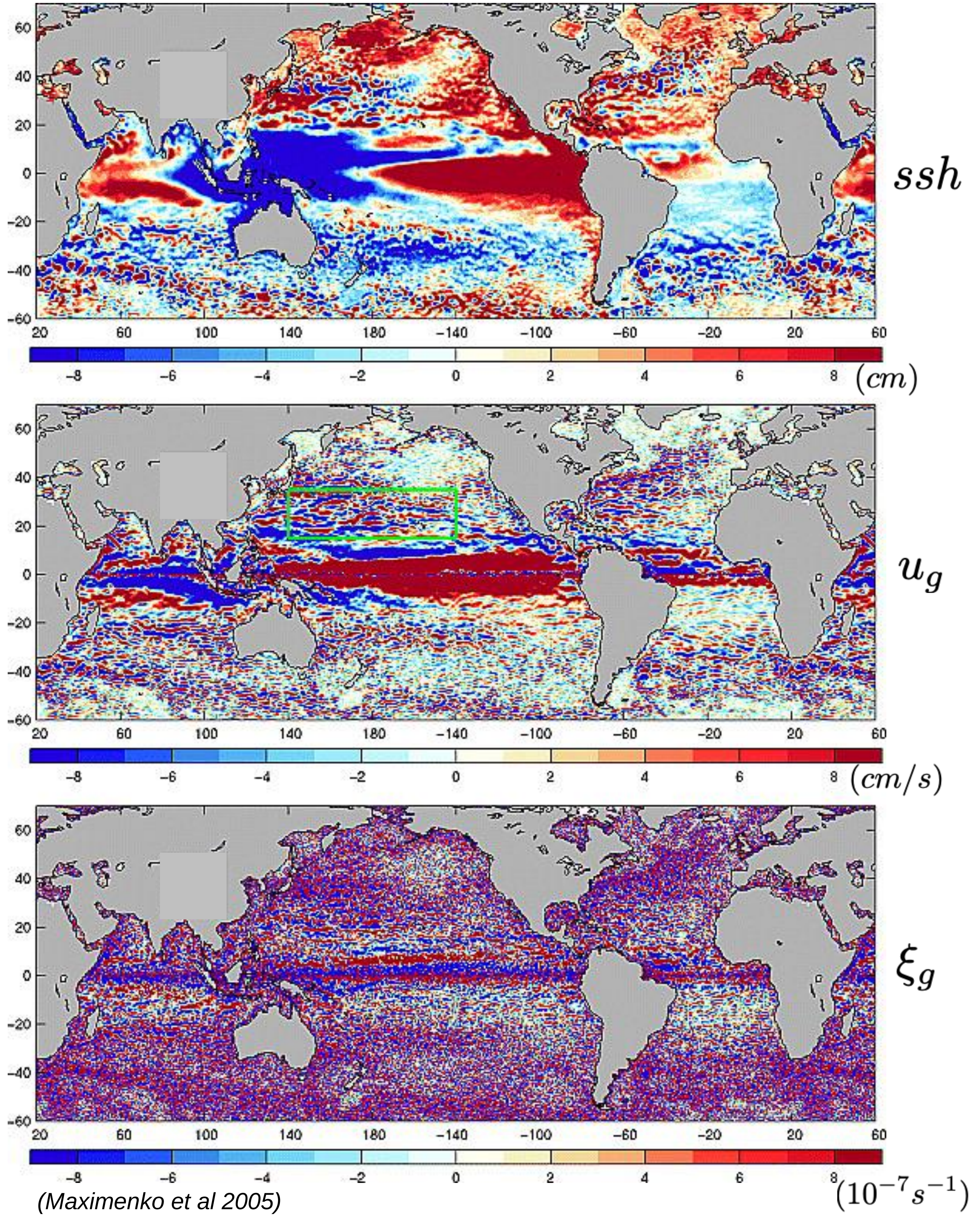
Altimetric observations and high resolution models have shown that the large scale ocean circulation on timescales of a few months is characterised by zonal jets of alternating sign.

Ocean model u_{400m}



(Richards et al 2006)

Satellite observations



(Maximenko et al 2005)

Wave-Turbulence crossover

Remember the Rossby radius ? The length scale on which relative vorticity and vortex stretching make equal contributions to potential vorticity:

$$\nabla^2\psi \sim \frac{f^2}{gH}\psi \quad \Rightarrow \quad L \sim \frac{\sqrt{gH}}{f}$$

Now let's consider larger scales. Compare advection of planetary and relative vorticity:

$$\frac{\partial \xi}{\partial t} + \mathbf{v}.\nabla\xi + \beta v = 0 \quad \rightarrow \quad \mathbf{v}.\nabla\xi \sim \beta v$$

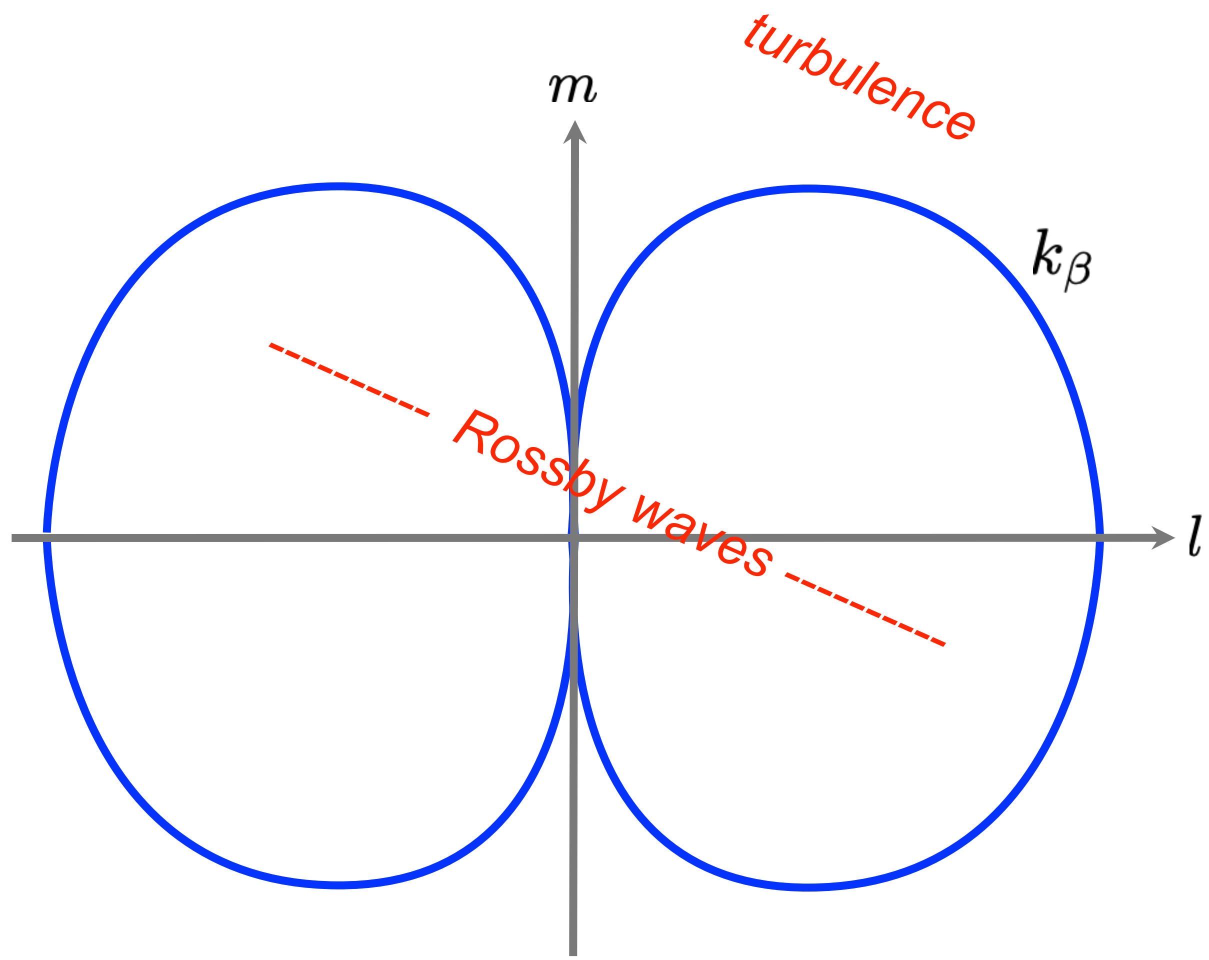
Scale analysis of this ->

$$U\frac{U}{L^2} \sim \beta U \quad \rightarrow \quad L \sim \sqrt{\frac{u}{\beta}}$$

This is called the "Rhines scale", where Rossby waves give way to turbulence.

Compare Rossby wave frequency with a typical turbulence inverse timescale

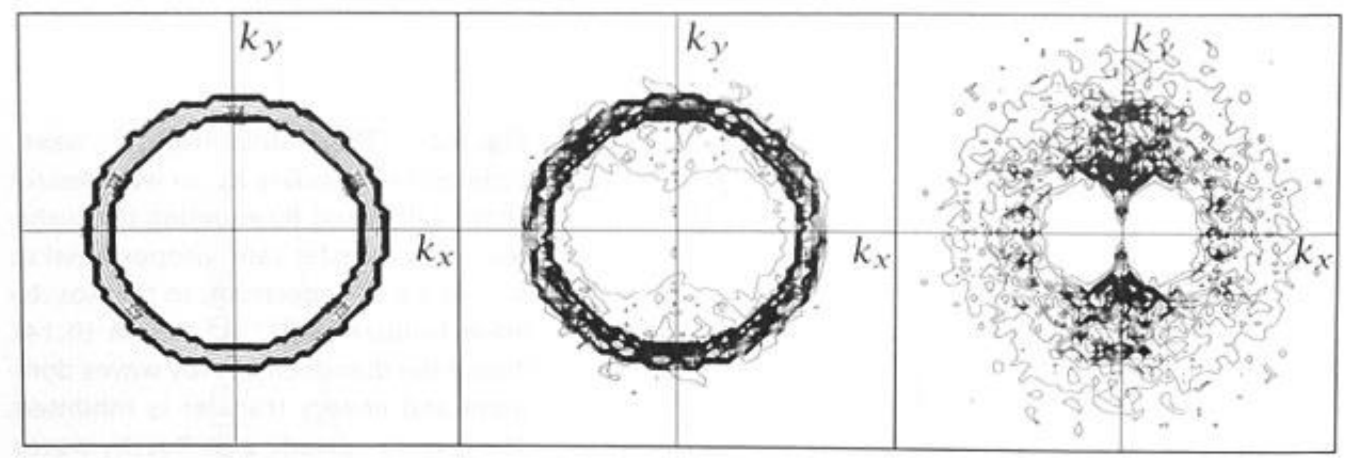
$$\omega = \frac{\beta l}{k^2} \sim u^* k \quad \rightarrow \quad k^2 = \frac{\beta}{u^*}\cos\theta$$

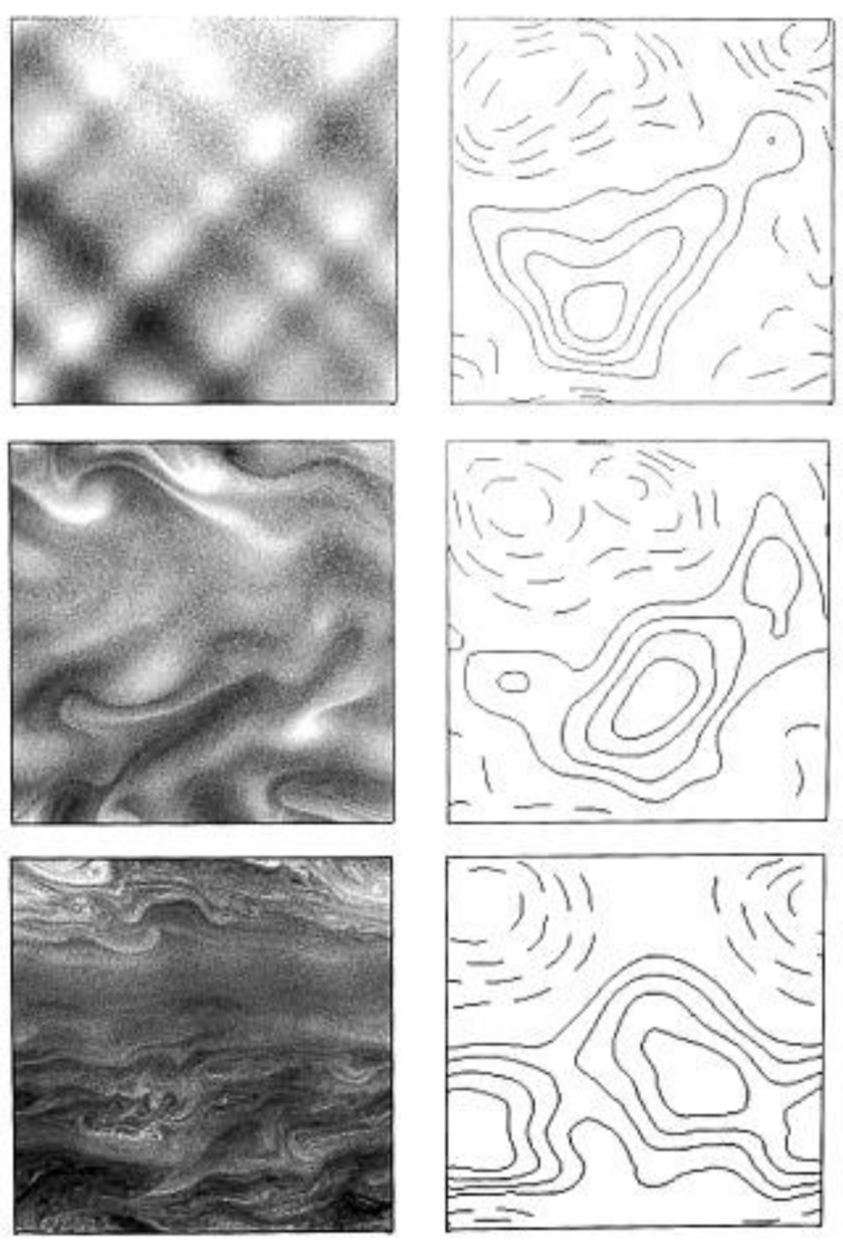


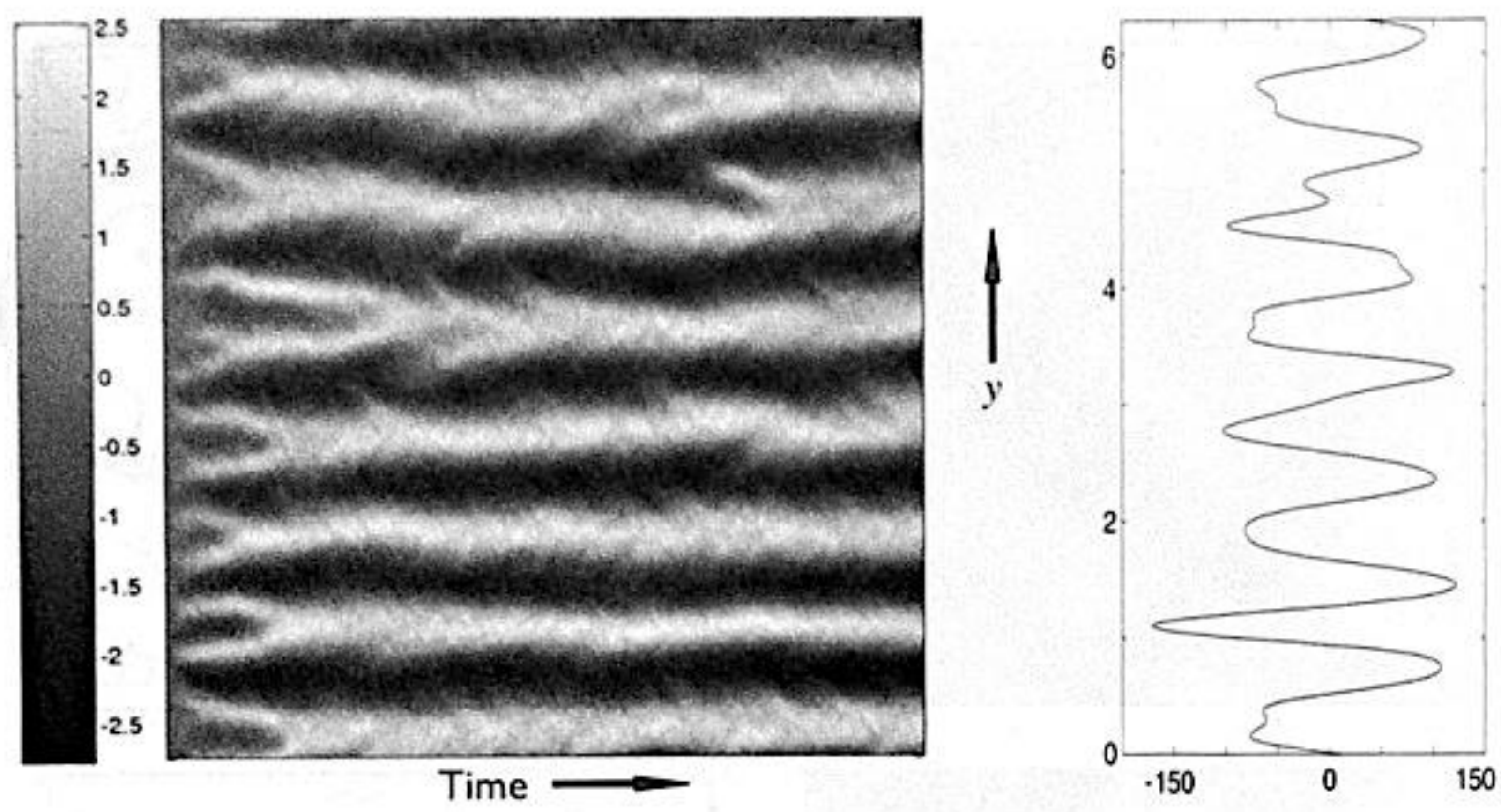
This leads to an anisotropic boundary in wavenumber space between waves and turbulence

Collapse to zonal jets

Physically, Rossby wave solutions exist inside the dumbell. Scale transfer is not possible in this region. Cascade is therefore towards $k_x = 0$, $k_y \neq 0$. This implies zonal jets separated in latitude by scale k_β.







Scale separation and boundary conditions