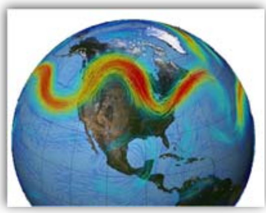
Chapter 3: Rossby waves and instability

- ➪ Parcel displacements and the conservation of potential vorticity
- ➪ The **Rossby wave** dispersion relation
- ➪ Topographic RW, baroclinic RW and vertical modes
- ➪ Parcel displacements in shear flow
- ➪ Barotropic **instability** and the necessary conditions for growth
- ➪ Scales and structures for baroclinic growth and the Eady problem



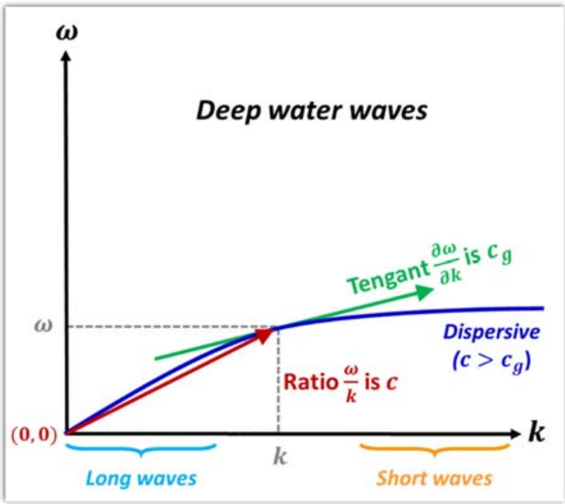
$$e^{i(lx+my-\omega t)}$$

Some Recap

$$e^{i(kx-\omega t)}$$

⇢ **the phase speed** (c) is the arrow that points from the origin toward the curve (the ratio $\frac{\omega}{k}$)

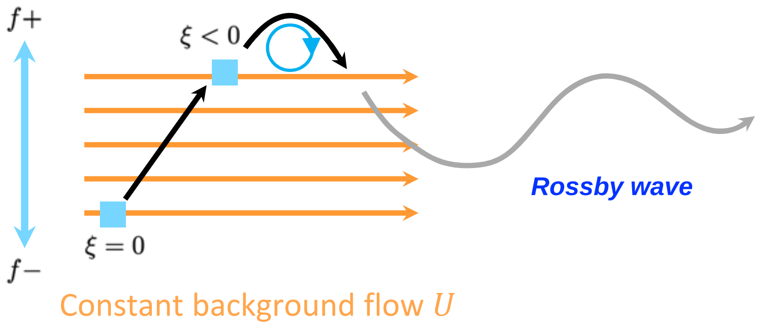
⇢ **the group speed** (c_g) is the tangent to the curve ($\frac{\partial\omega}{\partial k}$)



Parcel displacements in a vorticity gradient

⇨ Consider a parcel of fluid that **conserves** its **absolute vorticity** in a westerly current

$$f + \xi = \text{const}$$



The conservation of vorticity

⇨ *Let's look at **various forms** of the vorticity equation in a westerly flow* $\frac{Dq}{Dt} = 0$

where D/Dt is given by $\frac{D}{Dt} = \frac{\partial}{\partial t} + (U + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$ *(prime denotes small perturbation)*

*... and **q can take various forms** :*

1) Nondivergent barotropic

$$q = \beta y + \nabla^2 \psi$$

2) Single layer of variable thickness

$$q = \beta y + \nabla^2 \psi - L_R^{-2} \psi$$

3) Two active quasi-geostrophic layers with a flat bottom and a rigid lid

$$q = \beta y + \nabla^2 \psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial \psi}{\partial z}$$

$$\left(N^2 = \frac{g'}{H} \quad L_{1,2} = \frac{\sqrt{g' H_{1,2}}}{f} \right)$$

$\frac{\partial \psi}{\partial z} = 0$

H_1 $\quad q_1 = \beta y + \nabla^2 \psi_1 - L_1^{-2}(\psi_1 - \psi_2)$

H_2 $\quad q_2 = \beta y + \nabla^2 \psi_2 + L_2^{-2}(\psi_1 - \psi_2)$

$\frac{\partial \psi}{\partial z} = 0$

1) Nondivergent barotropic case

In the first case we write down the vorticity equation $\frac{Dq}{Dt} = 0$ as:

$$q = \beta y + \nabla^2 \psi$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (U + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$$

u=U+u'   v=v'

$$\frac{\partial}{\partial t}(\beta y + \nabla^2 \psi) + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}(\beta y + \nabla^2 \psi) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\beta y + \nabla^2 \psi) = 0$$

$$\frac{\partial}{\partial t}\nabla^2 \psi + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}\nabla^2 \psi + \beta\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2 \psi = 0$$

Note that q is conserved with the flow. $\Psi = \Psi_B + \Psi$, with $\Psi_B = -Uy$. $\nabla^2 \Psi_B = 0$ It is crossed out from the PV equation

The **linear** equation in perturbations ψ is

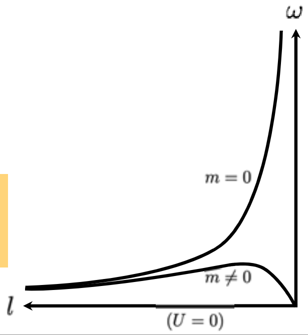
$$\frac{\partial}{\partial t}\nabla^2 \psi + U\frac{\partial}{\partial x}\nabla^2 \psi + \beta\frac{\partial \psi}{\partial x} = 0$$

Look for zonal wave solutions of the form $\psi = Re\, \tilde{\psi} e^{i(lx + my - \omega t)}$

Substitution into the derivatives gives algebraic expressions

$$\frac{\partial}{\partial t} \to -i\omega \qquad \frac{\partial}{x} \to il \qquad \nabla^2 \to -(l^2 + m^2)$$

⇨ Leads to the dispersion relation: $\omega = Ul - \frac{\beta l}{l^2 + m^2}$



- l is the **zonal wave number** (2π divided by the x-wavelength)
- m is the **meridional wave number** (2π divided by the y-wavelength)
- ω is the **angular frequency** (2π divided by the period)

details

$$\frac{\partial}{\partial t}\nabla^2 \psi + U\frac{\partial}{\partial x}\nabla^2 \psi + \beta\frac{\partial \psi}{\partial x} = 0 \qquad \frac{\partial}{\partial t} \to -i\omega \qquad \frac{\partial}{x} \to il \qquad \nabla^2 \to -(l^2 + m^2)$$

$$i\omega(l^2 + m^2) - il(l^2 + m^2)U + il\beta = 0$$

$$\omega(l^2 + m^2) = l(l^2 + m^2)U - l\beta$$

$$\omega = lU - \frac{l\beta}{(l^2 + m^2)}$$

$$\frac{\omega}{l} = c_x = U - \frac{\beta}{(l^2 + m^2)}$$

$$\frac{\partial \omega}{\partial l} = U - \beta\frac{\partial}{\partial l}(l(l^2 + m^2)^{-1})$$

$$\frac{1}{(l^2 + m^2)} + l(-(l^2 + m^2)^{-1} \times 2l) = \frac{(l^2 + m^2) - 2l^2}{(l^2 + m^2)^2} = -\frac{l^2 - m^2}{(l^2 + m^2)^2}$$

$$\frac{\partial \omega}{\partial l} = U + \beta\frac{l^2 - m^2}{(l^2 + m^2)^2}$$

Rossby wave dispersion

Dispersion relation

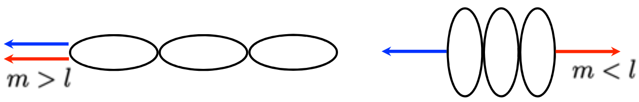
$$\omega = Ul - \frac{\beta l}{l^2 + m^2}$$

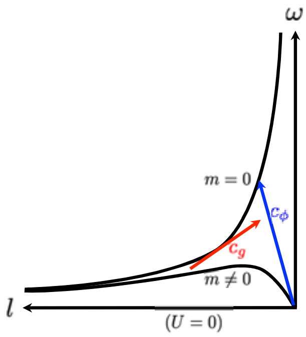
The phase speed and group speed in the x direction are given by

$$c = \frac{\omega}{l} = U - \frac{\beta}{l^2 + m^2} \qquad c_g = \frac{\partial \omega}{\partial l} = U + \frac{\beta(l^2 - m^2)}{(l^2 + m^2)^2}$$

The phase speed is westwards relative to the mean flow.

The group speed depends on the zonal and meridional scale of the wave.

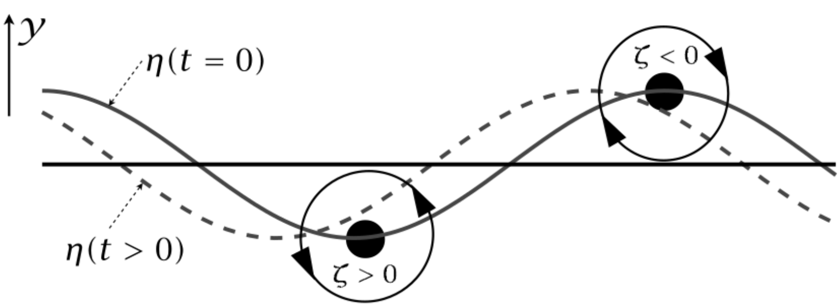




Longer waves (smaller k^2) travel faster.
Waves closer to the equator (bigger β) travel faster.

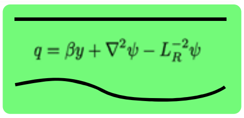
Rossby wave propagation mechanism

Can be understood in terms of the conservation of potential vorticity. When a parcel of fluid changes latitude, to compensate for its changing planetary vorticity, it must acquire either positive or negative relative vorticity. This induces a circulation that leads to the westward propagation of the disturbance.



2) Divergent case (variable layer thickness)

If we allow some vortex stretching in the conservation law, there is some modification of the Rossby wave characteristics. We linearize

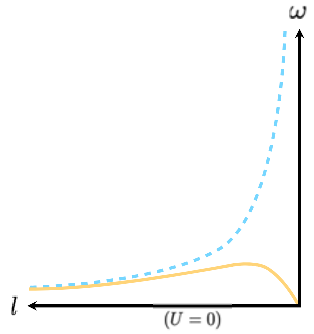


- In the PV conservation equation, the stream function is the summed-up contribution of:
  - the stream function associated with the perturbation ψ
  - the **background flow stream function** ψ_B=-*Uy*

$$\left(\frac{\partial}{\partial t} + (U+u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\left(\beta y + \nabla^2\psi - L_R^{-2}(\psi - Uy)\right) = 0$$

so
$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\nabla^2\psi - L_R^{-2}\psi) + (\beta + L_R^{-2}U)\frac{\partial\psi}{\partial x} = 0$$

material tendency of perturbation relative vorticity and vortex stretching term — *perturbation advection of planetary vorticity and basic state stretching term*



$\tilde{\psi}e^{i(lx+my-\omega t)}$ ⇨ *The dispersion relation is now* $\omega = Ul - l\dfrac{\beta + L_R^{-2}U}{l^2+m^2+L_R^{-2}}$

The current no longer just provides a simple doppler shift, but actively changes the basic state PV gradient, altering the propagation speed of the waves.

Note also that the denominator does not go to zero, so the phase speed is bounded and long waves are much less dispersive, with group speed to the west, even when m=0.

Topographic Rossby waves

Quasi-geostrophic potential vorticity
$$q = f_0 + \beta y + \xi - \frac{f_0}{H}\delta h$$

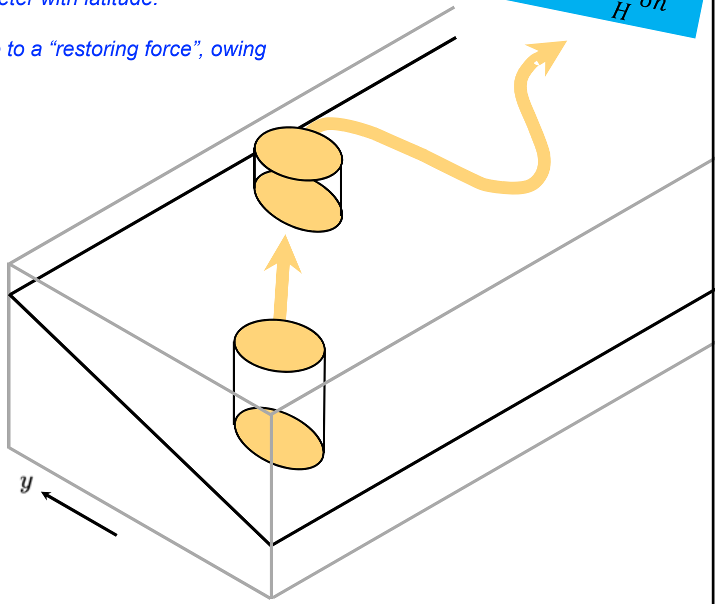
Vortex stretching can be important for Rossby waves in situations where there is a sloping bottom ($h_{OC} = \alpha y$). It is analogous to the effect of changing the Coriolis parameter with latitude.

Both are geometric effects giving rise to a "restoring force", owing to the generation of relative vorticity.

$$q = f_0 + \beta y + \xi - \frac{f_0}{H}(\alpha y + \eta)$$

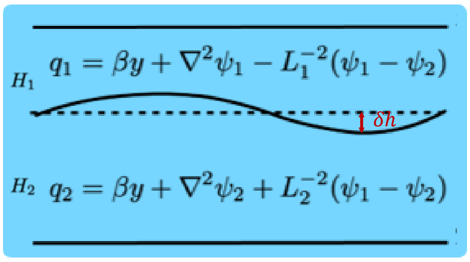
In the northern hemisphere, an ocean floor that is shallowing to the north will have the same effect as beta.

In the southern hemisphere the ocean floor must shallow to the south.



3) Two active layers

Two active quasi-geostrophic layers with a flat bottom and a rigid lid. Ignore advecting current for simplicity.



⇨ In this framework, we have two active layers in which the quasi-geostrophic potential vorticity is conserved (cf. **#GFD2.3h**):

$$q_1 = \frac{f+\xi_1}{h_1} \approx f + \xi_1 - \frac{f}{H_1}\delta h$$

$$q_2 = \frac{f+\xi_2}{h_2} \approx f + \xi_2 + \frac{f}{H_2}\delta h$$

⇨ We retrieve **geostrophic stream functions** for each layer (cf. **#GFD1.4e**):

$$\boldsymbol{f}_0 \times \boldsymbol{u}_1 = -\frac{1}{\rho_0}\nabla P_1 = -g\nabla(h_1 + h_2) \quad \text{and} \quad \boldsymbol{f}_0 \times \boldsymbol{u}_2 = -\frac{1}{\rho_0}\nabla P_2 = -g\nabla(h_1 + h_2) - g'\nabla h_2$$

$$\psi_1 = \frac{g}{f_0}(h_1 + h_2) \quad \text{and} \quad \psi_2 = \frac{g}{f_0}(h_1 + h_2) + \frac{g'}{f_0}h_2$$

↳ The interface displacements (from the rigid lid) are $\delta h = -h_2 = \frac{f_0}{g'}(\psi_1 - \psi_2)$. Therefore, **the vortex stretching term** is a **coupled term** defined in terms of the **difference between both stream functions**.

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 - L_1^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_1}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 + L_2^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_2}{\partial x} = 0$$

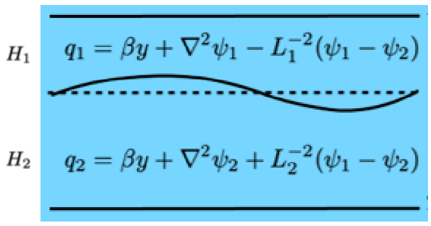
3) Two active layers

Two active quasi-geostrophic layers with a flat bottom and a rigid lid. Ignore advecting current for simplicity.

$$q = \beta y + \nabla^2\psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial \psi}{\partial z}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 - L_1^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_1}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 + L_2^{-2}(\psi_1 - \psi_2)\right] + \beta\frac{\partial \psi_2}{\partial x} = 0$$



We can uncouple these equations by subtraction and addition to find the "normal modes". We can then find independent solutions for the two modes.

$$\left(N^2 = \frac{g'}{H} \quad L_{1,2} = \frac{\sqrt{g' H_{1,2}}}{f}\right)$$

$$L_R^{-2} = L_1^{-2} + L_2^{-2}$$

$$\left(L_R = \frac{\sqrt{g'\hat{H}}}{f}, \quad \hat{H} = \frac{H_1 H_2}{H_1 + H_2}\right)$$

$$\overline{\psi} = \frac{L_2^{-2}\psi_1 + L_1^{-2}\psi_2}{L_1^{-2} + L_2^{-2}} = \frac{H_1\psi_1 + H_2\psi_2}{H_1 + H_2}$$

Barotropic Mode

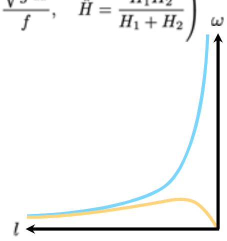
$$\hat{\psi} = \psi_1 - \psi_2$$

Baroclinic Mode

The equations become

$$\frac{\partial}{\partial t}\nabla^2\overline{\psi} + \beta\frac{\partial \overline{\psi}}{\partial x} = 0 \quad \rightarrow \quad \omega = -\frac{\beta l}{l^2 + m^2}$$

$$\frac{\partial}{\partial t}\left[(\nabla^2 - L_R^{-2})\hat{\psi}\right] + \beta\frac{\partial \hat{\psi}}{\partial x} = 0 \quad \rightarrow \quad \omega = -\frac{\beta l}{l^2 + m^2 + L_R^{-2}}$$



Extension to the vertical continuum

Consider quasi-geostrophic fluid bounded at top and bottom by rigid flat surfac
For simplicity we assume constant basic state stratification.

Quasi-geostrophic potential vorticit

$$q = \beta y + \nabla^2\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial\psi}{\partial z}\right)$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \left(\frac{f^2}{N^2}\right)\frac{\partial^2\psi}{\partial z^2}\right] + \beta\frac{\partial\psi}{\partial x} = 0$$ *with boundary condition* $$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial z}\right) = 0$$

seek wave solutions $\psi = Re\,\tilde{\psi}(z)e^{i(lx+my-\omega t)}$

with separable vertical dependence $\left(\frac{f^2}{N^2}\right)\frac{d^2\tilde{\psi}}{dz^2} = -\Gamma\tilde{\psi}$

In general, leads to dispersion relation

$$\omega = -\frac{\beta l}{k^2+\Gamma}$$

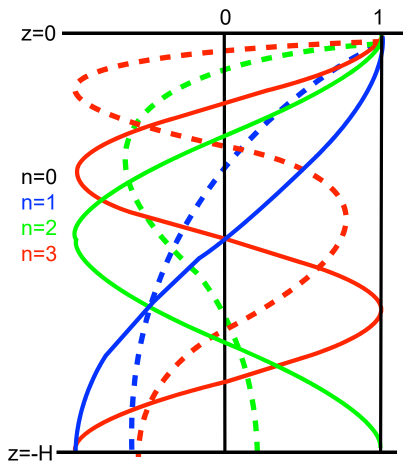
In this simple case the eigensolutions are cosines

$\tilde{\psi}_n(z) = \cos(n\pi z/H), \quad n = 1, 2..., \quad k_v = n\pi/H$

$$\omega = -\frac{\beta l}{k^2 + \underbrace{(f^2/N^2)k_v^2}_{\Gamma_n}}$$

$$L_n = \frac{NH}{n\pi f} = \frac{N}{fk_v} = \frac{c_n}{f}$$

$$\Rightarrow \quad c_n = \frac{N}{k_v}$$



Vertically propagating Rossby waves

Consider the vertical wavenumber for each mode

$k_{vn} = n\pi/H = N/c_n$

Remember c_n is the gravity wave speed associated with the vertical mode, not the phase speed of the Rossby wave !

Dispersion relation for long Rossby waves

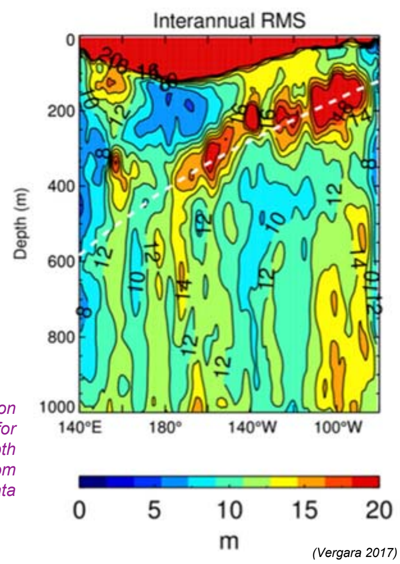
$$\omega = -\frac{\beta l}{k^2 + (f^2/N^2)k_v^2} \approx -\frac{\beta l N^2}{f^2 k_v^2} = -\frac{\beta l c_n^2}{f^2}$$

We can trace the signal path associated with vertical propagation in the x-z plane by calculating the ratio of components of the group velocity

$$\frac{\partial\omega}{\partial l} = -\frac{\beta N^2}{f^2 k_v^2} = -\frac{\beta c_n^2}{f^2}$$

$$\frac{\partial\omega}{\partial k_v} = \frac{2\beta l N^2}{f^2 k_v^3} = \frac{2\beta l c_n^3}{f^2 N}$$

So $$\frac{dz}{dx} = \frac{c_g^z}{c_g^x} = -\frac{2lc_n}{N} = \frac{2f^2\omega}{\beta N c_n}$$

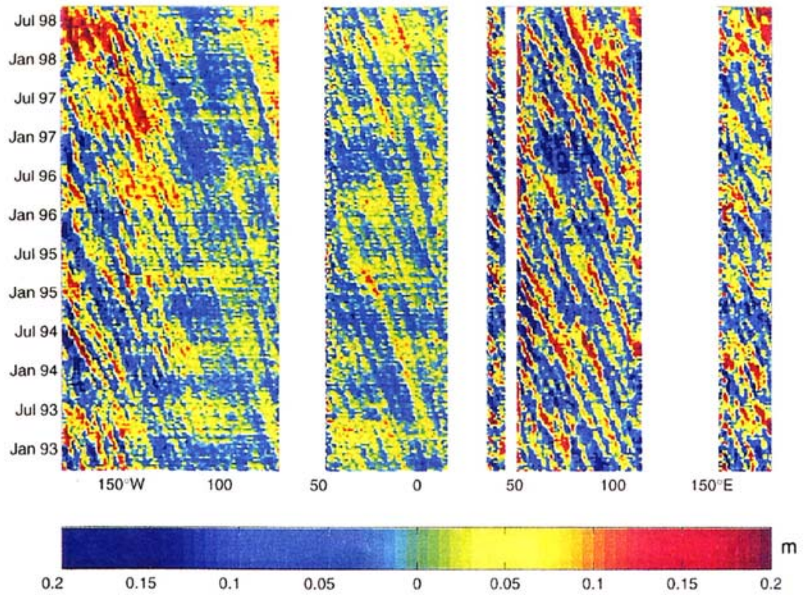


Propagation pathway for isotherm depth variability from ARGO data

(Vergara 2017)

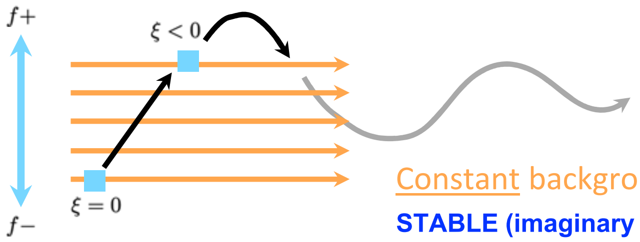
Observations

Evidence of Rossby wave propagation in satellite altimetry of the sea surface ?



Growing Rossby waves ?

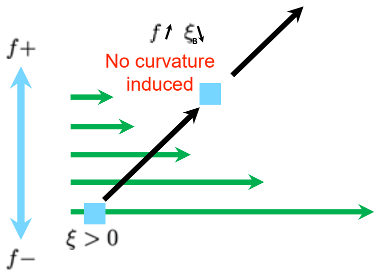
Let's revisit the mechanism for Rossby waves, but this time with horizontal shear



Constant background flow U

STABLE (imaginary exponential)

$f + \xi = \text{const}$

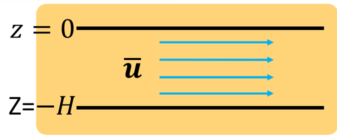


Shear background flow U

UNSTABLE (real exponential)

Perturbations on a shear flow

⇨ **Barotropic nondivergent flow**: *uniform in the vertical*

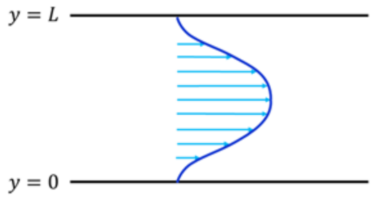


$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

choose a background flow that is a solution of the equations: $u=\overline{u}(y)=\frac{1}{\rho f}\frac{d\overline{p}}{dy}$

add perturbations: $u'=-\frac{\partial \psi}{\partial y},\quad v'=\frac{\partial \psi}{\partial x}$



⇨ *Leads to the perturbation barotropic vorticity equation:*

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi+\left(\beta-\frac{d^2\overline{u}}{dy^2}\right)\frac{\partial \psi}{\partial x}=0$$

advection of perturbation vorticity by basic state winds

advection of basic state absolute vorticity by perturbation winds

details

$$u'_t+\overline{u}u'_x+v'\overline{u}_y-fv'=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad -\psi_{yt}-\overline{u}\psi_{xy}+\overline{u}_y\psi_x-f\psi_x=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$v'_t+\overline{u}v'_x+\cancel{v'\overline{v}_y}+fu'=-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad -\psi_{xt}-\overline{u}\psi_{xx}\quad -f\psi_y=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-\psi_{yyt}-\overline{u}\psi_{xyy}-\cancel{\overline{u}_y\psi_{xy}}+\overline{u}_{yy}\psi_x+\cancel{\overline{u}_y\psi_{xy}}-\beta\psi_x-\cancel{f\psi_{xy}}-\psi_{xxt}-\overline{u}\psi_{xxx}+\cancel{f\psi_{xy}}=0$$

$$-\frac{\partial}{\partial t}\nabla^2\psi-\overline{u}\frac{\partial}{\partial x}\nabla^2\psi+\left(\overline{u}_{yy}-\beta\right)\psi_x=0$$

Stationary Rossby waves

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi+\left(\beta-\frac{d^2\overline{u}}{dy^2}\right)\frac{\partial\psi}{\partial x}=0$$

As before, we can derive a dispersion relation for Barotropic Rossby waves, this time on a shear flow, by introducing solutions of the form $\psi = Re\,\tilde{\psi}e^{i(lx+my-\omega t)}$

$$\omega = Ul - \frac{(\beta - U_{yy})l}{l^2+m^2}$$

Consider stationary waves: $\omega = 0$

$$\Rightarrow U(l^2+m^2) = (\beta - U_{yy})$$

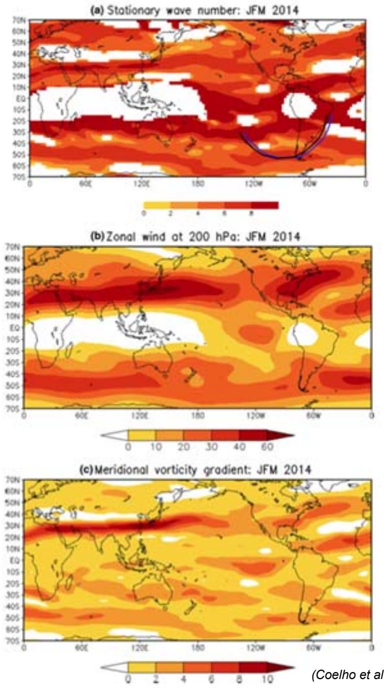
And the stationary wavenumber $k_s = \sqrt{(\beta - U_{yy})/U}$

For stationary Rossby waves to exist, $(\beta - U_{yy})$ *must have the same sign as* U *(which usually means both must be positive).*

Ray paths can be calculated as before from the ratio of components of the group velocity

$$\mathbf{c}_g = \left(U + \frac{\beta_*(l^2-m^2)}{k^4}, -\frac{2\beta_* lm}{k^4}\right)$$

$(\beta_* = \beta - U_{yy},\ k^2 = l^2+m^2)$



(Coelho et al 2016)

Growing solutions

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi+\left(\beta-\frac{d^2\overline{u}}{dy^2}\right)\frac{\partial\psi}{\partial x}=0$$

Now let's seek solutions in form of zonal wave with coefficients that depend on y

$$\psi(x,y,t) = \phi(y)e^{i(lx-\omega t)}$$

Substitute in, get

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta - d^2\overline{u}/dy^2}{\overline{u}(y)-c}\phi = 0$$

the "Rayleigh equation" (where c = ω / l). If we add channel boundary conditions Φ = 0 at y = 0, L, in general we get a set of solutions for Φ associated with complex conjugate pairs of values for c (or ω). The imaginary part of the solution is associated with growth or decay. The growth rate is the imaginary part of ω.

$$c = c_r + ic_i,\ \ c^* = c_r - ic_i$$
$$\omega = \omega_r + i\omega_i,\ \ \omega^* = \omega_r - i\omega_i$$

(note that the wavenumber l is real)

details

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0 \qquad \psi = \phi(y)e^{i(lx-\omega t)}$$

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = \frac{\partial}{\partial x}\left(\phi\, il\, e^{<>}\right) + \frac{\partial}{\partial y}\left(\phi_y e^{<>}\right) = \left(-\phi l^2 + \phi_{yy}\right)e^{<>}$$

$$\psi_x = \phi\, il\, e^{<>}$$

$$-\cancel{i}\omega(-\phi\, l^2 + \phi_{yy}) + \cancel{i}l\,\overline{u}(-\phi\, l^2 + \phi_{yy}) + (\beta - \overline{u}_{yy})\phi\cancel{i}l = 0$$

$$-\frac{\omega}{l}(\phi_{yy} - \phi\, l^2) + \overline{u}(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$-\left(\frac{\omega}{l} - \overline{u}\right)(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$(\overline{u} - c)(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$\phi_{yy} - l^2\phi + \left(\frac{\beta - \overline{u}_{yy}}{\overline{u} - c}\right)\phi = 0$$

Conditions for growth: the Rayleigh criterion

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta - d^2\overline{u}/dy^2}{\overline{u}(y) - c}\phi = 0$$

Multiply the Rayleigh equation by Φ and integrate across the domain:* (integrate by parts and apply boundary conditions)

$$-\int_0^L\left(\left|\frac{d\phi}{dy}\right|^2 + l^2|\phi|^2\right)dy + \int_0^L \frac{\beta - d^2\overline{u}/dy^2}{\overline{u} - c}|\phi|^2 dy = 0$$

The term on the left is real. If c is complex, and we multiply top and bottom by $(\overline{u} - c)^*$ *we can isolate the imaginary part:*

$$c_i \int_0^L \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{|\phi|^2}{|\overline{u} - c|^2}dy = 0$$

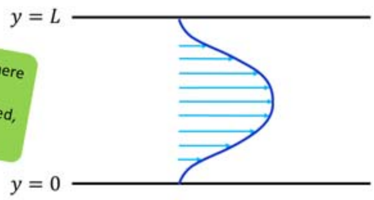
*If $c_i \neq 0$ then we have growth. So a **necessary condition** for growth is that the integral is zero.*
This means that $\beta - \overline{u}_{yy}$ *must change sign between y = 0 and y = L.*
To put it another way, the gradient of absolute vorticity of the background flow:

$$\frac{d}{dy}(f_0 + \beta y - \overline{u}_y)$$

must change sign in the domain.

$\beta - \overline{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
If the Rayleigh criterion is satisfied, we might have an instability.

So we require an extremum in absolute vorticity.



derivation: integrating by parts

$$\frac{d}{dy}(\phi\phi_y) = \phi_y^2 + \phi\phi_{yy} \quad \Longrightarrow \quad d(\phi\phi_y) = \phi_y^2\, dy + \phi\phi_{yy}\, dy$$

$$\int \phi(\phi_{yy} - l^2\phi)\, dy = \int (\phi\phi_{yy} - l^2\phi^2)\, dy = \int_0^L d(\phi\phi_y) - \int_0^L (\phi_y)^2\, dy - \int_0^L l^2\phi^2\, dy$$

$$= [\phi\phi_y]_0^L - \int_0^L |\phi_y|^2 + l^2\phi^2\, dy$$

More conditions for growth: the Fjørtoft criterion

The real part of the integral must also be zero. By the same manipulation as before this gives

$$-\int_0^L \left(\left| \frac{d\phi}{dy} \right|^2 + l^2|\phi|^2 \right) + \int_0^L (\overline{u} - c_r) \left(\beta - \frac{d^2\overline{u}}{dy^2} \right) \frac{|\phi|^2}{|\overline{u} - c|^2} = 0$$

thus $\int_0^L (\overline{u} - c_r) \left(\beta - \frac{d^2\overline{u}}{dy^2} \right) \frac{|\phi|^2}{|\overline{u} - c|^2} > 0$ $\quad \int A(u-c) > 0$

as we already know

$$\int_0^L \left(\beta - \frac{d^2\overline{u}}{dy^2} \right) \frac{|\phi|^2}{|\overline{u} - c|^2} = 0 \qquad \int A = 0$$

Fjørtoft Logic

$\int A = 0$
$\int A(u-c) > 0$
$\int A(u-u_0) = \int A(u-c) + \int A(c-u_0) > 0$ (constant)
$\int A(u-u_0) > 0$

we can deduce that $(\overline{u} - u_0)(\beta - \overline{u}_{yy})$ must be positive somewhere in the domain for any u_0 including the value for the background flow at the extremum.

The expression is obviously zero at the extremum but must be positive somewhere in the domain. The choice of the value of u_0 at the latitude of the vorticity extremum makes this criterion as stringent as possible.

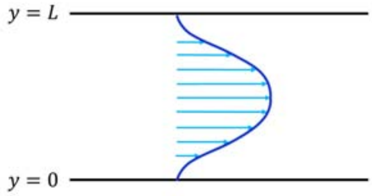
Note: a the non-satisfaction of a necessary condition for instability can also be seen as a sufficient condition for stability

$\forall u_0$, $(\overline{u} - u_0)(\beta - \overline{u}_{yy})$ must be positive somewhere in the domain.
If the Fjørtoft criterion is satisfied, we might have an instability.

Conditions for growth

$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
↳ If the Rayleigh criterion is satisfied, we might have an instability.

$\forall u_0, (\bar{u} - u_0)(\beta - \bar{u}_{yy})$ must be positive somewhere in the domain.
↳ If the Fjørtoft criterion is satisfied, we might have an instability.

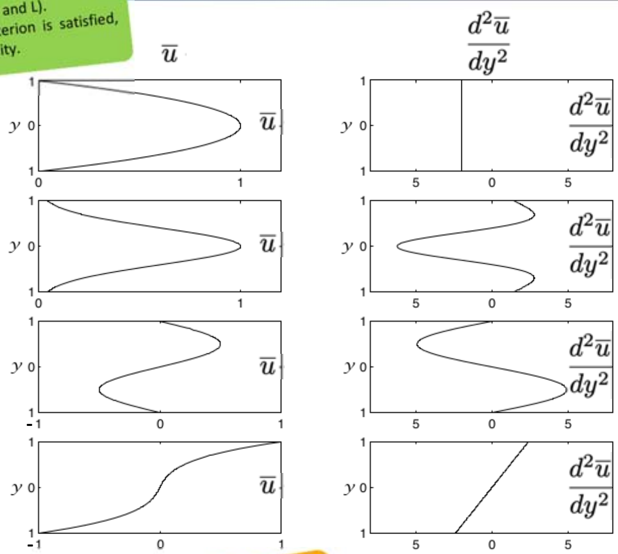


Both Rayleigh and Fjørtoft criteria are just ***necessary conditions***. They are not sufficient conditions. This means that, when analyzing a potential vorticity map, if one of these conditions is satisfied, it does not mean that the flow is unstable, it means that ***it is possible for the flow to be unstable***.

On the other hand, the non-satisfaction of a necessary condition is a sufficient condition, which means that ***if the Rayleigh or the Fjørtoft condition is not satisfied then the flow is stable***.

Stable and unstable profiles

$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
↳ If the Rayleigh criterion is satisfied, we might have an instability.



Poiseuille Flow ($u = 1 - y^2$) — stable (Rayleigh) - no change of sign

Gaussian jet — possibly unstable – change of sign

sinusoidal — possibly unstable – change of sign

polynomial (boundary extrema) — stable (Fjørtoft) – vorticity extrema at the boundaries

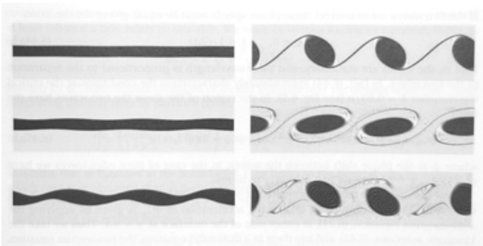
$\forall u_0, (\bar{u} - u_0)(\beta - \bar{u}_{yy})$ must be positive somewhere in the domain.
↳ If the Fjørtoft criterion is satisfied, we might have an instability.

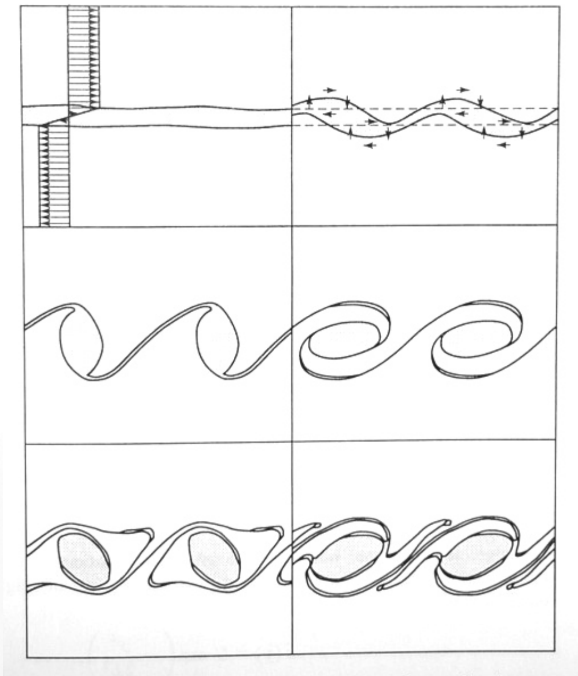
boundary extrema but u always has opposite sign to vorticity gradient (f plane example, beta might change this) so product always negative.

- Remember: Fjortoft criterion must work for any u_0

Physical mechanism

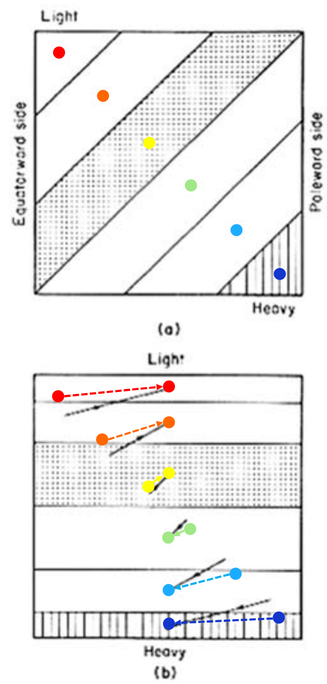
Take the example of an isolated shear layer. It has negative (clockwise) vorticity and is embedded in a flow that has no vorticity. So it represents an extremum. The perturbation meridional flow can export this vorticity into a region where there is none. At the same time, on the other side of the vorticity strip, but just out of phase, the same thing happens. The induced flow deforms the vorticity strip so that the situation amplifies and the deformation continues.



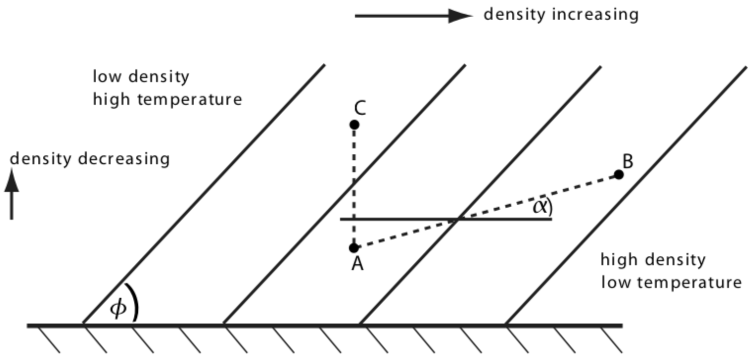


Baroclinic instability

Now we turn to a mechanism that can liberate stored potential energy in a system that may be barotropically (and statically) stable. Ultimately, work is done by gravity to provide growing kinetic energy. The perturbation must have the right structure to make the necessary rearrangements to tap this source of energy.



Sloping convection



In a rotating system we can imagine a steady basic state with inclined density contours (we need rotation to balance the pressure gradient forces). It can be statically stable. But a sloping parcel displacement can still leave a parcel in a situation where it is more buoyant. The displacement A-C is stable. But the exchange of the two parcels A and B will release energy stored in the density structure.

Optimal scales for growth

The mechanism relies on horizontal variations of density, and on a perturbation that has the right phase arrangement to amplify by vortex stretching. Consider the following scaling for the quasigeostrophic potential vorticity (on an f plane):

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right)$$

If distance scales as L, then $\nabla^2\psi \sim \dfrac{\Psi}{L^2}$

and if height scales as H then the vortex stretching term scales as $\sim \dfrac{f^2\Psi}{N^2H^2} = \dfrac{\Psi}{{L_R}^2}$

If L >> L_R then the relative vorticity is unable to balance stetching, so stretching is inhibited and we have vertically uniform disturbances - this is the barotropic limit.

If L<< L_R then relative vorticity dominates, and the layers become uncoupled, and thus unable to cooperate to produce the necessary structures to liberate the potential energy stored in the horizontal variations of stratification, or the vertical shear of the wind.

The optimal scale is thus the Rossby (internal) radius of deformation L_R = NH/f
Growing disturbances of this scale will be ***selected***

Physical mechanism

Consider a two-layered shear flow in thermal wind balance.

Introduce a positive PV anomaly in the upper layer, with associated cyclonic flow.

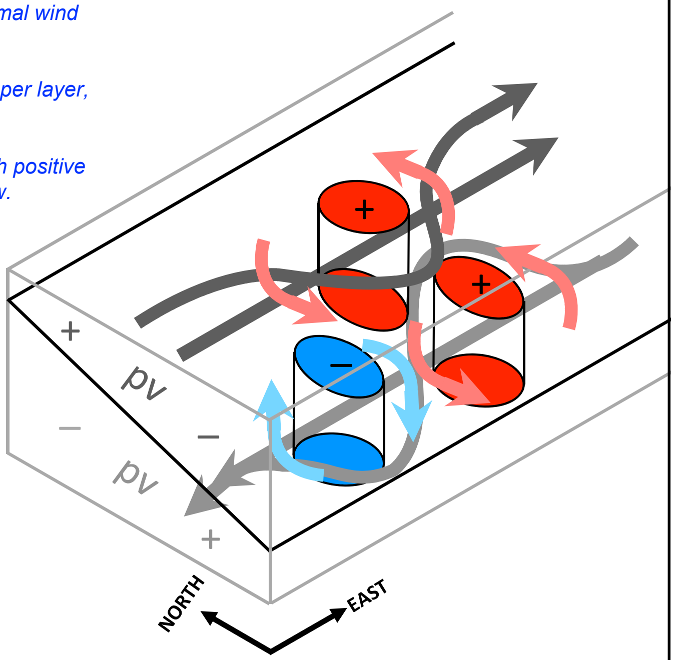
Positive relative vorticity is associated with positive layer thickness, squeezing the layer below.

Induced advection in the lower layer creates a dipole of PV anomalies with associated circulation patterns

This induces southward advection of more positive PV in the layer above, amplifying the original perturbation.

Note the westward tilt with height of the PV perturbations.

At the same time, due to the upper level PV gradient and the gradient of f, the entire structure propagates westwards as a Rossby wave.



Modal solutions

The linear perturbation potential vorticity equation is

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial Q}{\partial y} = 0$$

and as usual we seek wavelike solutions in x

$$\psi' = \tilde{\psi}(y,z)e^{i(lx-\omega t)}$$

substitution leads to the equation

$$(U-c)(\tilde{\psi}_{yy} + \frac{\partial}{\partial z}\frac{{f_0}^2}{N^2}\tilde{\psi}_z - l^2\tilde{\psi}) + Q_y\tilde{\psi} = 0$$

with boundary conditions at the top and bottom (z = 0, H)

$$(U-c)\tilde{\psi}_z - U_z\tilde{\psi} = 0$$

↳ *These are analogous to the Rayleigh equation for barotropic (shear) instability.*

Conditions for growth

We go through the same procedure as before with these equations: multiply by the complex conjugate and integrate over the domain. The "domain" is now in y and z.

This eventually leads to

$$\int_0^L \int_0^H |\Psi_y|^2 + f_0^2/N^2|\tilde{\psi}_z|^2 + l^2|\tilde{\psi}|^2 dz dy - \int_0^L \left\{ \int_0^H \frac{Q_y}{U-c}|\tilde{\psi}|^2 dz + \left[\frac{f_0^2/N^2 U_z|\tilde{\psi}|^2}{U-c} \right]_0^H \right\} dy = 0$$

the imaginary part of which is

$$-c_i \int_0^L \left\{ \int_0^H \frac{Q_y}{|U-c|^2}|\tilde{\psi}|^2 dz + \left[\frac{f_0^2/N^2 U_z|\tilde{\psi}|^2}{|U-c|^2} \right]_0^H \right\} dy = 0$$

If $c_i \neq 0$ then the integral must be zero. This means that at least one of the following conditions must be met (the "Charney - Stern - Pedlosky criteria")

- Q_y changes sign in the domain (there is a PV extremum)
- Q_y has the opposite sign to U_z at z = H
- Q_y has the same sign as U_z at z = 0
- if $Q_y = 0$, U_z has the same sign at z =0 and z = H

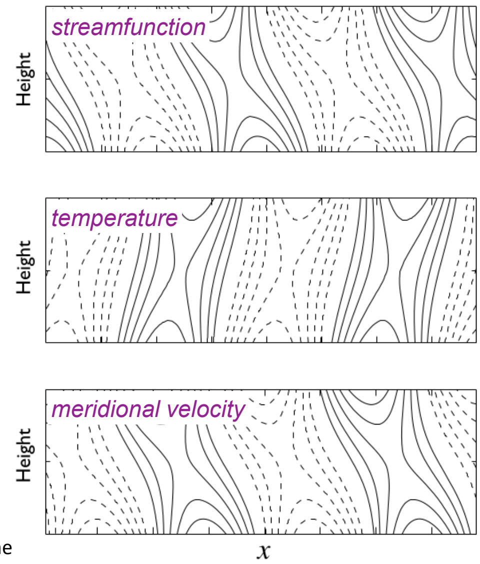
Note these are just necessary conditions for the integral to vanish.
Note that U_z is directly related to the basic state meridional temperature or density gradient.

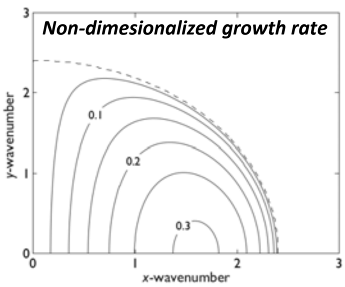
- Waves can grow in the interior of the fluid (on PV extrema) or as boundary phenomena (on boundary temperature gradients).

The Eady problem

Simplest archetype of baroclinic instability
- f plane
- N^2 constant
- constant vertical shear U(z) = U z/H
- motion is between two rigid flat surfaces at z = 0, H

- Uniform vertical shear means the basic state PV = 0.
- The procedure for solving the problem is the same as before - substitute wave functions into the PV equation to produce a Rayleigh-type equation, and apply the boundary conditions w = 0 at z = 0, H.
- Instability arises from boundary temperature gradients.





Max. growth rate

$$\sim 0.31 \frac{U}{L_R}$$

Wavenumber and wavelength at which the instability is the greatest are:

$$k_m = \frac{1.61}{L_R} \qquad \lambda_m = \frac{2\pi}{k_m} = \frac{2\pi}{1.6} L_R$$

What we learn from the Eady problem

- length scale of maximum instability characterised by the deformation scale (factor of about four times)
- the most unstable growth rate is 0.3 U/L_R = 0.3 f_0/N du/dz
- there is a short wave cutoff - short waves are not unstable
- the circulation (meridional current, streamfunction) must slope westwards with height in westerly shear to extract energy from the basic state.

Some results of the Eady calculation applied in an oceanic context:

H ~ 1 km, U ~ 0.1 m/s, N ~ 10^{-2} s^{-1} leads to

deformation radius L_R = NH/f = 10^{-2} x 1000 / 10^{-4} = 100 km
scale of maximum instability = 3.9 L_R ~ 400 km
growth rate = 0.3 U/L_R ~ 0.3 x 0.1 / 10^5 ~ 0.026 $days^{-1}$ (period ~ 40 days)

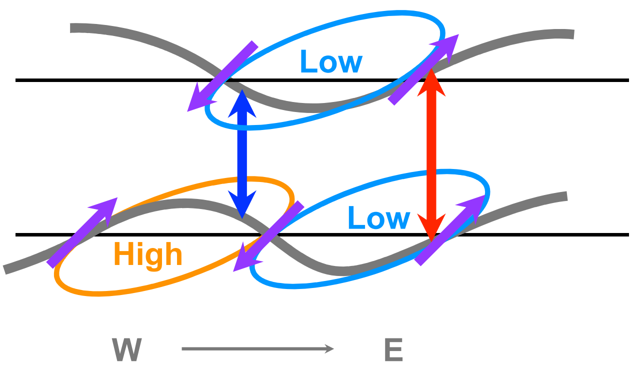
Compare with the atmosphere

H ~ 10 km, U ~ 10 m/s, N ~ 10^{-2} s^{-1} leads to
L_R ~ 1000 km, instability scale ~ 4000 km, growth rate ~ 0.26 $days^{-1}$ (period 4 days)

In the Eady problem is theoretical, the instability relies on an interaction between waves at the upper and lower boundaries. If either boundary is removed, the instability dies.

Heat transport in a baroclinic system

Growing structures tilt westwards with height.
Consistent with thermal wind balance this structure also transports heat polewards.



Baroclinic instability: summary

- There is clear evidence of a preferred scale for turbulent motions in the ocean

- Simple scaling arguments and more sophisticated stability analyses show that there is a preferred scale on which growth can occur.

- If this growth depends on extracting energy from sloping density surfaces (or equivalently, vertical wind shear, or horizontal temperature gradients), then there must be an interplay between vortex stretching and relative vorticity terms in the conservation of PV.

- This naturally selects structures around the Rossby deformation scale.

- These structures can grow exponentially provided certain criteria are met: notably if extrema exist in the potential vorticity of the basic state.