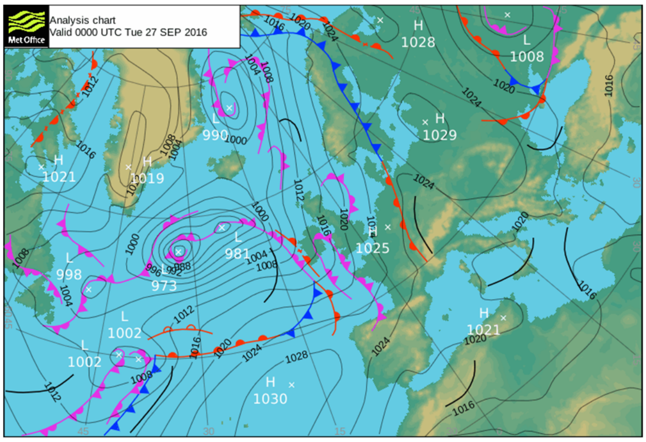
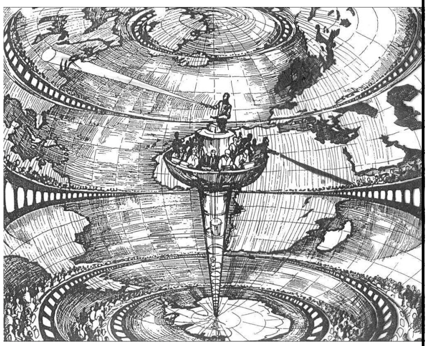
Chapter 2: Quasi-geostrophic theory

⇨ Steady departures from geostrophy: nonlinearity and drag.
Ageostrophy, divergence and potential vorticity.
⇨ *f*-plane quasi-geostrophy in shallow water.
Quasi-geostrophy on a curved planet.
⇨ Continuous stratification
Development and vertical motion.





"Richardson's dream"

Gradient wind balance

Recall x-momentum equation (for a single layer)

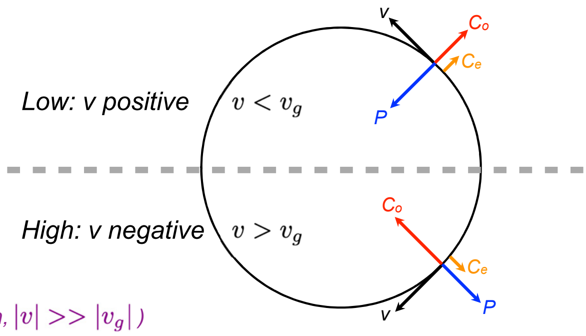
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

Now assume time independent uniform flow in a circle. The nonlinear (advection) terms express centrifugal force - this is gradient wind balance (without the Coriolis force it is "cyclostrophic" balance).

$$fv + \frac{v^2}{r} = g\frac{dh}{dr} = fv_g$$

so $$v\left(1 + \frac{v}{fr}\right) = v_g$$

or $$|v| = \frac{|v_g|}{1 \pm R_o}$$



("anomalous" cases have v and v_g opposite sign, $|v| >> |v_g|$)

solution for v

$$v = -\frac{fr}{2} \pm \sqrt{\frac{f^2r^2}{4} + rg\frac{dh}{dr}}$$

so flow around a high limited by

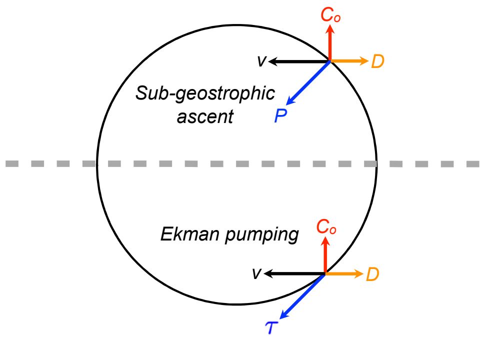
$$\left|\frac{dh}{dr}\right| \le \frac{f^2 r}{4g}$$ *(no such limitation for flow round a low)*

Boundary friction

Adding surface stress can alter the balance in a linear framework, leading to convergence or divergence.

This is the reason air ascends in lows (cloudy weather) and descends in highs (clear sky).

It is also the basis of the way the ocean driven the wind through Ekman pumping and Ekman suction.



But now we need to move away from these anecdotal cases, and put together a system with advection and time dependence that is almost, but not quite, geostrophic.

We do this essentially by separating the flow into a geostrophically balanced, nondivergent part, and the ageostrophic plus the divergent parts as a small perturbation. This small perturbation allows prognostic equations that lead to the evolution of the flow.

Ageostrophic perturbation

Start with the shallow water momentum equations in a single layer

$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0$$
$$\frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\left\{ \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right\}$$

What happens if we substitute in geostrophic velocity ? $u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \quad v_g = \frac{g}{f}\frac{\partial h}{\partial x}$

Redefine the material tendency to advect with the geostrophic wind

$$\frac{D}{Dt} \to \frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$$

What if we use the geostrophic value in the Coriolis term as well ?
Obviously this can't work because it leads to zero tendency ! $\frac{D_g}{Dt}u_g - fv_g + g\frac{\partial h}{\partial x} = 0$

Instead we make sure the equation is linear in the ageostrophic part $v_{ag} = v - v_g$

So the full flow is used in the linear Coriolis terms and we advect with the geostrophic flow.
This is consistent with the idea that the ageostrophic part of the flow is small.

Quasi-geostrophic f-plane vorticity equation

$$\frac{D_g}{Dt}u_g - fv + g\frac{\partial h}{\partial x} = 0 \quad (1)$$
$$\frac{D_g}{Dt}v_g + fu + g\frac{\partial h}{\partial y} = 0 \quad (2)$$

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$$ -> vorticity equation

$$\frac{\partial}{\partial t}\xi_g + u_g\frac{\partial}{\partial x}\xi_g + v_g\frac{\partial}{\partial y}\xi_g + \xi_g\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = 0$$

If we are on an f-plane the last term disappears and the geostrophic flow is nondivergent

$$\frac{df}{dy} = 0, \quad \nabla.\mathbf{v}_g = 0$$

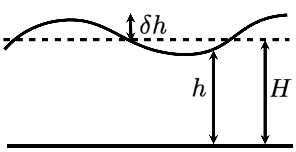
so we can write this as

$$\frac{D_g}{Dt}(f + \xi_g) = -f\nabla.\mathbf{v}$$

Continuity equation

What happens when we try this with the continuity equation ?

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \qquad \Rightarrow \frac{Dh}{Dt} + h\nabla.\mathbf{v} = 0$$



Replace this D/Dt with the geostrophic operator D_g/Dt

$$\rightarrow \frac{\partial h}{\partial t} + \mathbf{v}_g.\nabla h + h\nabla.\mathbf{v} = 0$$

But v depends on h. Terms involving v_{ag} were linear in the momentum equations so they must be linear here too. For consistency we must therefore write:

$$h = H + \delta h, \quad \delta h \ll H$$

$$\rightarrow \frac{\partial h}{\partial t} + \mathbf{v}_g.\nabla h + H\nabla.\mathbf{v} = 0$$

(this is equivalent to the approximation $\mathbf{v}_g h \approx \mathbf{v}_g \delta h + \mathbf{v}H$)

The ageostrophic term is now linear.

$$\rightarrow \frac{\partial}{\partial t}\delta h + \mathbf{v}_g.\nabla\delta h + H\nabla.\mathbf{v} = 0 \quad \textit{or} \quad \frac{D_g}{Dt}\delta h + H\nabla.\mathbf{v} = 0$$

In order to keep the ageostrophic term linear we had to make a strong assumption about the stratification.

Quasi-geostrophic potential vorticity

We can rewrite the continuity equation as $\frac{f}{H}\frac{D_g}{Dt}\delta h = -f\nabla.\mathbf{v}$

which as before has the same right hand side as the vorticity equation so we get

$$\frac{D_g}{Dt}\left(f\frac{\delta h}{H}\right) = \frac{D_g}{Dt}(f+\xi_g), \quad \rightarrow \frac{D_g}{Dt}\left\{f+\xi_g - f\frac{\delta h}{H}\right\} = 0$$

This is the conservation law for quasi-geostrophic potential vorticity:

$$\frac{D_g}{Dt}q = 0, \quad q = f+\xi_g - f\frac{\delta h}{H}$$

q is the linearised form of the full "Ertel" potential vorticity (but note that the units are different)

$$\frac{f+\xi}{h} = (f+\xi)(H+\delta h)^{-1} \approx \left(\frac{f+\xi}{H}\right)\left(1-\frac{\delta h}{H}\right) \Rightarrow q = f+\xi - f\frac{\delta h}{H} - \xi\frac{\delta h}{H}$$

Geostrophic scaling

$$Ro \ll 1, \quad \frac{U}{fL} \ll 1 \Rightarrow f \gg \xi, \quad \xi = \xi_g, \quad \Rightarrow \boxed{q = f+\xi_g - f\frac{\delta h}{H}}$$

This linearisation of layer thickness variations is a surprising consequence of our insistence that the flow be close to geostrophic.

In a vertically continuous framework it means that the stratification is uniform in the horizontal

Adding curvature to the earth

That was all pretty straightforward because we assumed that *f* was constant.

But for many important dynamical phenomena the variation of *f* is important (Rossby waves, for example).

On an *f*-plane the geostrophic flow is strictly nondivergent.

If *f* varies then we have to deal with the divergent part of the geostrophic flow as well as the ageostrophic flow.

We will assume that both these components are small compared to the nondivergent part of the geotrophic flow.

To proceed, we must derive the quasi-geostrophic set as an expansion of this perturbation in a small parameter.

We naturally choose the Rossby number for this small parameter.

Derivation of the quasi-geostrophic set for a shallow water layer

Recall full momentum and continuity equations:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} + f\hat{\mathbf{k}}_\wedge\mathbf{v} + g\nabla h = 0, \quad \frac{\partial h}{\partial t} + \mathbf{v}.\nabla h + h\nabla.\mathbf{v} = 0$$

Introduce scaling for non-dimensionalisation:

$$x' = x/L, \quad u' = u/U, \quad t' = t/T, \quad \eta' = \delta h/\Delta h, \quad (h = H + \delta h)$$

So the full equations become

$$\frac{U}{T}\frac{\partial \mathbf{v}'}{\partial t'} + \frac{U^2}{L}\mathbf{v}'.\nabla\mathbf{v}' + Uf\hat{\mathbf{k}}_\wedge\mathbf{v}' + g\frac{\Delta h}{L}\nabla\eta' = 0 \quad (1)$$

$$\frac{\Delta h}{T}\frac{\partial \eta'}{\partial t'} + \frac{U}{L}\Delta h\mathbf{v}'.\nabla\eta' + \frac{U}{L}(H + \Delta h\eta')\nabla.\mathbf{v}' = 0 \quad (2)$$

If the basic scalings conform to geostrophic balance (f_0 is the value of f at a reference latitude)

$$f\mathbf{v} \sim g\nabla h \quad \rightarrow \quad Uf_0 \sim g\frac{\Delta h}{L} \quad \rightarrow \quad \Delta h = \frac{Uf_0L}{g} \quad (3)$$

Define the Rossby number and temporal Rossby number as $\epsilon = \dfrac{U}{f_0L} \quad (4)$ and $\epsilon_T = \dfrac{1}{f_0T} \quad (5)$

Dropping primes, (1) / f_0U, (3), (4) and (5) -> $$\epsilon_T\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + \frac{f}{f_0}\hat{\mathbf{k}}_\wedge\mathbf{v} + \nabla\eta = 0$$

Quasi-geostrophic continuity equation

Again dropping primes, L/UH x (2) -> $$\epsilon_T\left(\frac{L^2f_0^2}{gH}\right)\frac{\partial\eta}{\partial t} + \epsilon\left(\frac{L^2f_0^2}{gH}\right)(\mathbf{v}.\nabla\eta + \eta\nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

The non-dimensional constant that appears in brackets in this equation is B_u^{-1}, or L^2/L_R^2. Call it F.
Remember, when F~1, Coriolis and gravity / buoyancy effects are comparable.

So the non-dimensional continuity equation is

$$\epsilon_T F\frac{\partial\eta}{\partial t} + \epsilon F(\mathbf{v}.\nabla\eta + \eta\nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

So far we haven't made any approximations.

But we can already see from these two equations that to zero order in our Rossby number parameters, the flow is geostrophic and nondivergent.

First order terms concern advection, divergence and time development.

Before doing a formal expansion in the Rossby number, we will set out our assumptions in detail.

The assumptions of quasi-geostrophic theory

1) small Rossby number, i.e. close to geostrophy $\epsilon \ll 1$

2) small temporal Rossby number, i.e. timescales slow compared to local rotation rate. $\epsilon_T \ll 1$
- no fast moving linear waves
- nonlinear (advection terms important for time development)

In fact we assume that $\epsilon_T = \epsilon$

3) buoyancy/gravity - stratification effects as important as Coriolis effects $L \sim L_R, \quad F = O(1)$

A consequence of (3) and (1) is that δh << H
- linearisation of continuity equation and q.g.p.v. as we saw before.
In a continuously stratified case this is equivalent to saying that N^2 is a function of z but not of x and y.

4) scales of motion small compared to the radius of the earth $\frac{L}{r_e} \ll 1$

In fact we assume that $\frac{L}{r_e} = \epsilon$

Assumptions (3) and (4) have nothing to do with geostrophy !
They are necessary for our expansion to be self-consistent.

The beta effect

Is a consequence of assumption (4), that the scale of motion is small compared to the radius of the earth.

$$f = 2\Omega \sin\phi$$

Taylor expansion about reference latitude Φ_0 (where y' = y/r_e)

$$f = f_0 + \beta_0 y + \ldots \quad = f_0 + \left.\frac{df}{dy}\right|_0 y + \left.\frac{d^2 f}{dy^2}\right|_0 \frac{y^2}{2} + \ldots \quad = 2\Omega\sin\phi_0 + \frac{y'L}{r_e} 2\Omega\cos\phi_0 + \ldots$$

set $\beta' = \cot\phi_0 = \frac{\beta_0 L}{f_0 r_e} \sim 1$ *then provided* $\frac{L}{r_e} = \frac{U}{f_0 L}$

we can write, to first order $\frac{f}{f_0} = 1 + \epsilon\beta' y'$ *(as long as we stay away from the equator where* $\cot\phi_0 \to \infty$ *)*

Henceforth drop primes on nondimensional y' and β'

This is often referred to as the "beta-plane" approximation, because the function f describes a plane in x-y space.
Not to be confused with the actual shape of the surface of the earth !
When we add the beta term, the surface of the earth ceases to be a plane and becomes a curve.
If we approximate the surface of the earth as a plane, then f is constant: the f-plane.

The expansion

Our equations are now non-dimensional.
$u, v, \eta, \beta, F \sim 1; \varepsilon << 1$
Expand variables in increasing powers of ε.

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + (1+\epsilon\beta y)\hat{\mathbf{k}}_\wedge\mathbf{v} + \nabla\eta = 0$$

$$\epsilon F\frac{\partial \eta}{\partial t} + \epsilon F(\mathbf{v}.\nabla\eta + \eta\nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + ...$$

$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + ...$$

Substitute this into the equations and compare coefficients of ε^0 (zero order) and ε^1 (first order).

Zero order $\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_0 = 0 \quad (1) \qquad \nabla.\mathbf{v}_0 = 0 \quad (2)$

Note that the curl of (1) gives (2). The two equations are equivalent. No development. Degenerate dynamics. Geostrophic nondivergent flow can only change in time with the help of the first order (divergent) flow.

We can say η_0 acts as a streamfunction for v_0, i.e.

$$v_0 = \frac{\partial \eta_0}{\partial x}, \quad u_0 = -\frac{\partial \eta_0}{\partial y}$$

Note that since v_0 is nondivergent, it is not the total geostrophic flow, just its nondivergent part. It represents the geostrophic flow on the f-plane at $f = f_0$.

First order in ε

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla\mathbf{v}_0 + \hat{\mathbf{k}}_\wedge\mathbf{v}_1 + \beta y\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_1 = 0 \quad (1)$$

$$F\frac{\partial \eta_0}{\partial t} + F(\mathbf{v}_0.\nabla\eta_0 + \eta_0\nabla.\mathbf{v}_0) + \nabla.\mathbf{v}_1 = 0 \quad (2) \qquad \rightarrow F\frac{\partial \eta_0}{\partial t} = -\nabla.\mathbf{v}_1$$

The second term in (2) is zero because $\mathbf{v}_0$ is perpendicular to $\nabla\eta_0$, and $\mathbf{v}_0$ is nondivergent.

The local tendency of zero order height comes from the divergence of the first order flow. Note that this divergence comes from the ageostrophic flow and the divergent part of the geostrophic flow.

Take the curl of (1) to eliminate η_1 and form the first order vorticity equation:

$$\frac{\partial \xi_0}{\partial t} + \xi_0\nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta y\nabla.\mathbf{v}_0 + \beta v_0 = 0$$

The second and fifth terms are zero (nondivergent $\mathbf{v}_0$). Combining this with (2) gives:

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \beta v_0 = -\nabla.\mathbf{v}_1 = F\frac{\partial \eta_0}{\partial t}$$

then using $\frac{\partial}{\partial t}(\beta y) = 0, \quad \mathbf{v}_0.\nabla(\beta y) = \beta v_0, \quad \mathbf{v}_0.\nabla\eta_0 = 0$ we can write

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = F\left[\frac{\partial \eta_0}{\partial t} + \mathbf{v}_0.\nabla\eta_0\right]$$

or

$$\boxed{\left[\frac{\partial}{\partial t} + \mathbf{v}_0.\nabla\right][\beta y + \xi_0 - F\eta_0] = 0}$$

Quasi-geostrophic potential vorticity again

Now, using

$$v_0 = \frac{\partial \eta_0}{\partial x}, \quad u_0 = -\frac{\partial \eta_0}{\partial y}, \qquad \xi_0 = \nabla^2 \eta_0$$

$$\rightarrow \mathbf{v}_0.\nabla = \frac{\partial \eta_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial}{\partial x} \quad \rightarrow \mathbf{v}_0.\nabla(q) = J(\eta_0, q)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_0.\nabla\right][\beta y + \xi_0 - F\eta_0] = 0$$

and we can write our prognostic equation as

$$\left[\frac{\partial}{\partial t} + \frac{\partial \eta_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial}{\partial x}\right][\beta y + \nabla^2 \eta_0 - F\eta_0] = 0$$

or

$$\frac{\partial q}{\partial t} + J(\eta_0, q) = 0$$

$$q = \beta y + \nabla^2 \eta_0 - F\eta_0$$

One equation, one variable.

q is now the non-dimensional q.g.p.v.

<u>Re-dimonsionalise q:</u>

Using previously defined scalings, working back to dimensional equations leads to

$$q = \beta y + \nabla^2 \psi - \frac{f_0}{H}\delta h$$

and if we define the quasi-geostrophic streamfunction $\psi = \frac{g}{f_0}\delta h$

we get $q = \beta y + \nabla^2 \psi - \left(\frac{f_0^2}{gH}\right)\psi$ or

$$q = \beta y + \nabla^2 \psi - \frac{1}{L_R^2}\psi$$

and

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

Continuously stratified fluid

Up until now we have worked with discrete layers, each of which is homogeneous (constant density). The extension to continuous stratification requires that we abandon this formulation and reintroduce a vertical coordinate. The expansion around small Rossby number is very similar so it is shown in appendix slides. The result is once again a conservation law for potential vorticity, which is defined entirely in terms of a streamfunction, so one equation, one variable.

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

where now, ψ is defined as $\psi = \frac{p_0}{\rho_s f_0}$

This is an anelastic fluid, which allows large variations of density with height, accounting for the static compressibility of the atmosphere. In this case the q.g.p.v. is

$$q = \beta y + \nabla^2 \psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s \frac{\partial \psi}{\partial z}\right)$$

$$p = p_s(z) + \tilde{p}(x,y,z,t)$$
$$\rho = \rho_s(z) + \tilde{\rho}(x,y,z,t)$$

Only the vortex stretching component has changed.

In a Boussinesq fluid, where ρ_s is a constant (independent of z), this simplifies to

$$q = \beta y + \nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)$$

$$p = p_s + \tilde{p}(x,y,z,t)$$
$$\rho = \rho_s + \tilde{\rho}(x,y,z,t)$$

details

IV) EXTENSION TO A CONTINUOUSLY STRATIFIED FLUID (with non-Boussinesq, static compressibility effects)

Three dimensional scalings for a compressible, baroclinic stratified fluid:

$$x, y \to L, \quad u, v \to U, \quad z \to H, \quad w \to \frac{UH}{L}, \quad t \to \frac{L}{U}$$

$$p = p_s(z) + \tilde{p}(x, y, z, t)$$
$$\rho = \rho_s(z) + \tilde{\rho}(x, y, z, t)$$

Geostrophic scaling for pressure

$$fv \sim \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x}$$

so

$$\tilde{p} \to f_0 U L \rho_s$$

Hydrostatic scaling for density

$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g$$

so

$$\tilde{\rho} \to \frac{f_0 U \rho_s L}{Hg} = \rho_s \epsilon F$$

so

$$\boxed{\rho = \rho_s(1 + \epsilon F \rho')}$$

recall

$$F = \frac{f_0^2 L^2}{gH}$$
$$\epsilon = \frac{U}{f_0 L}$$

also

$$\frac{f}{f_0} = 1 + \epsilon \beta' y'$$

where

details

$$\beta' = \frac{\beta_0 L^2}{U} = \cot \phi_0$$

as before.

Non-dimensional momentum equation:

$$\frac{\partial \mathbf{v}'}{\partial t}\frac{U^2}{L} + \mathbf{v}'.\nabla \mathbf{v}'\frac{U^2}{L} + w'\frac{\partial \mathbf{v}'}{\partial z}\frac{HU^2}{LH} + fU\hat{\mathbf{k}}_\wedge \mathbf{v}' = -\frac{1}{\rho_s(1+\epsilon F\rho')}\frac{\nabla p'}{L}ULf_0\rho_s$$

$$= -Uf_0\nabla p'(1 - \epsilon F\rho')$$

(to first order)

Divide by Uf_0, drop primes

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + \epsilon w\frac{\partial \mathbf{v}}{\partial z} + (1 + \epsilon\beta y)\hat{\mathbf{k}}_\wedge\mathbf{v} = -(1 - \epsilon F\rho)\nabla p$$

Non-dimensional continuity equation (non-Boussinesq)

$$\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla\rho + w\frac{\partial \rho}{\partial z} + \rho\nabla.\mathbf{v} + \rho\frac{\partial w}{\partial z} = 0$$

$$\rho_s\epsilon F\frac{U}{L}\frac{\partial \rho'}{\partial t'} + \rho_s\epsilon F\frac{U}{L}\mathbf{v}'.\nabla\rho' + \rho_s\epsilon F\frac{UH}{LH}w'\frac{\partial \rho'}{\partial z'}$$

$$+\frac{UH}{L}w'\left[\frac{\partial \rho_s}{\partial z}\right] + \rho_s(1 + \epsilon F\rho')\left[\frac{U}{L}\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right)\right] = 0$$

$$\times\frac{L}{\rho_s U} \to$$

$$\epsilon F\frac{\partial \rho'}{\partial t'} + \epsilon F\mathbf{v}'.\nabla\rho' + \epsilon Fw'\frac{\partial \rho'}{\partial z} + Hw'\left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z}\right] + (1 + \epsilon F\rho')\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right) = 0$$

Note that the expression in square brackets resembles N^2, and note that z is dimensionless.

$$N^2 = \frac{g}{\theta_s}\frac{\partial \theta_s}{\partial z}$$

Define

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

then the fourth term above becomes

$$\left(\frac{HS^2}{g}\right)w'$$

This is the non-Boussinesq term.

So dropping primes

$$\epsilon F\left(\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla\rho + w\frac{\partial \rho}{\partial z}\right) + \frac{HS^2}{g}w + (1 + \epsilon F\rho)\left(\nabla.\mathbf{v} + \frac{\partial w}{\partial z}\right) = 0$$

Expansion of non-dimensional variables

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + ...$$
$$w = w_0 + \epsilon w_1 + ...$$
$$\tilde{p} = p_0 + \epsilon p_1 + ...$$
$$\tilde{\rho} = \rho_0 + \epsilon\rho_1 + ...$$

Momentum equation to zero order

Geostrophic balance

$$\hat{\mathbf{k}}_\wedge\mathbf{v}_0 = -\nabla p_0$$

and

$$\nabla.\mathbf{v}_0 = 0$$

Continuity equation to zero order

$$\frac{HS^2}{g}w_0 + \nabla.\mathbf{v}_0 + \frac{\partial w_0}{\partial z} = 0$$

Therefore we can't generate w_0 in the body of the fluid by horizontal motion. At zero order, vertical motion can only be generated at the boundary.

Assume that the bottom vertical velocity

details

$$w_b = 0 + \epsilon w_{1b} + ...$$

(this is assumption (3): weak orography)
Integrate upwards, this implies

$$w_0 = 0$$

everywhere, so

$$w = \epsilon w_1 + ...$$

Momentum equation to first order

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla\mathbf{v}_0 + \hat{\mathbf{k}}_\wedge\mathbf{v}_1 + \beta y\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla p_1 - F\rho_0\nabla p_0 = 0$$

$$\hat{\mathbf{k}}.\nabla_\wedge(\text{this}) \rightarrow$$

vorticity equation:

$$\frac{\partial \xi_0}{\partial t} + \xi_0\nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta y\nabla.\mathbf{v}_0 + \beta\mathbf{v}_0 - F\left[\frac{\partial \rho_0}{\partial x}\frac{\partial p_0}{\partial y} - \frac{\partial \rho_0}{\partial y}\frac{\partial p_0}{\partial x}\right] = 0$$

Second and fifth terms disappear by nondivergence of the zero order flow, and the last term can be rewritten using geostrophy of the zero order flow to give

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta\mathbf{v}_0 + F\mathbf{v}_0.\nabla\rho_0 = 0$$

Continuity equation to first order

$$F\frac{\partial \rho_0}{\partial t} + F\mathbf{v}_0.\nabla\rho_0 + \left(\frac{HS^2}{g}\right)w_1 + \nabla.\mathbf{v}_1 + \frac{\partial w_1}{\partial z} = 0$$

Note that the second and fourth terms have just appeard in the vorticity equation. So we can eliminate them by combining the continuity and vorticity equations:

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \beta\mathbf{v}_0 = -F\mathbf{v}_0.\nabla\rho_0 - \nabla.\mathbf{v}_1$$

$$= F\frac{\partial \rho_0}{\partial t} + \left(\frac{HS^2}{g}\right)w_1 + \frac{\partial w_1}{\partial z}$$

At this stage we note that for synoptic scales $F \sim 0.1$ so we neglect the first term on the right hand side. This is because we have set

$$F = \frac{f_0^2L^2}{gH} = \frac{L^2}{R^2}$$

(remember, for the atmosphere:

$$R_{ext} \sim \frac{\sqrt{gH}}{f_0} = \frac{\sqrt{10 \times 10^4}}{10^{-4}} \sim 3 \times 10^6\,\mathrm{m} = 3000\,\mathrm{km}$$

$$F = \frac{L^2}{R^2} \sim \frac{1000^2}{3000^2} \sim 10^{-1}$$

)

So the vorticity equation is now

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = \frac{HS^2}{g}w_1 + \frac{\partial w_1}{\partial z}$$

Using

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z_*}$$

the right hand side can be written

$$= \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1)$$

We can evaluate the right hand side using the ...

Thermodynamic equation

$$\frac{D\theta}{Dt} = 0$$

scale

$$\theta = \theta_s(1 + \epsilon F(\theta_0 + ...))$$

as we did for density, so

$$\frac{U}{L}\frac{\partial \theta'}{\partial t'}\epsilon F\theta_s + \frac{U}{L}\mathbf{v}'.\nabla\theta'\epsilon F\theta_s + w'\frac{\partial \theta'}{\partial z'}\epsilon F\theta_s\frac{HU}{LH} + w\frac{\partial \theta_s}{\partial z_*}\frac{UH}{L} = 0$$

drop primes, get

$$\epsilon F\left(\frac{\partial \theta}{\partial t} + \mathbf{v}.\nabla\theta + w\frac{\partial \theta}{\partial z}\right) + w\frac{N^2H}{g} = 0$$

At zero order we recover

$$w_0 = 0$$

details

At first order:

$$F\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0.\nabla\theta_0\right) + w_1\frac{N^2H}{g} = 0$$

$$\rightarrow w_1 = -\frac{f_0^2L^2}{N^2H^2}\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0.\nabla\theta_0\right)$$

introduce

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_0.\nabla$$

so

$$w_1 = -\frac{f_0^2L^2}{N^2H^2}\left[\frac{D_0\theta_0}{Dt}\right]$$

Multiply by ρ_s, take vertical derivative and then divide by ρ_s, and exchange derivatives when possible. This gives

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1) = -\frac{D_0}{Dt}\left[\frac{f_0^2L^2}{H^2\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\theta_0}{N^2}\right)\right]$$

and we can use this to rewrite the vorticity equation as

$$\frac{D_0}{Dt}\left[\beta y + \xi_0 + \frac{f_0^2L^2}{H^2\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\theta_0}{N^2}\right)\right] = 0$$

Now we have one last thing to do...

Hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g$$

$$p = p_s + p_0\rho_s f_0 UL$$
$$\rho = \rho_s + \rho_0\rho_s\epsilon F$$
$$\rightarrow \frac{\partial}{\partial z}(p_o\rho_s) = -\rho_0\rho_s$$

or

$$\rho_0 = -\frac{1}{\rho_s}\frac{\partial}{\partial z}(p_0\rho_s)$$

Now, define

$$\theta_* = \theta_s(z)(1 + \epsilon F\theta)$$

and define

$$\theta_0 = -\rho_0 + \frac{1}{\gamma}\left(\frac{\rho_s gH}{p_s}\right)p_0$$

ASIDE: So where does this come from ?
Its needed to ensure

$$\frac{d\theta}{\theta} = \frac{1}{\gamma}\frac{dp}{p} - \frac{d\rho}{\rho}$$

(where

$$\gamma = \frac{c_p}{c_v}$$

)

PROOF:
integrate this, gives

$$\log\theta_* = \frac{1}{\gamma}\log p_* - \log\rho_* + \text{const}$$

but

$$\theta_* = \theta_s(1 + \epsilon F\theta_0)$$
$$\rho_* = \rho_s(1 + \epsilon F\rho_0)$$
$$p_* = p_s + \rho_s f_0 ULp_0 = p_s\left(1 + f_0UL\frac{\rho_s}{p_s}p_0\right)$$
$$= p_s\left(1 + \epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0\right)$$

The inner term in brackets is the reference hydrostatic scaling, ~ 1.
Substitute these expressions for ϑ, ρ* and p* into the log expression using the fact that to first order*

$$\log(1 + \epsilon x) = \epsilon x$$

$$\rightarrow \epsilon F\theta_0 = \frac{1}{\gamma}\epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0 - \epsilon F\rho_0$$

$$\rightarrow \theta_0 = \frac{1}{\gamma}\left(\frac{gH\rho_s}{p_s}\right)p_0 - \rho_0$$

END OF ASIDE

details

Substitute this into the hydrostatic relation to elimintate density

$$\rho_0 = -\theta_0 + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right) p_0 = -\frac{\partial p_0}{\partial z} - \frac{p_0}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

$$\theta_0 = \frac{\partial p_0}{\partial z} + p_0\left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right)\right]$$

From the reference hydrostatic relation, the second term in square brackets can be written

$$= -\frac{1}{\gamma}\frac{\partial p_s}{\partial z}$$

but this is just

$$\theta_0 = \frac{\partial p_0}{\partial z} - p_0\left(\frac{1}{\theta_s}\frac{\partial \theta_s}{\partial z}\right)$$

and the term in brackets

$$= \frac{N^2 H}{g} \sim \frac{g'}{g} \sim \epsilon$$

so we can write the perturbation hydrostatic relation in terms of perturbation potential temperature:

$$\boxed{\theta_0 = \frac{\partial p_0}{\partial z}}$$

... put this back into the vorticity equation:

$$\frac{D_0}{Dt}\left[\beta y + \xi_o + \frac{f_0^2 L^2}{H^2 \rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial p_0}{\partial z}\right)\right] = 0$$

This is the non-d quasi-geostrophic potential vorticity.

Redimensionalise:

$$q = \beta y + \xi_0 + \frac{f_0}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial}{\partial z}\left(\frac{p_0}{\rho_s}\right)\right)$$

introduce a dimensional geostrophic streamfunction

$$\psi = \frac{p_0}{\rho_s f_0}$$

get

$$\boxed{q = \beta y + \nabla^2\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial \psi}{\partial z}\right)}$$

This is the full quasi-geostrophic potential vorticity for a compressible stratified fluid.

Note: for stratified Boussinesq fluids this form reduces to

$$\boxed{q = \beta y + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)}$$

(this is OK for the ocean).

q is conserved following the flow:

$$\boxed{\frac{\partial q}{\partial t} + J(\psi, q) = 0}$$

Everything is represented in terms of one prognostic equation in one variable (the streamfunction).

One variable to rule them all

Since ψ is the only variable in the system, it must be possible to express anything in term of ψ, and is !

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \quad w = \frac{f}{N^2}\left[\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right]\frac{\partial \psi}{\partial z}$$

$$p' = \rho_0 f_0 \psi, \quad \rho' = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}$$

Knowledge of q plus boundary conditions leads to knowledge of ψ, and hence the advecting flow.

Prediction becomes a sequence of operations:

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = 0,$$

1) diagnose q
2) integrate the prognostic equation forward in time to find the next values of q
3) apply boundary conditions and invert the elliptic operator to find ψ
4) rinse and repeat

$$q = \beta y + \nabla^2\psi + \mathrm{fn}(\psi_z)$$

Development

We approximate the quasi-geostrophic potential vorticity equation as

$$\frac{\partial}{\partial t}\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) + \mathbf{v}.\nabla\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) = 0$$

rearrange the derivatives on the variable ψ:

$$\left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial\psi}{\partial t} = -\mathbf{v}.\nabla\left(\nabla^2\psi + f\right) - \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

Now assume ψ is a wavelike disturbance with a sign change in the vertical (first baroclinic mode)

$$\psi \propto \sin lx\ \sin my\ \cos \pi z/H$$

$$\rightarrow \left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right) = -\left(l^2 + m^2 + \frac{f^2\pi^2}{N^2H^2}\right)$$

so

$$\frac{\partial\psi}{\partial t} \propto +\mathbf{v}.\nabla(\nabla^2\psi + f) + \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

The terms on the right hand side generate the tendency in ψ. So a local rate of change of ψ, or equivalently a change of pressure or geopotential, is proportional to...

Advection of absolute vorticity

$$\frac{\partial\psi}{\partial t} \propto \mathbf{v}.\nabla(\nabla^2\psi + f)$$

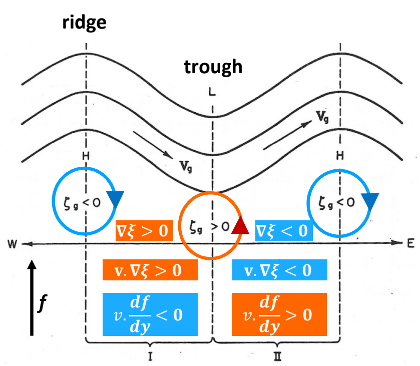
We see from the picture that zonal advection of relative vorticity sends troughs and ridges east. Meridional advection of planetary vorticity sends troughs and ridges west. Which process wins ?

$$\nabla^2\psi = -(l^2 + m^2)\psi$$

Long waves go west, f dominates (Rossby waves).
Short waves go east, ξ dominates

For short waves $+\mathbf{v}.\nabla\xi$ positive $\Rightarrow \frac{\partial\psi}{\partial t}$ positive

so a ridge in region I propagates east. But the tendency is zero at the axes of the ridges and troughs, so no amplification.



Vertical gradient of temperature advection

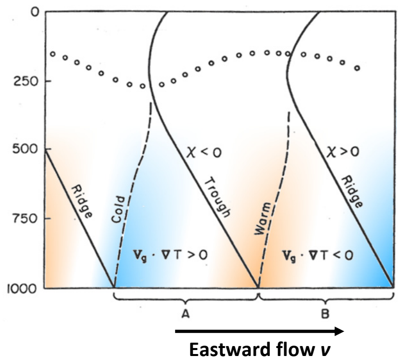
$$\frac{\partial \psi}{\partial t} \propto \frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial \psi}{\partial z}\right) \propto \frac{\partial}{\partial z}(\mathbf{v}.\nabla\theta) \quad \left[\theta_0 = \frac{\partial p_0}{\partial z}\right]$$

This is sometimes called the "differential thickness advection"

(ever noticed how synopticians love talking in multiple derivatives ?)

If we have warm advection at low levels then this term is positive and a ridge is created.

If we have cold advection at low levels this term is negative and a trough is created.



Vertical velocity

The quasi-geostrophic system allows us to do a more accurate diagnosis than we can do with 3-d nondivergence which suffers from large cancellation.

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

The first order Boussinesq thermodynamic equation yields

$$w = -B_u\left(\frac{\partial}{\partial t}\frac{\partial p_o}{\partial z} + \mathbf{v}_0.\nabla\frac{\partial p_0}{\partial z}\right)$$

Compare the Laplacian of this equation with the vertical derivative of the vorticity equation

$$\nabla^2\left(\frac{\partial}{\partial z}\frac{\partial p_o}{\partial t}\right) = -\nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_0}{\partial z}\right) - B_u^{-1}\nabla^2 w_1$$

(using $\xi_0 = \nabla^2 p_0$)

$$\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial t}(f+\xi_0) + \mathbf{v}_0.\nabla(f+\xi_0) = \frac{\partial w_1}{\partial z}\right\} \rightarrow \nabla^2\left(\frac{\partial}{\partial z}\frac{\partial p_0}{\partial t}\right) = \frac{\partial^2 w_1}{\partial z^2} - \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f+\xi_0))$$

equate right hand sides

$$\left(B_u^{-1}\nabla^2 + \frac{\partial^2}{\partial z^2}\right)w_1 = \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f+\xi_0)) - \nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_o}{\partial z}\right)$$

Note that this time we have eliminated the tendency term (rather than the vertical velocity term) between the vorticity and thermodynamic equations and obtained a diagnostic equation for w (rather than a prognostic equation for ψ). It's an elliptic equation for vertical velocity in terms of the geostrophic streamfunction. It's often called the Omega Equation (usually derived in pressure coordinates).

Application of the omega equation

$$\left(Bu^{-1}\nabla^2 + \frac{\partial^2}{\partial z^2}\right) w_1 = \frac{\partial}{\partial z}\left(\mathbf{v}_0.\nabla(f+\xi_0)\right) - \nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_0}{\partial z}\right)$$

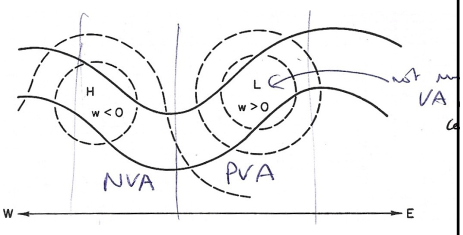
Again we assume a wavelike disturbance with simple baroclinic vertical structure.

$$\rightarrow (\text{elliptic operator}) w \propto -w$$

First term

$$w \propto -\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla(f+\xi)\right)$$

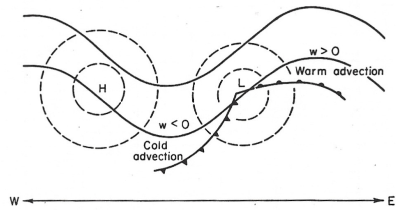
w is proportional to the rate of increase with height of positive vorticity advection. Over the low, we have vorticity increasing, so the tendency of ψ is negative, so the geopotential surface is falling (reducing thickness) so we have adiabatic cooling and thus upward motion.



Second term

$$w \propto -\mathbf{v}.\nabla\theta$$

w is proportional to warm advection. So we have upward motion wherever we have ∂ (PVA)/∂z or warm advection (positive temperature advection). Warm advection increases thickness, so this increases ψ, decreases ξ, which implies divergence, so mass continuity implies downward motion.



Recap

Tendency equation: $\left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial \psi}{\partial t} = -\mathbf{v}.\nabla\left(\nabla^2\psi + f\right) - \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial \psi}{\partial z}\right)$

Geopotential (fall/rise) proportional to:

A) (+/-) vorticity advection
B) rate of decrease with height of (cold/warm) advection.

Omega equation: $\left(B_u^{-1}\nabla^2 + \frac{\partial^2}{\partial z^2}\right) w_1 = \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f+\xi_0)) - \nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_o}{\partial z}\right)$

(rising/sinking) motion proportional to:

A) rate of increase with height of (+/-) vorticity advection
B) (warm/cold) advection