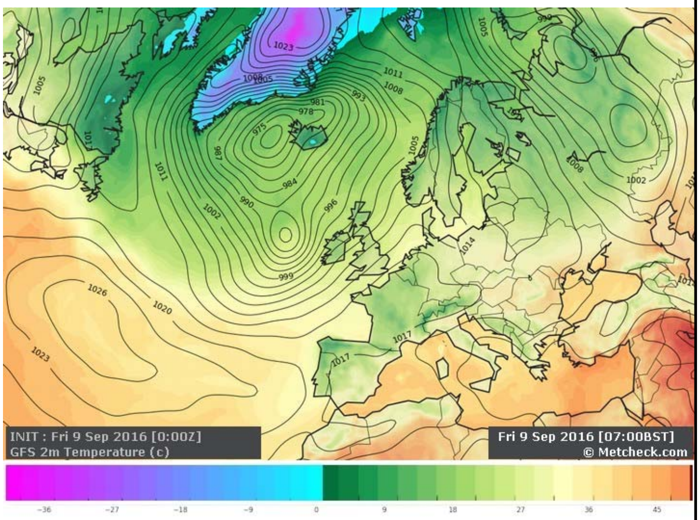
Chapter 1: Shallow water and vorticity

Some concepts to discuss
rotation, stratification,
Development, balance, nonlinearity,
homogeneous-boussinesq-anelastic,
barotropic-baroclinic,
stationary-transient

The variables we use
wind/current, pressure, density...
layer thickness, vorticity, divergence,
streamfunction, velocity potential

The shallow water equations
coordinate transformation
reduced gravity,
external and internal modes

Circulation and vorticity
the circulation theorem
the vorticity equation
potential vorticity



The momentum equations

The familiar (!) equations for x and y momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

development   nonlinear   rotation   balance

$$\frac{\partial p}{\partial z} = -\rho g$$ *hydrostatic*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ *continuity*

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = k\frac{\partial^2 \rho}{\partial z^2}$$ *density (thermodynamic)*

notes:

Strictly speaking, if the flow is balanced then it will not develop. Not much use for prediction.

The nonlinear terms are the advection terms. A linear system can still develop through wave solutions, and can still transport perturbation properties with the flow.

We have five variables: three wind components, density and pressure.

There are five equations if we add hydrostatic balance, continuity and a thermodynamic/density equation. (if we add temperature as a variable then it is linked to density and pressure by the equation of state).

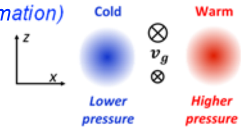
For an incompressible hydrostatic fluid, only the momentum and thermodynamic/density equations are prognostic. Continuity and hydrostatic balance are diagnostic.

Geostrophic hydrostatic flow

Geostrophic flow is possible with no density variations (homogeneous). But when density varies, things get interesting:

Geostrophic - hydrostatic flow -> Thermal Wind Balance (with the Boussinesq approximation)

$$\frac{\partial p}{\partial x} = \rho_0 f v_g, \quad \frac{\partial p}{\partial z} = -\rho g, \quad \frac{\partial^2 p}{\partial x \partial z} = \rho_0 f \frac{\partial v_g}{\partial z} = -g \frac{\partial \rho}{\partial x}$$



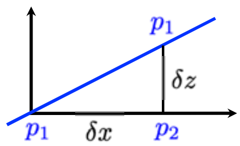
So horizontal gradients of density (or temperature) are related to vertical gradients of geostrophic flow.

And now in pressure coordinates without the approximation on density:

$$\left.\frac{\partial p}{\partial x}\right|_z = -\frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_p = \rho f v_g, \quad \frac{\partial p}{\partial z} = -\rho g$$

$$\left.\frac{\partial z}{\partial x}\right|_p = \frac{f v_g}{g}, \quad \frac{\partial z}{\partial p} = -\frac{1}{\rho g}$$

$$\frac{\partial^2 z}{\partial x \partial p} = \frac{f}{g}\frac{\partial v_g}{\partial p} = -\frac{1}{g}\left.\frac{\partial}{\partial x}\right|_p \left(\frac{1}{\rho}\right)$$



So without approximation we can say that vertical (pressure) gradients of the geostrophic wind depend on horizontal gradients of density.
This analysis also reveals that if density is a function of pressure, then the geostrophic wind must be vertically uniform -> "Barotropic flow".

Density and its variations

The way in which density varies can have important consequences for the flow. Here are the definitions for various levels of approximation:

Homogeneous: $\rho = \rho_0 \quad (\rho' = 0), \quad p = p_0(z) + p'(x,y,t) \quad \Rightarrow \frac{\partial \mathbf{v}}{\partial z} = 0$

Boussinesq: $\rho = \rho_0 + \rho'(x,y,z,t), \quad p = p_0(z) + p'(x,y,z,t)$

Anelastic: $\rho = \rho_0(z) + \rho'(x,y,z,t)$

Barotropic: $\rho = \rho(p) \quad \Rightarrow \quad \frac{\partial \mathbf{v}_g}{\partial z} = 0$

Baroclinic: $\rho \neq \rho(p)$

Later we will use the *shallow water* model.
This represents a Boussinesq fluid with a set of homogeneous layers. Density is piecewise constant. Pressure varies continuously in the vertical and in the horizontal (but horizonal gradients of pressure will be piecewise constant).

Barotropic and baroclinic flow

We have seen that in some circumstances the flow is vertically coherent. Depth independent flow is associated with the “barotropic” component, also referred to as the “external” mode (more on this later). Bartotropic flow can exhibit many phenomena: vortices, Rossby waves, jets and instability. It is a good starting point for theories of the large scale ocean circulation.

When density surfaces cross pressure surfaces the flow is “baroclinic”. The baroclinic component is associated with horizontal temperature gradients: fronts and developing cyclones; ocean eddies on the thermocline. Baroclinic processes are necessary to liberate potential energy and generate *circulation*. Baroclinic instability occurs on a preferred scale (the Rossby radius) and is important for generating geostrophic turbulence.

Stationary and transient flow

Stationary waves $\phi = [\phi] + \phi^*, \quad [vT] = [v][T] + [v^*T^*]$

This is the departure from the zonal mean. The flux produced by any flow structure, the time mean for example, can be decomposed into components effected by the zonal mean (Hadley, Ferrel cells) and by the stationary waves.

Transient eddies $\phi = \overline{\phi} + \phi', \quad \overline{vT} = \overline{v}\overline{T} + \overline{v'T'}$

This is the departure from the time mean. The flux due to time variations is an important part of the mean flux.

Transient eddy “forcing”

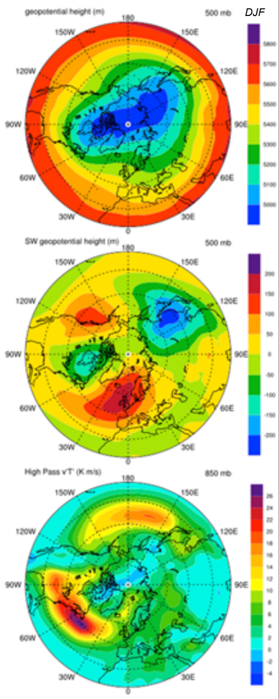
Consider the maintenance of the time mean flow:

$$\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T = \mathcal{F} - \mathcal{D}, \quad \overline{\mathbf{v}}.\nabla\overline{T} = -\overline{\mathbf{v}'.\nabla T'} + \overline{\mathcal{F}} - \overline{\mathcal{D}}$$

The mean effect of transient eddies is sometimes viewed as a forcing term the contributes to to the maintenance of the time-mean state.

These mechanisms depend on nonlinear terms, and may lead to nonlinear behaviour on longer timescales (but not necessarily).

Nonlinearity often manifests as asymmetry



Some scaling parameters

The importance of rotation: the Rossby number

Compare the advection term with the Coriolis force $u\frac{\partial u}{\partial x}\Big/ fu \rightarrow R_o = \frac{U}{fL}$

When the Rossby number is small the flow is close to geostrophic

The importance of stratification: the Froude number

For nonrotating steady flow $u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ *compare vertical divergence with horizontal divergence*

$$\frac{\partial w}{\partial z}\Big/\frac{\partial u}{\partial x} \rightarrow F_r = \frac{U}{NH} = \frac{U}{\sqrt{g'H}}$$

The Froude number is the ratio of flow speed to internal gravity wave speed.

When the Froude number is small stratification is important, vertical excursions of the flow are limited.

Rotation vs Stratification: the Burger number $(B_u)^{\frac{1}{2}} = \frac{R_o}{F_r} = \frac{NH}{fL} = \frac{\sqrt{g'H}}{fL}$

When the Burger number is ~1, vorticity advection balances vortex stretching.
This occurs at a special spatial scale...

The Rossby radius: $L_R = \frac{NH}{f} = \frac{\sqrt{g'H}}{f}$

Recall
$$N^2 = \frac{g}{\rho}\frac{\partial \rho}{\partial z} = \frac{g'}{H}$$
$$(g' = g\Delta\rho/\rho)$$

Equation sets and variables

The primitive equations u, v, w, p, ρ
essentially five variables, three prognostic equations and two diagnostic equations

The shallow water equations u, v, h
three variables, three prognostic equations

The quasi-geostrophic equations ψ, q
one variable, one prognostic equation, one definition

Streamfunction and velocity potential (revision)

The vector horizontal velocity can be written as two scalars $\mathbf{v} = (u, v) = -\nabla\phi + \hat{\mathbf{k}}_\wedge\nabla\psi$

$$\Rightarrow u = -\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}$$

Φ is the velocity potential. Divergent flow emanates from maxima of Φ.
ψ is the streamfunction. Nondivergent flow circulates clockwise round maxima of ψ.

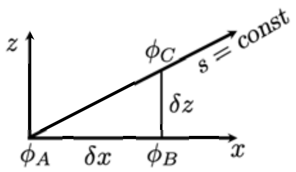
If the flow is either nondivergent or irrotational we can economise one variable.
Quasi-geostrophic flow is nondivergent so we only need ψ.

Furthermore, divergence, $D = \nabla.\mathbf{v} = -\nabla^2\phi$ *and relative vorticity,* $\xi = \hat{\mathbf{k}}.\nabla_\wedge\mathbf{v} = \nabla^2\psi$

Alternative vertical coordinates

We can simplify the equations if we use a conserved quantity as the vertical coordinate. In this frame of reference there is no "vertical velocity", rendering the system two-dimensional. So we can reduce our equation set by using coordinate systems based on density in the ocean or potential temperature in the atmosphere. But the price we pay for this simplification is to complicate the boundary conditions: coordinate surfaces outcrop, they move in time, and our coordinates are no longer orthogonal.

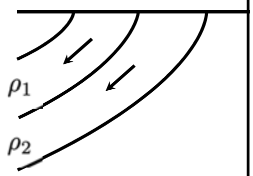
General coordinate transformation:



$$\frac{\phi_C - \phi_A}{\delta x} = \frac{\phi_C - \phi_B}{\delta z}\left(\frac{\delta z}{\delta x}\right) + \frac{\phi_B - \phi_A}{\delta x}$$

$$\delta x,\ \delta z \to 0$$

$$\left.\frac{\partial \phi}{\partial x}\right|_s = \frac{\partial \phi}{\partial z}\left(\left.\frac{\partial z}{\partial x}\right|_s\right) + \left.\frac{\partial \phi}{\partial x}\right|_z$$

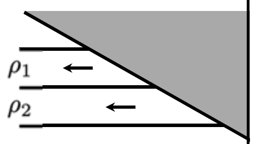


$$\Rightarrow \left.\frac{\partial \phi}{\partial x}\right|_z = \left.\frac{\partial \phi}{\partial x}\right|_s - \frac{\partial \phi}{\partial z}\left.\frac{\partial z}{\partial x}\right|_s \quad [1]$$

rule 1

$$\frac{\partial \phi}{\partial z} = \frac{\partial s}{\partial z}\frac{\partial \phi}{\partial s} \quad [2]$$

rule 2



Density coordinates

Let's transform the primitive equations to density coordinates for isopycnal flow in a Boussinesq fluid:

Hydrostatic equation: $\frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad \frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$ *(rule 2)*

Define "Montgomery potential" as $P = p + \rho g z, \quad \Rightarrow \quad \frac{\partial P}{\partial \rho} = gz$

Momentum equations: $\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\left.\frac{\partial p}{\partial x}\right|_z = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$ *(rule 1)*

and if ρ is conserved, no equivalent of vertical velocity so

$$\frac{\partial u}{\partial t} + u\left.\frac{\partial u}{\partial x}\right|_\rho + v\left.\frac{\partial u}{\partial y}\right|_\rho - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\left.\frac{\partial v}{\partial x}\right|_\rho + v\left.\frac{\partial v}{\partial y}\right|_\rho + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

Continuity: $\left.\frac{\partial u}{\partial x}\right|_z + \left.\frac{\partial v}{\partial y}\right|_z + \frac{\partial w}{\partial z} = 0$

apply rules 1 and 2 and after some manipulation:

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial x}\right|_\rho\left(u\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial y}\right|_\rho\left(v\frac{\partial z}{\partial \rho}\right) = 0$$

This is mass conservation expressed in terms of a flux of layer thickness. The final step to a layer model is to discretize: $\frac{\partial z}{\partial \rho} = \frac{h}{\Delta \rho}$

details

1) Hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g$$

$$[2] \Rightarrow \frac{\partial p}{\partial z} = \frac{\partial \rho}{\partial z}\frac{\partial p}{\partial \rho} = -\rho g$$

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$$

Define

$$P = p + \rho g z$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + gz + \rho g \frac{\partial z}{\partial \rho}$$

$$\frac{\partial P}{\partial \rho} = gz$$

Hydrostatic equation in terms of "Montgomery potential" *P*.

2) Thermodynamic equation (density equation)

in *any* coordinate system (Boussinesq fluid - incompressible)

$$\frac{D\rho}{Dt} = 0$$

in *z* coordinates

$$\frac{\partial \rho}{\partial t} + u \left.\frac{\partial \rho}{\partial x}\right|_z + v \left.\frac{\partial \rho}{\partial y}\right|_z + w \frac{\partial \rho}{\partial z} = 0$$

i.e.

$$\frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} + \frac{dy}{dt}\frac{\partial \rho}{\partial y} + \frac{dz}{dt}\frac{\partial \rho}{\partial z} = 0$$

On a surface of constant ρ, *z* varies. To make *z* the variable and ρ the coordinate, we rewrite this equation swapping the variables:

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + \frac{d\rho}{dt}\frac{\partial z}{\partial \rho} = w$$

and the last term is zero because ρ is conserved. This gives us an equation for *w*.

3) *x* momentum equation

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\left.\frac{\partial p}{\partial x}\right|_z$$

$$[1] \Rightarrow = -\frac{1}{\rho_0}\left[\left.\frac{\partial p}{\partial x}\right|_\rho - \frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\rho\right]$$

$$= -\frac{1}{\rho_0}\left.\frac{\partial}{\partial x}\right|_\rho (p + \rho g z) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

since

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left.\frac{\partial}{\partial x}\right|_\rho + v\left.\frac{\partial}{\partial y}\right|_\rho$$

(no vertical term because ρ is conserved)

$$\rightarrow \frac{\partial u}{\partial t} + u\left.\frac{\partial u}{\partial x}\right|_\rho + v\left.\frac{\partial u}{\partial y}\right|_\rho - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

likewise

$$\frac{\partial v}{\partial t} + u\left.\frac{\partial v}{\partial x}\right|_\rho + v\left.\frac{\partial v}{\partial y}\right|_\rho + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

4) continuity equation

z coordinates:

$$\left.\frac{\partial u}{\partial x}\right|_z + \left.\frac{\partial v}{\partial y}\right|_z + \frac{\partial w}{\partial z} = 0$$

(ignore *dv/dy* term for the moment)

$$[1] \text{ and } [2] \Rightarrow \left.\frac{\partial u}{\partial x}\right|_\rho - \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\rho + \frac{\partial w}{\partial \rho}\frac{\partial \rho}{\partial z} = 0$$

$$\text{multiply by } \frac{\partial z}{\partial \rho}$$

$$\frac{\partial z}{\partial \rho}\left.\frac{\partial u}{\partial x}\right|_\rho - \frac{\partial u}{\partial \rho}\left.\frac{\partial z}{\partial x}\right|_\rho + \frac{\partial w}{\partial \rho} = 0$$

details

the last term can be expanded

$$\frac{\partial w}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{Dz}{Dt}\right) = \frac{\partial}{\partial \rho}\left[\frac{\partial z}{\partial t} + u\left.\frac{\partial z}{\partial x}\right|_\rho\right]$$

$$= \frac{\partial^2 z}{\partial \rho \partial t} + \frac{\partial u}{\partial \rho}\left.\frac{\partial z}{\partial x}\right|_\rho + u\frac{\partial^2 z}{\partial \rho \partial x}$$

which leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \frac{\partial z}{\partial \rho}\left.\frac{\partial u}{\partial x}\right|_\rho + u\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \rho}\right) = 0$$

Putting the *y* term back in gives

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial x}\right|_\rho\left(u\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial y}\right|_\rho\left(v\frac{\partial z}{\partial \rho}\right) = 0$$

This is the flux form of the continuity equation. The tendency of *dz/dp* is given in terms of its flux along density surfaces. *dz/dp* is a continuous form but this can be identified with mass conservation in terms of a flux of layer thickness

$$\frac{\partial z}{\partial \rho} = \frac{h}{\Delta \rho}$$

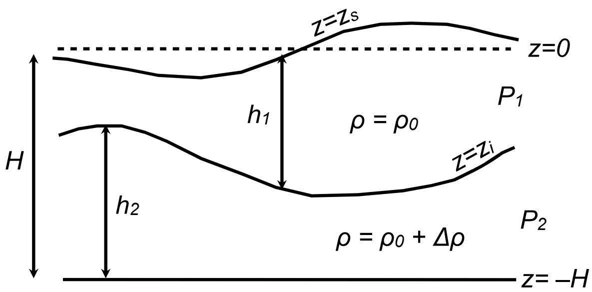
Shallow water layers

Apply the hydrostatic equation across the layer interface z_i (ignoring atmospheric pressure gradients)

$$P_1 = \rho_0 g(-H + h_2 + h_1) \ \{+p_a\}$$

$$\frac{\partial P}{\partial \rho} = \frac{\Delta P}{\Delta \rho} = \frac{P_2 - P_1}{\Delta \rho} = g z_i = g(-H + h_2)$$

$$P_2 - P_1 = \Delta \rho g(-H + h_2)$$



Horizontal gradients of P take the following forms (where $D = h_1 + h_2$)

$$\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} = g\frac{\partial D}{\partial x}$$

$$\frac{1}{\rho_0}\frac{\partial P_2}{\partial x} = g\frac{\partial D}{\partial x} + g'\frac{\partial h_2}{\partial x}$$

and in general, for N layers

$$\frac{1}{\rho_0}\frac{\partial \mathbf{P}}{\partial x} = g\frac{\partial D}{\partial x} + g'\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}$$

For two layers

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For the general N-layer case

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \\ & & & N-1 \end{pmatrix}$$

*The first term on the right is the "external mode", associated with fast surface waves. The terms involving the matrix **C** are the "internal modes" associated with slow waves on the layer interfaces. We have a set of linear expressions for the horizontal pressure gradient that we can decouple by finding the eigenvectors of **C**.*

The shallow water equations

Now we have expressions for the Montgomery potential we can eliminate it, discretize the stratification and write the equation set in terms of *u*,*v* and *h*: first for two layers, *i*=*1,2*

$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} + v_i\frac{\partial u_i}{\partial y} - f v_i = -g\frac{\partial}{\partial x}(h_1 + h_2) - g'\frac{\partial h_2}{\partial x}$$

$$\frac{\partial v_i}{\partial t} + u_i\frac{\partial v_i}{\partial x} + v_i\frac{\partial v_i}{\partial y} + f u_i = -g\frac{\partial}{\partial y}(h_1 + h_2) - g'\frac{\partial h_2}{\partial y}$$

this term just for i=2

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x}(u_i h_i) + \frac{\partial}{\partial y}(v_i h_i) = 0$$

And for *N* layers the momentum equations are

$$\frac{Du_i}{Dt} - f v_i = -g\frac{\partial D}{\partial x} - g'\left[\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}\right]_i$$

*where **h** is the column vector (h_1, h_2, ...)*

$$\frac{Dv_i}{Dt} + f u_i = -g\frac{\partial D}{\partial y} - g'\left[\mathbf{C}\frac{\partial \mathbf{h}}{\partial y}\right]_i$$

The thermocline and the abyss

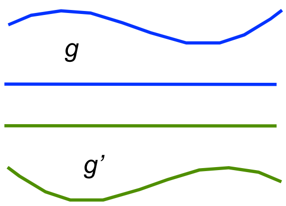
Instead of having a free surface and a flat bottom, we can reconfigure to have a rigid lid and a motionless abyss. This is sometimes called a 1$^1/_2$ layer model.

The equations are the same except we replace g with g'

$$\frac{Du}{Dt} - fv = -g^{(\prime)}\frac{\partial h}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -g^{(\prime)}\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$



With a rigid lid we lose the external mode. In the general case (*N* layers) the *x*-momentum equation becomes

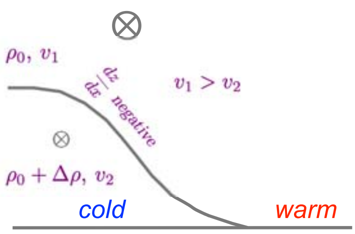
$$\frac{Du_i}{Dt} - fv_i = -g'\left[\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}\right]_i \qquad \mathbf{C} = \begin{pmatrix} N & N-1 & N-2 & \dots & 1 \\ N-1 & N-1 & N-2 & \dots & 1 \\ N-2 & N-2 & N-2 & \dots & 1 \\ \vdots & \vdots & \vdots & & \\ 1 & 1 & 1 & & 1 \end{pmatrix}$$

*Note that **C** has been flipped, and stripped of its zeros. One extra internal mode replaces the external mode associated with the free surface in the previous system. All the gravity waves are slow.*

Thermal wind revisited

Geostrophic hydrostatic balance is quite elegant in density coordinates

$$\frac{\partial^2 P}{\partial x \partial \rho} = \rho_0 f \frac{\partial v}{\partial \rho} = g \left.\frac{\partial z}{\partial x}\right|_\rho$$



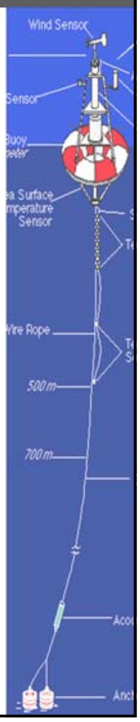
Application: fronts in the atmosphere

$$\frac{\partial v}{\partial \rho} = \frac{g}{\rho_0 f}\left.\frac{\partial z}{\partial x}\right|_\rho, \qquad v_1 - v_2 = -\frac{g'}{f}\left.\frac{\partial z}{\partial x}\right|_\rho$$

"Margules relation", southerlies increase with height

Application: currents in the ocean
When you're floating on a free surface it's impossible to measure pressure independently of depth. Since the density of water is 1000x the density of air, pressure surfaces are almost flat, making it very difficult to measure horizontal gradients. You have to make do with temperature and salinity (and thence density).
Measure vertical profiles at two points to see how the position of the thermocline varies horizontally.
The slope of the thermocline gives you the difference in current across it.
Sometimes oceanographers call this the "geostrophic current".
This assumes that the abyssal flow is weak, or that there is a "level of no motion".

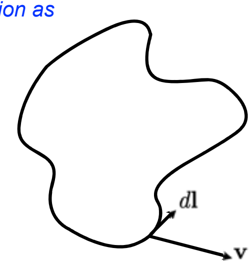
Final note: Thermal wind balance is nothing more than the horizontal component of the ***vorticity equation…***



Circulation and vorticity

Circulation *is the fluid equivalent of angular momentum. It is defined over a region as*

$$C = \oint \mathbf{v}.dl$$



Taking the time derivative gives $$\frac{dC}{dt} = \oint \frac{d\mathbf{v}}{dt}.d\mathbf{l} + \cancel{\frac{1}{2}\oint d(\mathbf{v}.\mathbf{v})}$$

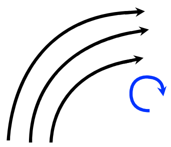
and if $\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p$ *then* $\frac{dC}{dt} = -\oint \frac{dp}{\rho}$

So over a fixed region, circulation can only be generated by baroclinic processes $\rho \neq \rho(p)$

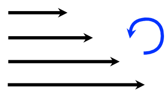
Vorticity *is the point quantity of which circulation is the integral*

$$C = \oint \mathbf{v}.d\mathbf{l} = \iint_A (\nabla_\wedge \mathbf{v}) dA$$

$$\nabla_\wedge \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi = \nabla^2 \psi$$



curvature



shear

Positive vorticity is anticlockwise

For solid body rotation $C = 2\pi\Omega r^2, \quad \xi = 2\Omega$

The vorticity equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = \mathcal{F}\left[\frac{\partial \mathbf{h}}{\partial x}\right] + \tau_x - \mathcal{D}_x \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = \mathcal{F}\left[\frac{\partial \mathbf{h}}{\partial y}\right] + \tau_y - \mathcal{D}_y \quad (2)$$

τ and $\mathcal{D}$ are sources and sinks of momentum

d/dx(2) - d/dy(1) =>

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left[f + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = \nabla_\wedge(\tau - \mathcal{D})$$

Sverdrup balance

or to put it another way

$$\left[\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right](f + \xi) = -(f + \xi)\nabla.\mathbf{v} + \nabla_\wedge(\tau - \mathcal{D})$$

This is the barotropic vorticity equation.

If the flow is nondivergent it works in individual layers, and absolute vorticity is conserved.

Vorticity can be generated and dissipated by mechanical stress at the boundaries (this is the basis of ocean circulation theory).

Vorticity can be generated by divergence, and divergence is associated with vertical displacement of layer boundaries - otherwise known as "vortex stretching" - which leads to coupling between the layers.

Vorticity can also be generated by "solenoidal" processes, which we have neglected in our Boussinesq fluid (cf circulation theorem)

$$\left(\frac{1}{\rho_0}\frac{\partial p}{\partial x} \rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x}\right) \Longrightarrow \frac{1}{\rho^2}J(\rho, p)$$

details

The vorticity equation

Effectively take the curl of the momentum equation. We'll do it by components:

$$-\frac{\partial}{\partial y} \quad : \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$+\frac{\partial}{\partial x} \quad : \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Rearrange the left hand side assuming we can swap the order of derivatives where necessary (smooth functions):

$$\begin{aligned}
\rightarrow \frac{\partial}{\partial t}\left[-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] & - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) & + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) \\
+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) & - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - w\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) \\
& + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - w\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right) \\
& + \frac{\partial f}{\partial y} + f\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] = RHS
\end{aligned}$$

Terms that cancel have been highlighted, along with necessary swapping of coordinates in the derivatives. This leads to

$$\frac{\partial \xi}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\xi) + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + w\frac{\partial \xi}{\partial z} + v\frac{\partial f}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} = RHS$$

we now evaluate the right hand side

$$\begin{aligned}
RHS = & \frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x} \\
& - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial y}
\end{aligned}$$

$$= \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

Rearranging a bit:

$$\frac{\partial}{\partial t}(f+\xi) + u\frac{\partial}{\partial x}(f+\xi) + v\frac{\partial}{\partial y}(f+\xi) + w\frac{\partial}{\partial z}(f+\xi)$$

$$= -(f+\xi)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

In vector form we take

$$\nabla_\wedge(\text{momentum equation})$$

which gives

$$\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v} - \hat{\mathbf{k}}.\nabla w_\wedge \frac{\partial \mathbf{v}}{\partial z} + \frac{\hat{\mathbf{k}}}{\rho^2}.\nabla\rho_\wedge\nabla p$$

The term on the left is the material tendency of absolute vorticity.
The first term on the right is the divergence (or vortex stretching) term.
The second term on the right is the tilting / twisting term.
The third term on the right is the baroclinic "solenoidal" term.

Associated phenomena

Advection / conservation of absolute vorticity: Planetary waves, large scale ocean circulation.
Divergence term: flow over mountains, ocean topography, tropical atmospheric circulation.
Tilting / twisting term: flow with large vertical motion, convective storms, fronts.
Solenoidal term: flow resulting from local differential heating, sea breeze circulations.

Recall barotropic flow

$$\frac{\partial}{\partial z}(\mathbf{v}, p) = 0$$

leads to

$$\boxed{\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v}}$$

the "Barotropic vorticity equation"

Generation of vorticity by divergence

Let's transform continuity equation from flux form to material tendency form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

So $\frac{Dh}{Dt} = -h\nabla.\mathbf{v}$ *looks remarkably similar to* $\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v}$

Clearly the layer thickness tendency, through mass conservation, is generated by the divergent flow.
Similarly, the tendency of absolute vorticity is generated by the divergent flow. If we eliminate the divergence:

$$-\nabla.\mathbf{v} = \frac{1}{h}\frac{Dh}{Dt} = \frac{1}{(f+\xi)}\frac{D}{Dt}(f+\xi)$$

$$\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = (f+\xi)\frac{D}{Dt}\left(\frac{1}{h}\right) + \frac{1}{h}\frac{D}{Dt}(f+\xi)$$

$$= -\frac{(f+\xi)}{h^2}\left(\frac{Dh}{Dt}\right) + \frac{1}{h}\frac{D}{Dt}(f+\xi) = 0$$

we get a new conservation law

$$\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = 0$$

This is the "potential vorticity"

In this form, potential vorticity is conserved on density layers.

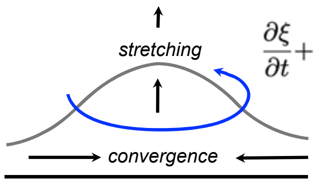
More generally, PV is the ratio of the absolute vorticity to the stratification, $(f+\xi)\frac{\partial\theta}{\partial p}$ *and it is conserved on isentropic surfaces (constant potential temperature).*

It's also a very compact convenient way to express the dynamics

Potential vorticity conservation

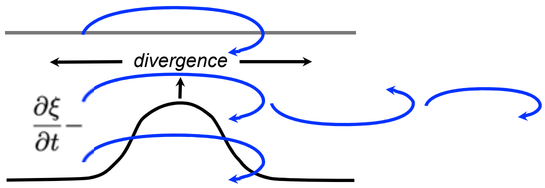
To conserve PV, changes in h are are compensated by changes in either f or ξ. This is another way to understand the link between divergent flow, mass conservation, vortex stretching and the generation of rotational flow.

Example: Cold air mass



$$\frac{f+\xi}{h}, \quad h+ \implies \xi+$$

Example: Mountains, Taylor columns and Rossby waves



$\frac{f}{h}$ *constant => flow does not cross h contours*

$f = f(y)$ *=> Rossby waves*

Conservation laws and potential quantities

The name "potential" vorticity gives a clue as to why it is conserved.

This is the relative vorticity the fluid parcel **would have** if stretched to the mean layer thickness and brought to the equator.

As such, it is like an *address label* that we attach to a parcel of fluid. The label refers to the state a parcel would have in reference conditions.

The vorticity of the fluid might change as it shifts latitude or stratification, but this label is a constant reference.

The same principle applies to potential temperature: it's the temperature a parcel would have if brought adiabatically to 1000mb.