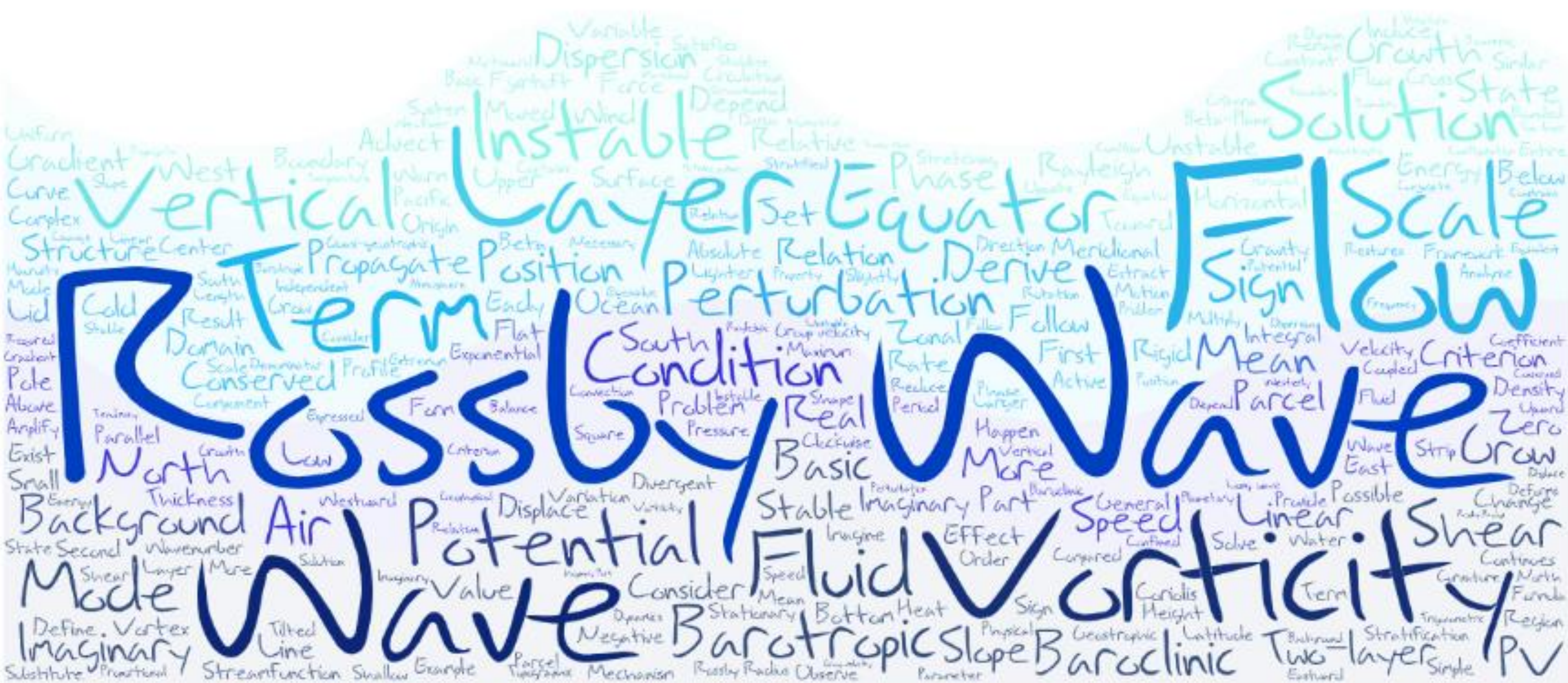
Rossby wave and instability



Instability and Rayleigh criterion

Question 3

A zonal jet has a profile between latitudes y = -L and y = L given by

$$u = U\cos\left(\frac{\pi y}{L}\right)$$

1) Show that for a planet that has no north-south surface curvature, this profile satisfies the **Rayleigh criterion** for barotropic instability.

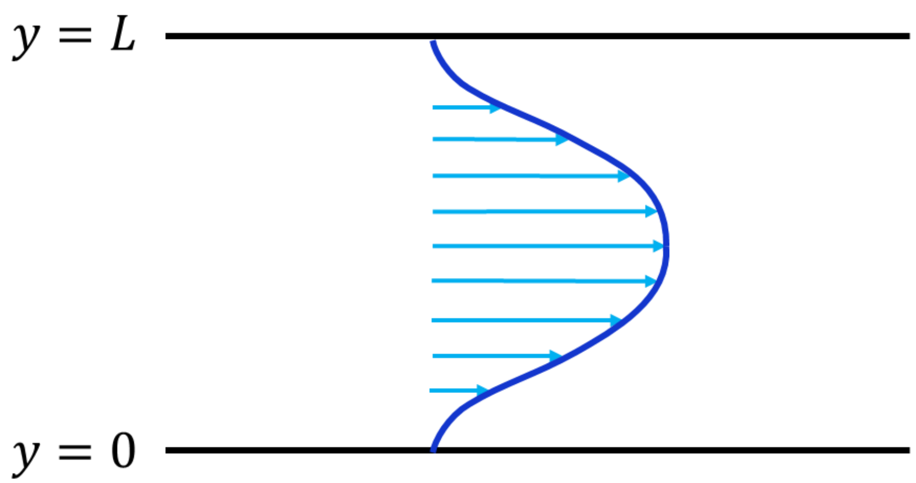
2) For a planet that does have north-south curvature, for what value of β is this profile stabilized ?

3) Show that for a planet that has no north-south surface curvature, this profile satisfies the **Fjørtoft criterion** for barotropic instability.

Conditions for growth

$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
➯ If the Rayleigh criterion is satisfied, we might have an instability.

$\forall u_0, (\bar{u} - u_0)(\beta - \bar{u}_{yy})$ must be positive somewhere in the domain.
➯ If the Fjørtoft criterion is satisfied, we might have an instability.



✋ Both Rayleigh and Fjørtoft criteria are just ***necessary conditions***. They are not sufficient conditions. This means that, when analyzing a potential vorticity map, if one of these conditions is satisfied, it does not mean that the flow is unstable, it means that ***it is possible for the flow to be unstable***.

On the other hand, the non-satisfaction of a necessary condition is a sufficient condition, which means that ***if the Rayleigh or the Fjørtoft condition is not satisfied then the flow is stable***.

Instability and Rayleigh criterion

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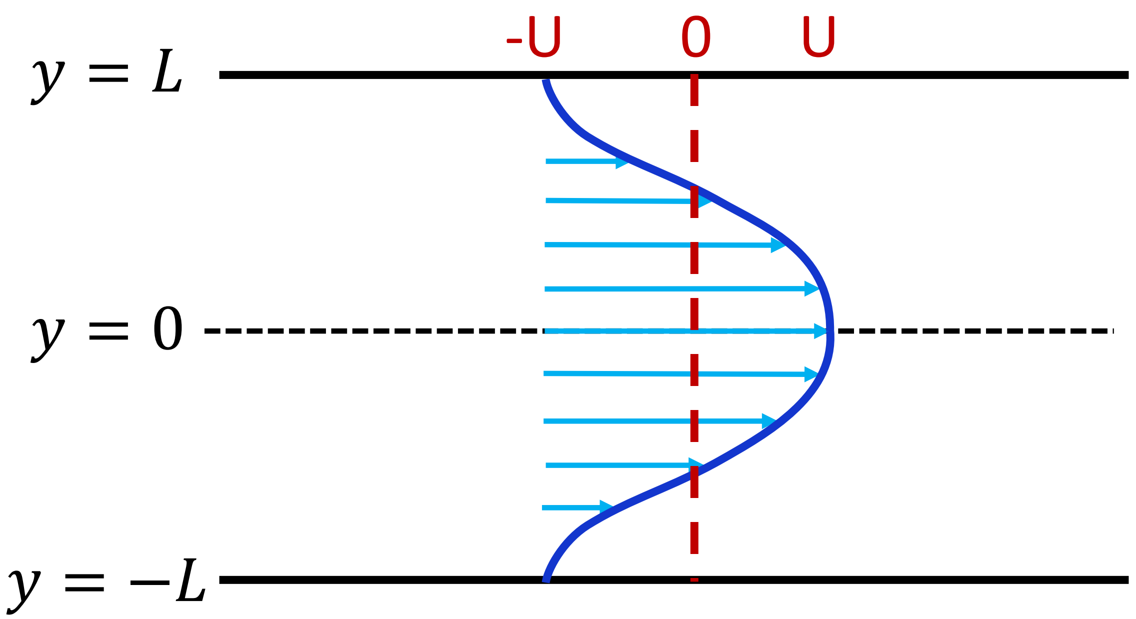

$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (-L and L).

↳ If the Rayleigh criterion is satisfied, we might have an instability.

Instability and Rayleigh criterion

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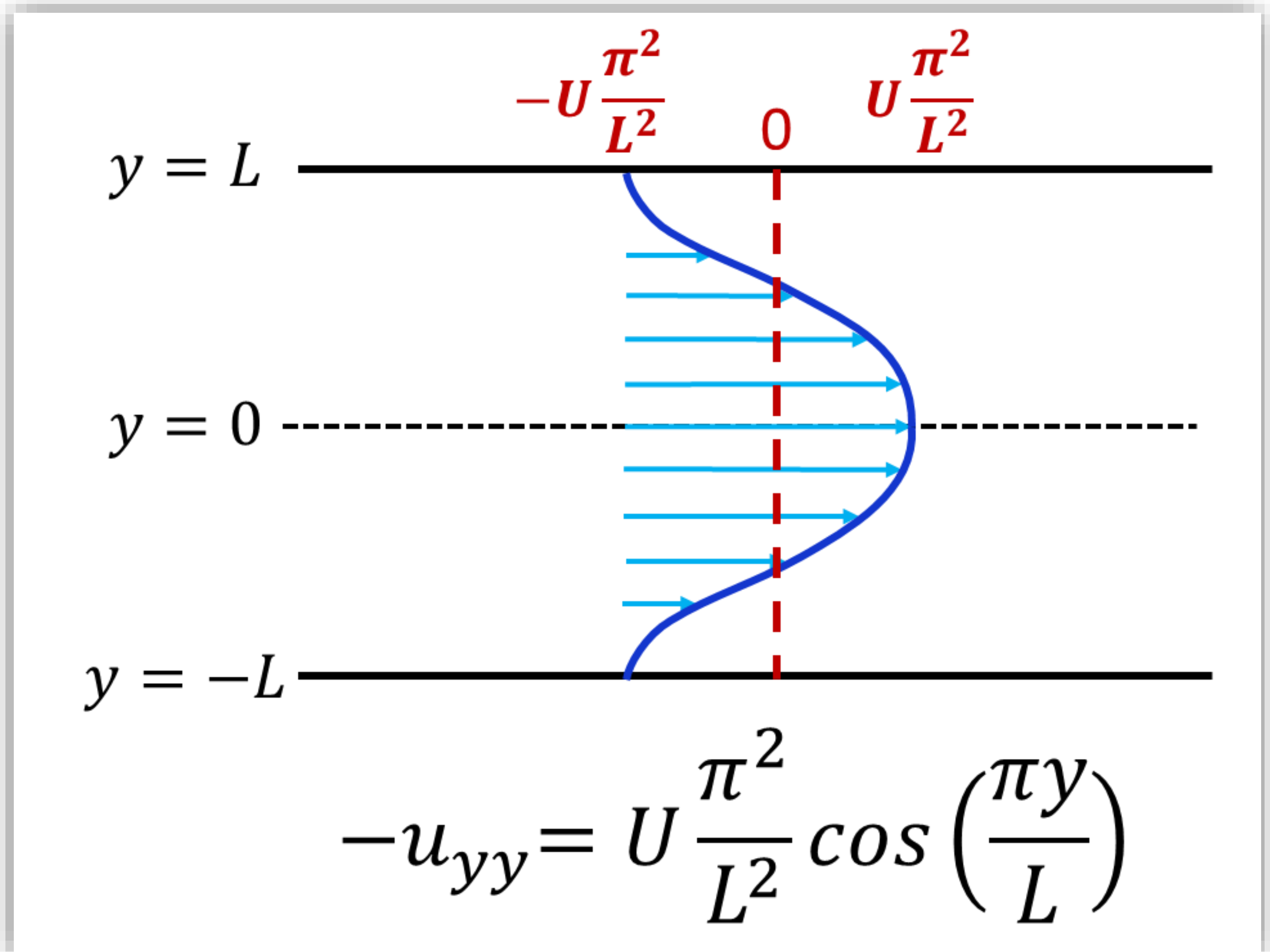
1) Show that for a planet that has no north-south surface curvature, this profile satisfies the Rayleigh criterion for barotropic instability.



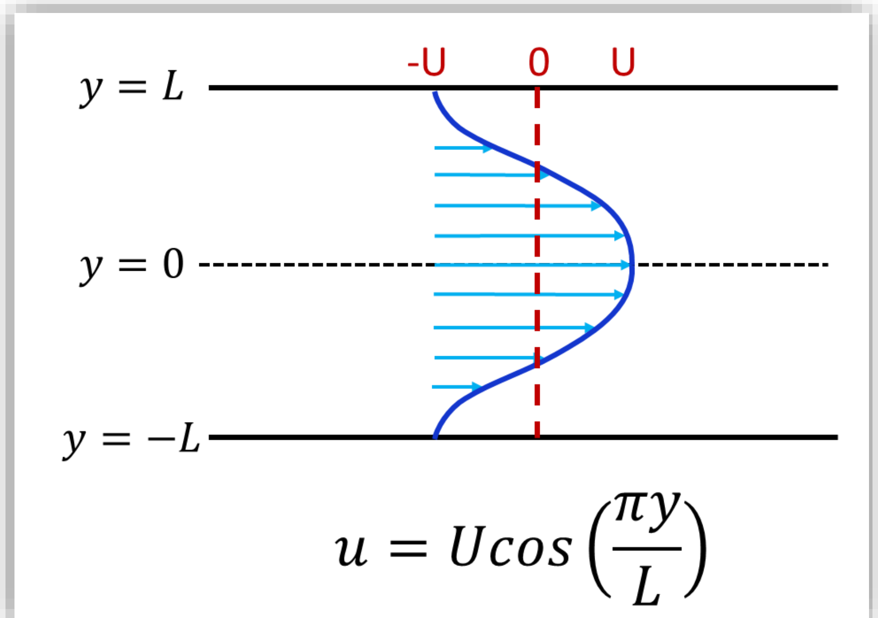
$$u_y = -U\frac{\pi}{L}sin\left(\frac{\pi y}{L}\right)$$

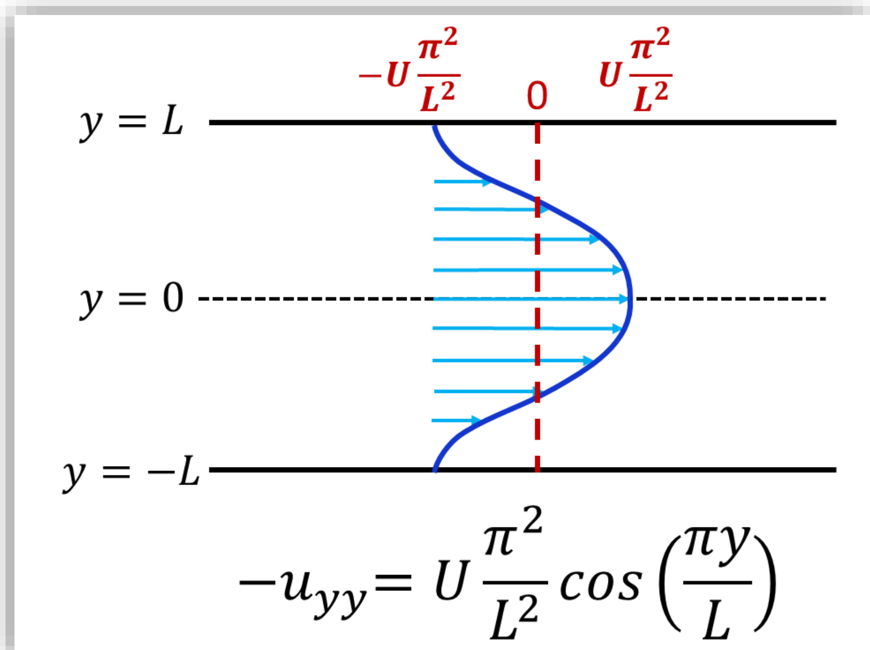
$$-u_{yy} = U\frac{\pi^2}{L^2}cos\left(\frac{\pi y}{L}\right)$$

- no curvature $\Rightarrow \beta = 0$
- $-u_{yy}$ changes sign in the domain
  - Possibly **unstable**

Instability and Rayleigh criterion

2) For a planet that <u>does have</u> north-south curvature, for what value of β is this profile stabilized ?





$\beta - \bar{u}_{yy}$ must change sign somewhere in the domain between (-L and L).
↳ If the Rayleigh criterion is satisfied, we might have an instability.

To guarantee no change sign in $\beta - \bar{u}_{yy}$:

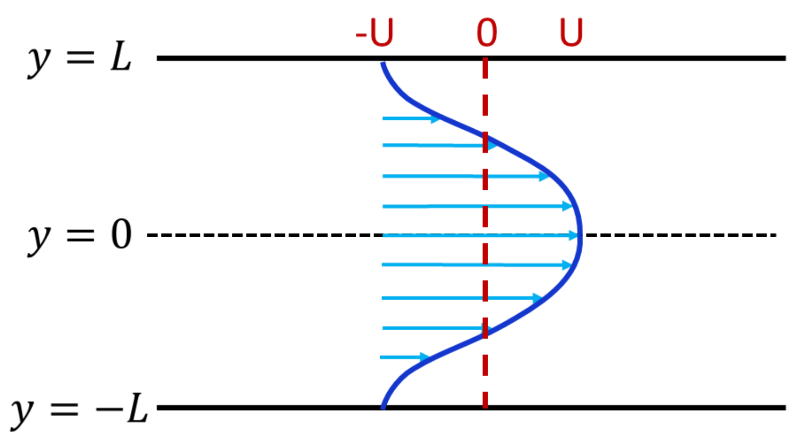
⇨ $\forall y, \beta - u_{yy} \geq 0 \;\Rightarrow \beta \geq min_y(u_{yy})$

$$\beta \geq U\frac{\pi^2}{L^2}$$

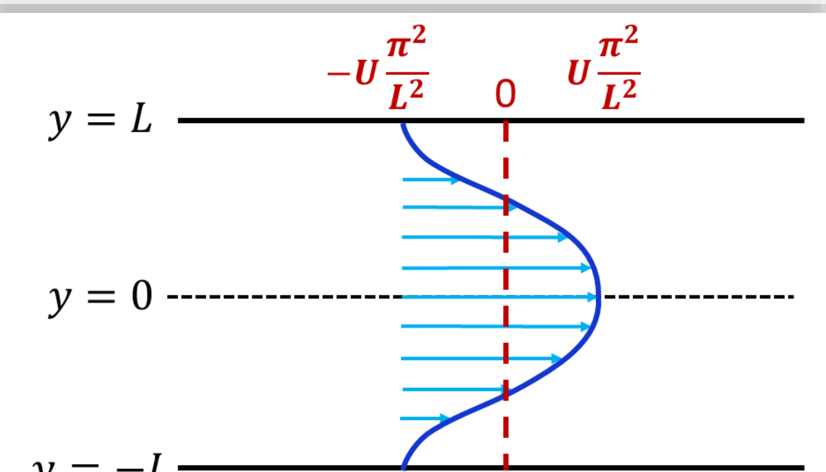
↳ **stable**

Instability and Fjortoft criterion

3) Show that for a planet that has no north-south surface curvature, this profile satisfies the **Fjørtoft criterion** for barotropic instability.



$$u = U\cos\left(\frac{\pi y}{L}\right)$$



$$-u_{yy} = U\frac{\pi^2}{L^2}\cos\left(\frac{\pi y}{L}\right)$$

$\forall u_0$, $(\bar{u} - u_0)(\beta - \bar{u}_{yy})$ must be positive somewhere in the domain.

If the Fjørtoft criterion is satisfied, we might have an instability.

$$\int_{y_1}^{y_2} (\beta - U_{yy})\frac{(U - U_s)}{|U - c|^2}|\tilde{\psi}|^2 \,\mathrm{d}y > 0$$

where U_s is *any* real constant. It is most useful to choose this constant to be the value of $U(y)$ at which $\beta - U_{yy}$ vanishes. This leads directly to the criterion

> *A necessary condition for instability is that the expression*
> $$(\beta - U_{yy})(U - U_s)$$
> *where U_s is the value of $U(y)$ at which $\beta - U_{yy}$ vanishes, be positive somewhere in the domain.*

This criterion is satisfied if the magnitude of the vorticity has an extremum inside the domain, and not at the boundary or at infinity (Fig. 6.8). Why choose U_s in the manner we did? Suppose we chose U_s to have a large positive value, so that $U - U_s$ is negative everywhere. Then (6.59) just implies that $\beta - U_{yy}$ must be negative somewhere, and this is already known from Rayleigh's criterion. If we choose U_s to be large and negative, we simply find that $\beta - U_{yy}$ must be positive somewhere. The most stringent criterion is obtained by choosing U_s to be the value of $U(y)$ at which $\beta - U_{yy}$ vanishes.

⇨ $(u - 0)(-u_{yy})$ is positive everywhere in the domain

Possibly **unstable**