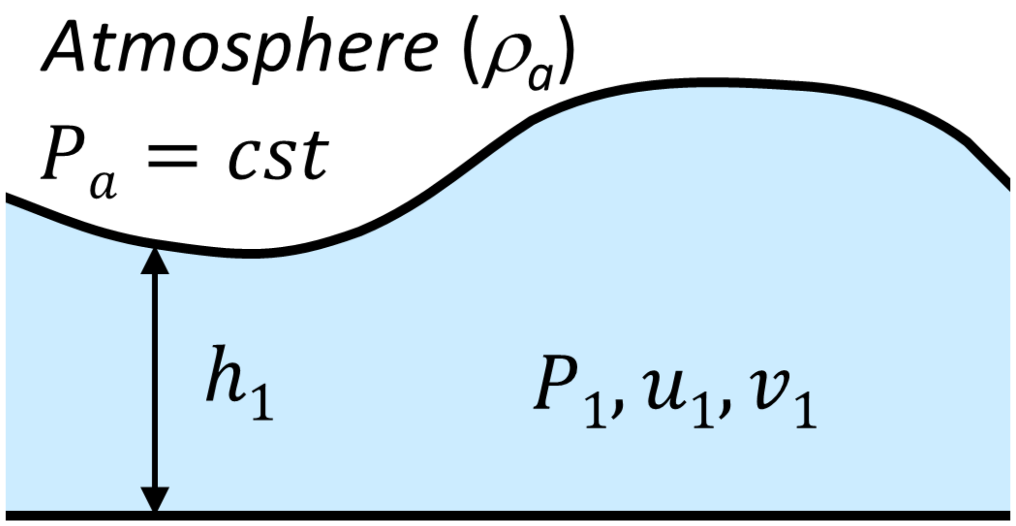
Shallow water and vorticity

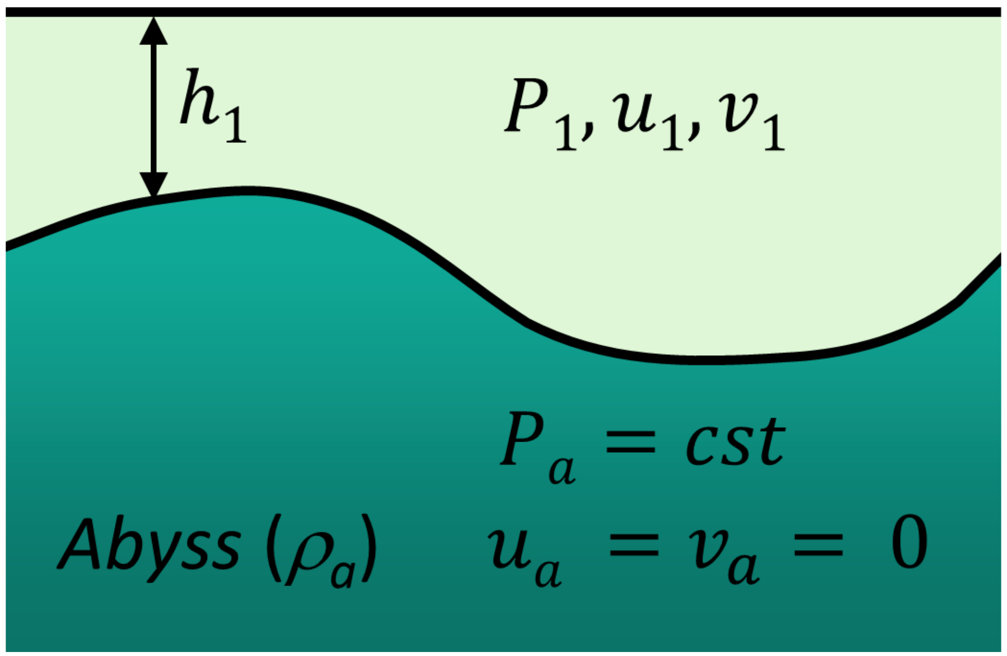
Answering some questions…



$z = -H$ *(flat bottom)*

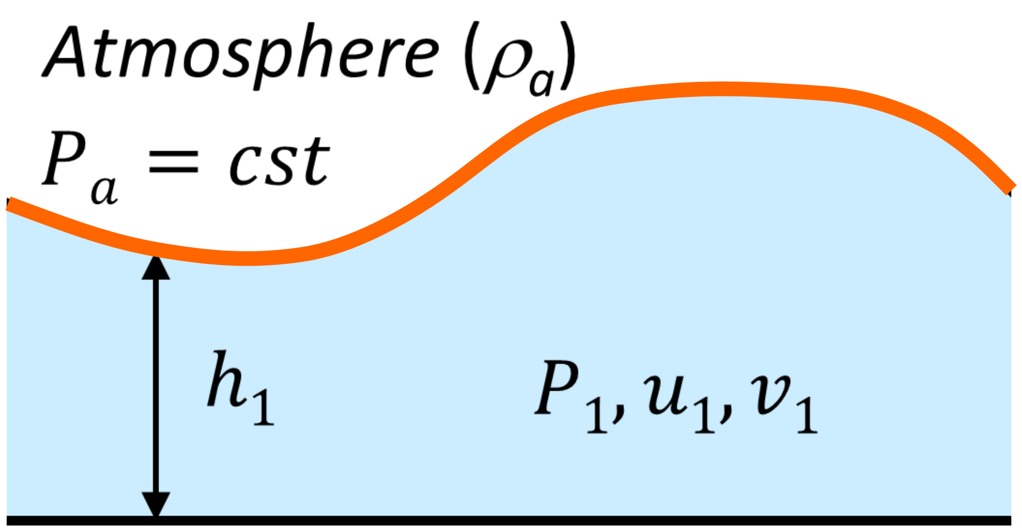
1 shallow water layer
over a flat bottom

$z = 0$ *(lid)*



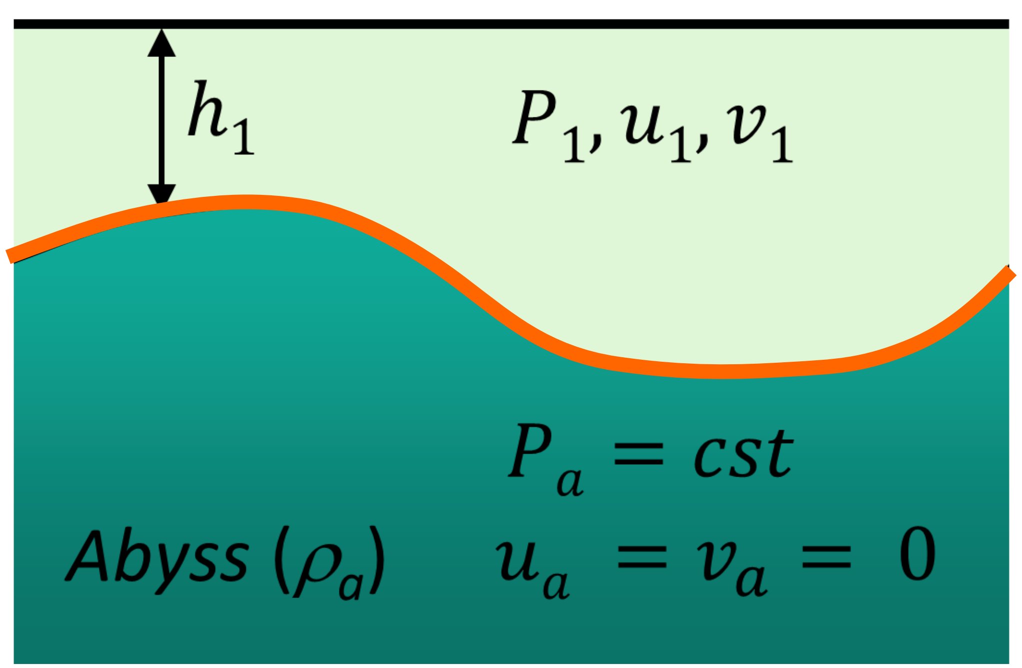
1 shallow water layer
topped **by a rigid lid**
overlying **a motionless abyss**

Answering some questions…



interface at
$z_i = -H + h_1$

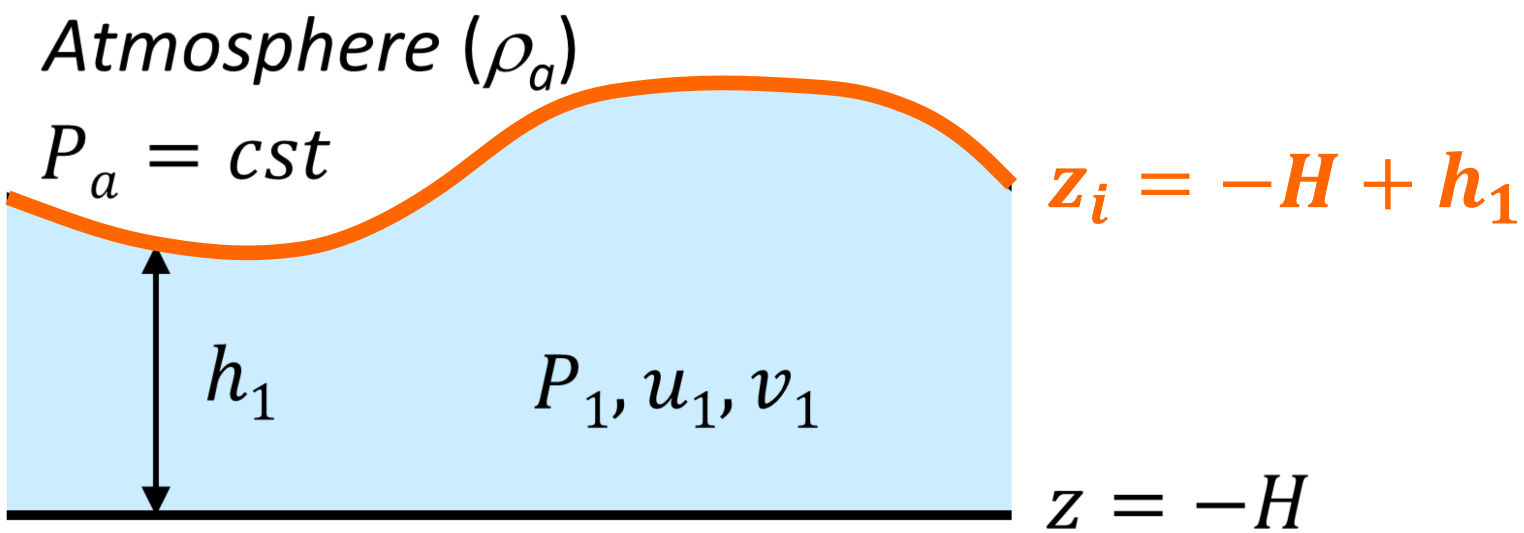
$z = -H$ *(flat bottom)*



$z = 0$ *(lid)*

interface at
$z_i = -h_1$

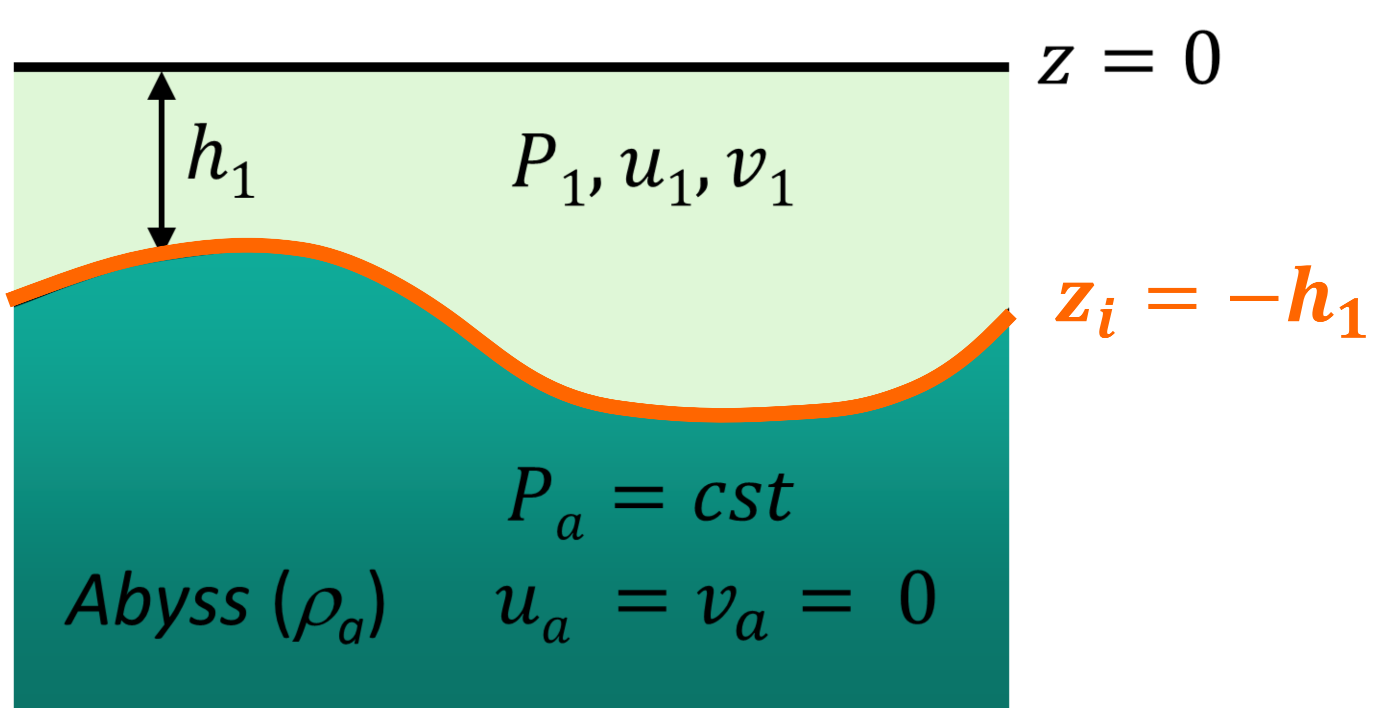
Answering some questions…



Applying hydrostatic balance at the interface

$$\frac{\Delta P}{\Delta \rho} = g z_i$$

$$\frac{P_1 - P_a}{\rho_1 - \rho_a} = g(-\boldsymbol{H} + \boldsymbol{h_1})$$

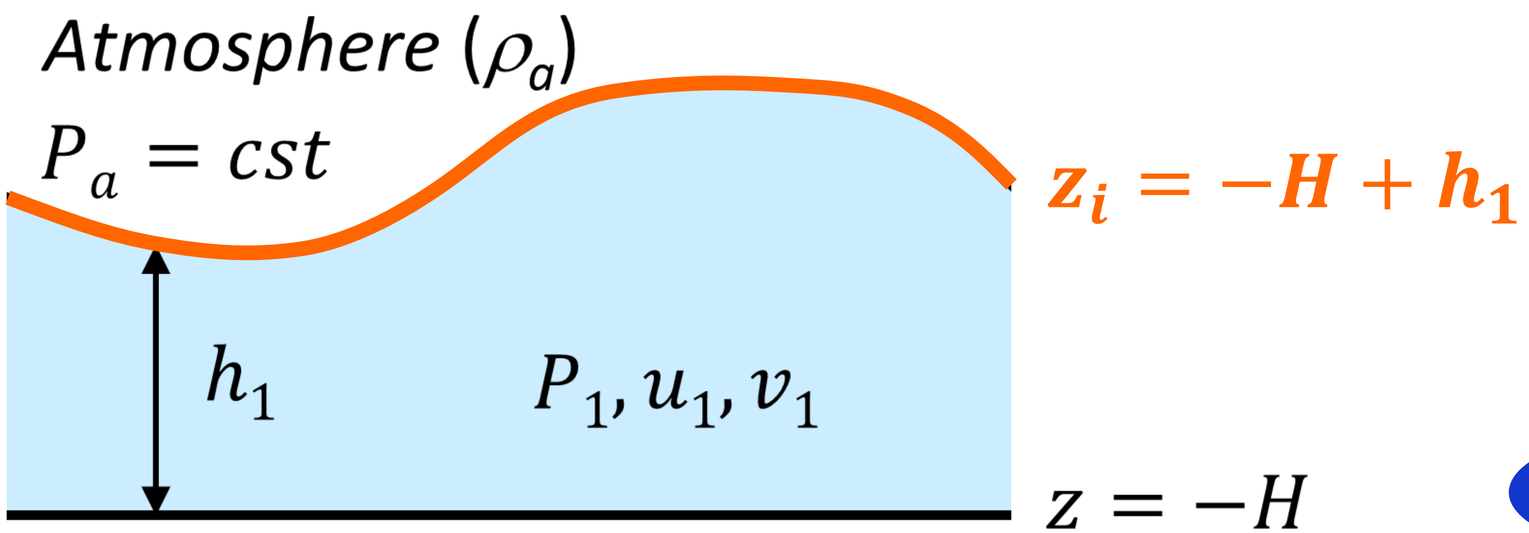


$$\frac{\Delta P}{\Delta \rho} = g z_i$$

$$\frac{P_1 - P_a}{\rho_1 - \rho_a} = g(-\boldsymbol{h_1})$$

Answering some questions…

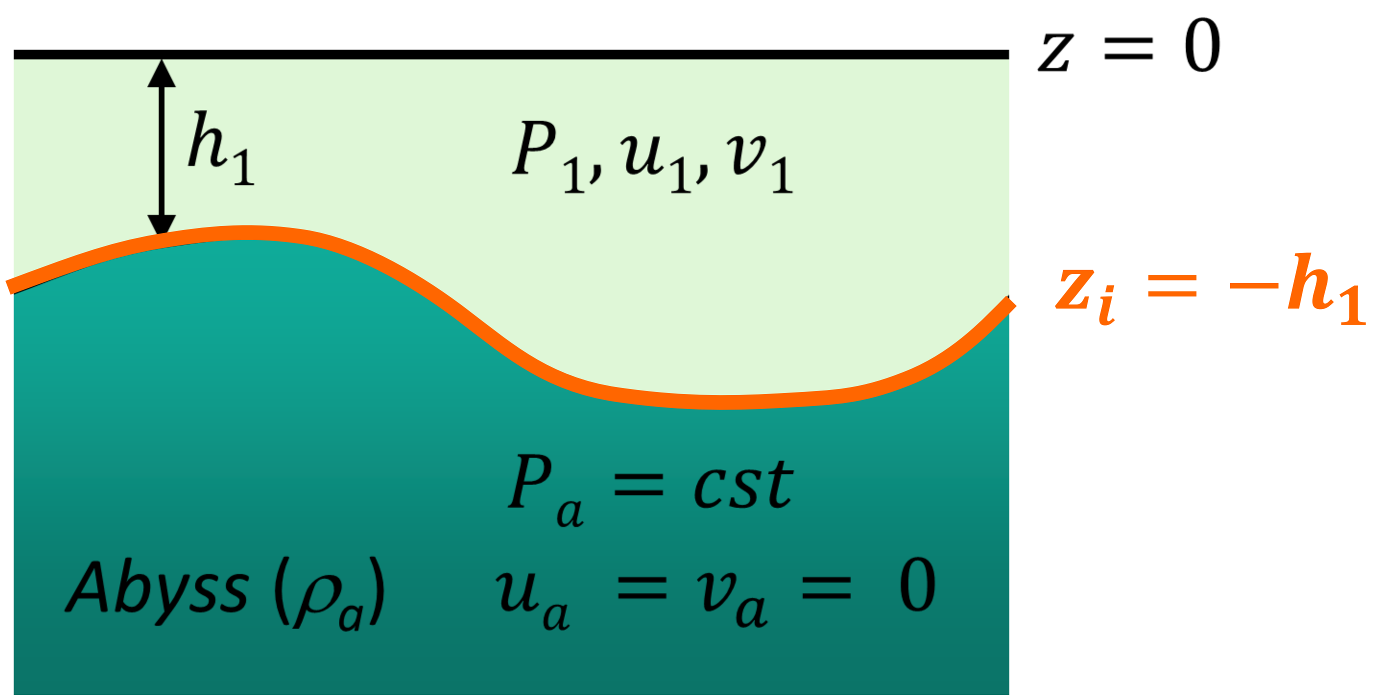
Applying hydrostatic balance at the interface



$$\frac{P_1 - P_a}{\rho_1 - \rho_a} = g(-H + h_1)$$

$\rho_a \ll \rho_1$

$$\frac{1}{\rho}\frac{\partial P_1}{\partial x} = g\left(\frac{\partial h_1}{\partial x}\right)$$



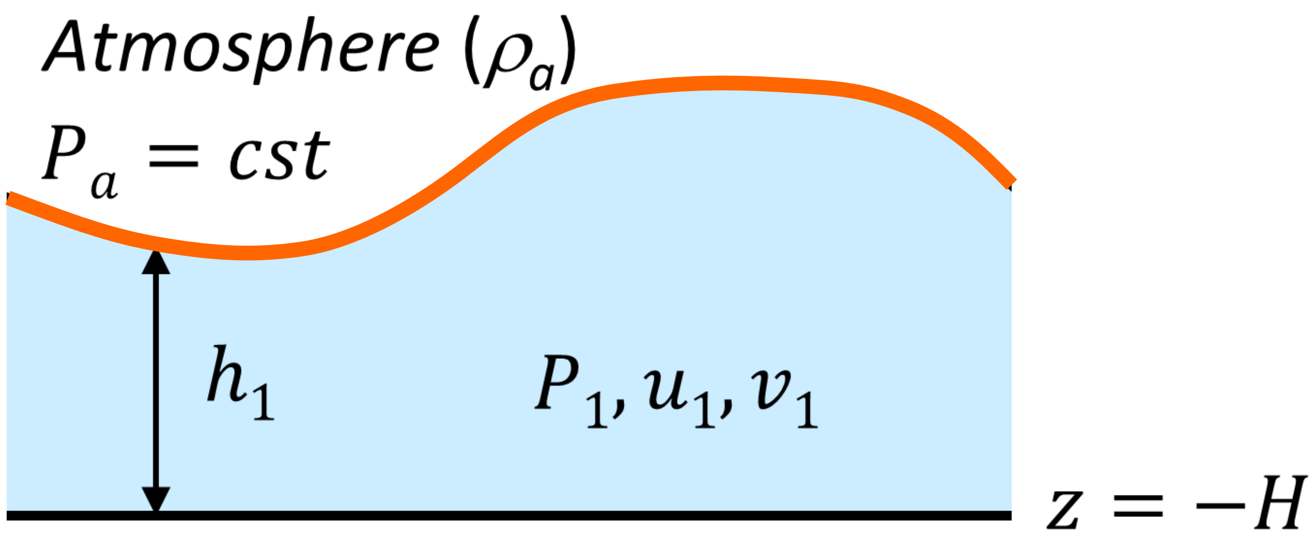
$$\frac{P_1 - P_a}{\rho_1 - \rho_a} = g(-h_1)$$

$\rho_a = \rho_1 + \Delta\rho$

$$\frac{P_1 - P_a}{-\Delta\rho} = g(-h_1)$$

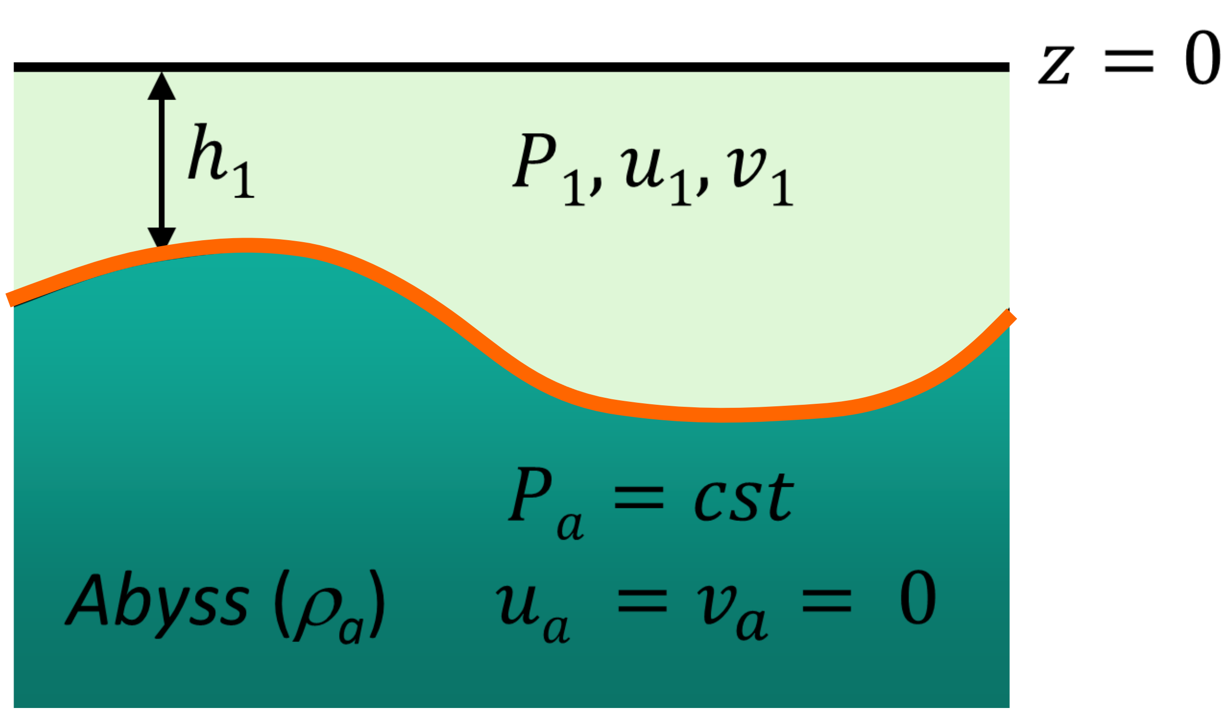
$$\frac{1}{\rho}\frac{\partial P_1}{\partial x} = g\frac{\Delta\rho}{\rho}\left(\frac{\partial h_1}{\partial x}\right)$$

Answering some questions…



1 shallow water layer over a flat bottom

$$\frac{1}{\rho}\frac{\partial P_1}{\partial x} = \boldsymbol{g}\left(\frac{\partial \boldsymbol{h_1}}{\partial x}\right)$$



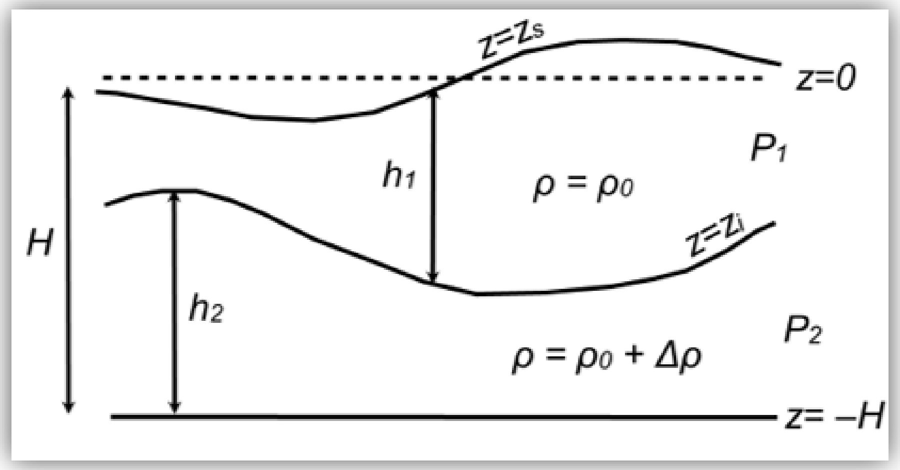
1.5 reduced-gravity shallow water model

$$\frac{1}{\rho}\frac{\partial P_1}{\partial x} = \boldsymbol{g'}\left(\frac{\partial \boldsymbol{h_1}}{\partial x}\right)$$

$$g' = g\frac{\Delta\rho}{\rho}$$

Some precisions about course…

For a two-layer system:



$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial u_i}{\partial y} - f v_i = \boxed{-g \frac{\partial}{\partial x}(h_1 + h_2)} - g' \frac{\partial h_2}{\partial x}$$

Barotropic mode because it triggers a flow modulation that is identical for each layer

$$-g\frac{\partial D}{\partial x}$$

➥ Throughout the fluid, the flow undergoes the effect of the free surface variations, a **barotropic external mode**, and going down layer by layer different contributions from the stratification (the **baroclinic part**) add up.

$$\frac{Du_i}{Dt} - f v_i = -g \frac{\partial D}{\partial x} - g' \left[\mathbf{C} \frac{\partial \mathbf{h}}{\partial x}\right]_{i,\, i>1}$$

➥ These N equations are **strongly coupled**. One cannot take one layer and solve for the flow in this particular layer. We need to know about the thicknesses of every other layer above/below. We solve the system of equation mode by mode. This involves finding the eigenvectors of the matrix **C** and transforming the variables to get a set of **decoupled equations**.

Exercise

Question 1

1) Draw a diagram to represent two shallow water layers topped by a rigid lid and overlying a motionless abyss. The difference in layer density is always $\Delta\rho$.

2) Derive expressions for the depth of the layer interfaces in terms of the layer thicknesses.

$zi_{12}=$ **?** – $zi_{2a}=$ **?**

3) Using the hydrostatic relation $\boldsymbol{\Delta P/\Delta\rho = gz}$, derive expressions for the Montgomery potential $\boldsymbol{P}$ in the two layers.

$P_1=$ **?** – $P_2=$ **?**

4) Write down the linear x-momentum equation in each layer (just the x-momentum)

$\frac{Du_1}{Dt}=$**?** - $\frac{Du_2}{Dt}=$**?**

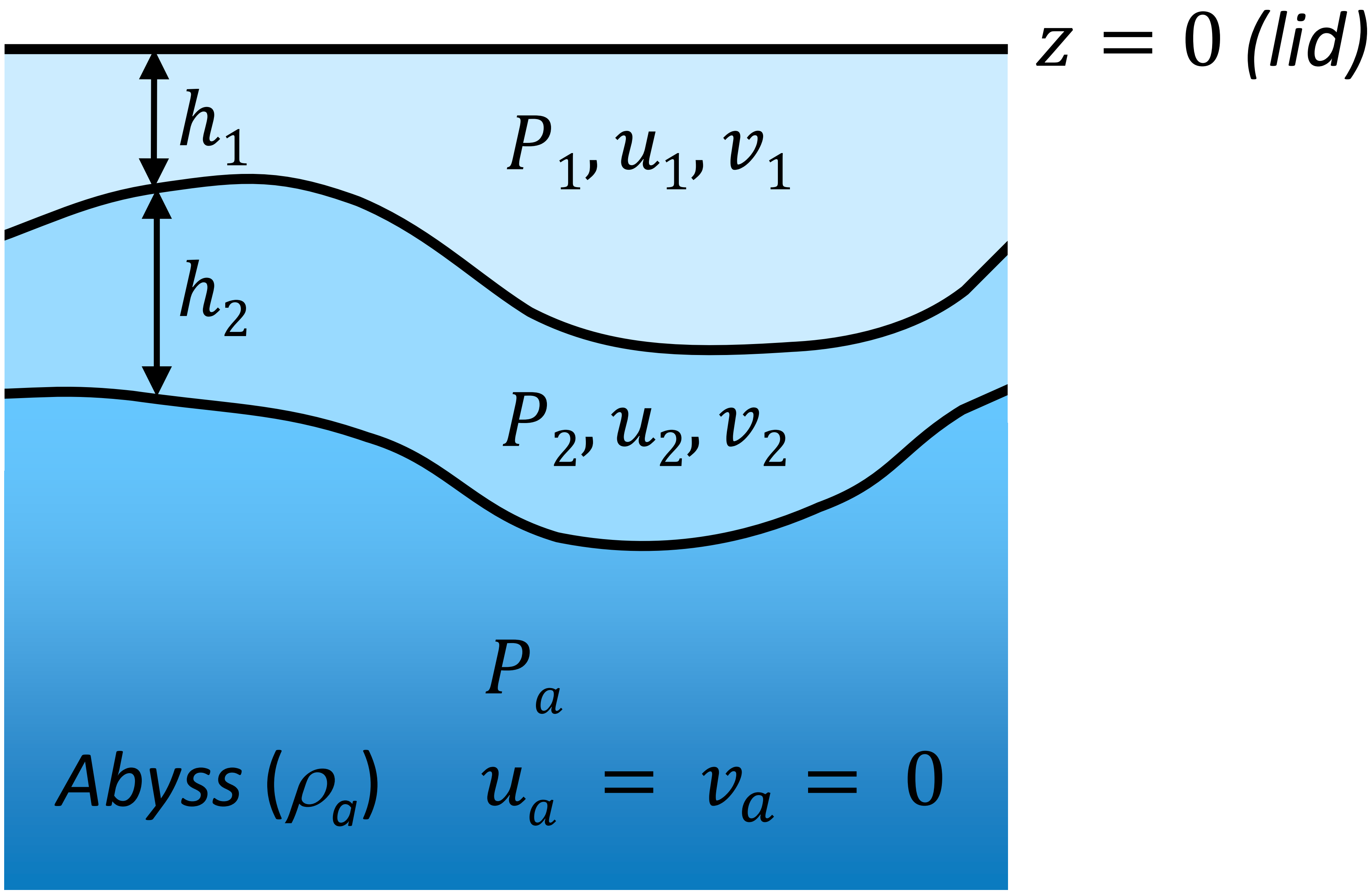
5) Write the x-momentum linear equations (for **u**) as a single column vector equation in **u**$=(u_1,u_2)$, **v**$=(v_1,v_2)$ and **h**$=(h_1,h_2)$ and the matrix **C**.

6) Find the eigenvalues and eigenvectors of **C**.

7) Find the variable transformation that gives two independent equations, and write down the two equations.

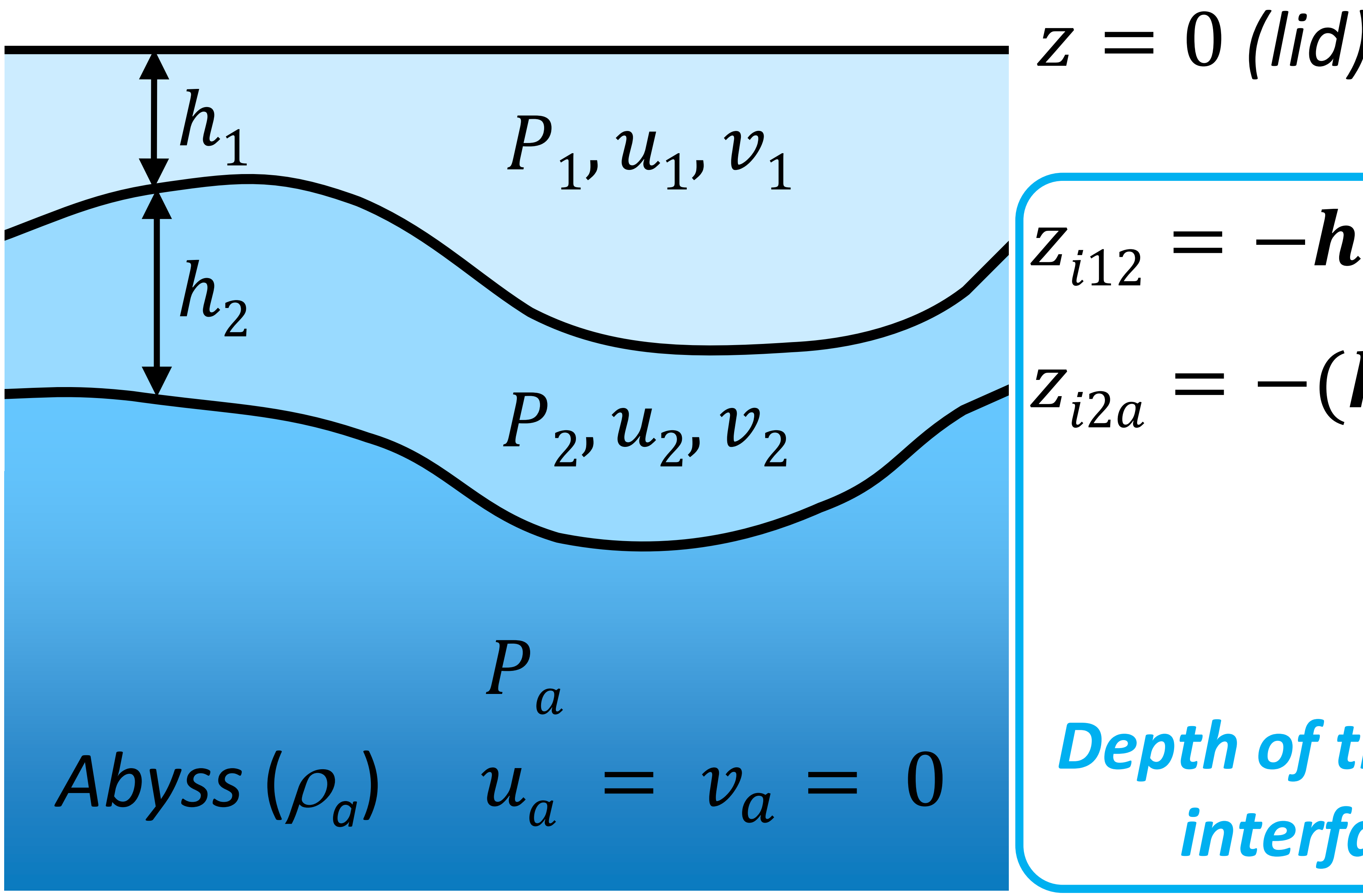
Solution – Question 1

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Solution – Question 2

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



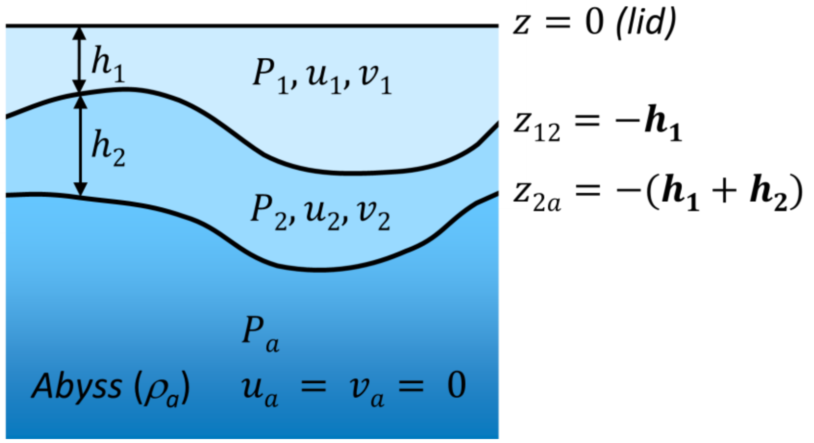
$z_{i12} = -\boldsymbol{h_1}$

$z_{i2a} = -(\boldsymbol{h_1} + \boldsymbol{h_2})$

Depth of the layer interfaces

Solution – Question 3

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Montgomery Potential (P) in each active layer

⇨ Applying the **hydrostatic equation** across the layer interfaces z_i

$$\frac{\Delta P}{\Delta \rho} = g z_i$$

Lower layer

$$\frac{P_a - P_2}{\Delta \rho} = g z_{i2a}$$

$$P_2 = -g \Delta \rho z_{i2a}$$

$$P_2 = g \Delta \rho (h_1 + h_2) \; \{+P_a\}$$

Upper layer

$$\frac{P_2 - P_1}{\Delta \rho} = g z_{i12}$$

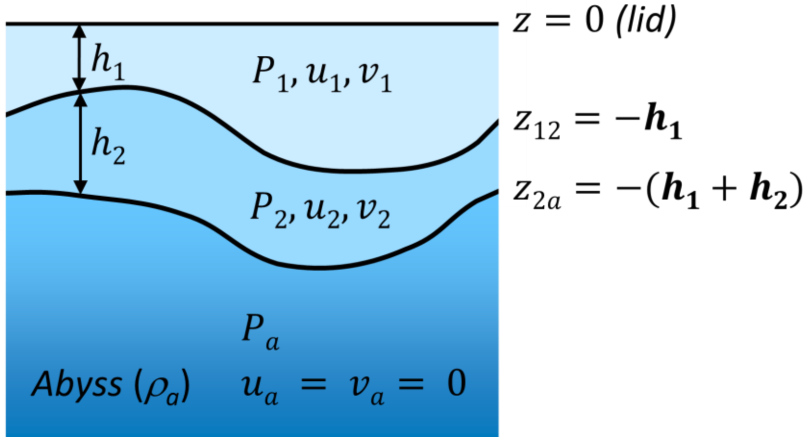
$$P_1 = P_2 - g \Delta \rho z_{i12}$$

$$P_1 = g \Delta \rho (h_1 + h_2) + g \Delta \rho h_1$$

$$P_1 = g \Delta \rho (2h_1 + h_2)$$

Solution – Question 4

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Linear x-momentum equation in each active layer

$$\frac{\partial u_i}{\partial t} - f v_i = \frac{-1}{\rho_0}\frac{\partial P_i}{\partial x}$$

Upper layer

$$P_1 = g\Delta\rho(2h_1 + h_2)$$

$$\frac{\partial u_1}{\partial t} - f v_1 = \frac{-1}{\rho_0}\frac{\partial P_1}{\partial x}$$

$$\frac{\partial u_1}{\partial t} - f v_1 = -g'\frac{\partial}{\partial x}(2h_1 + h_2)$$

Lower layer

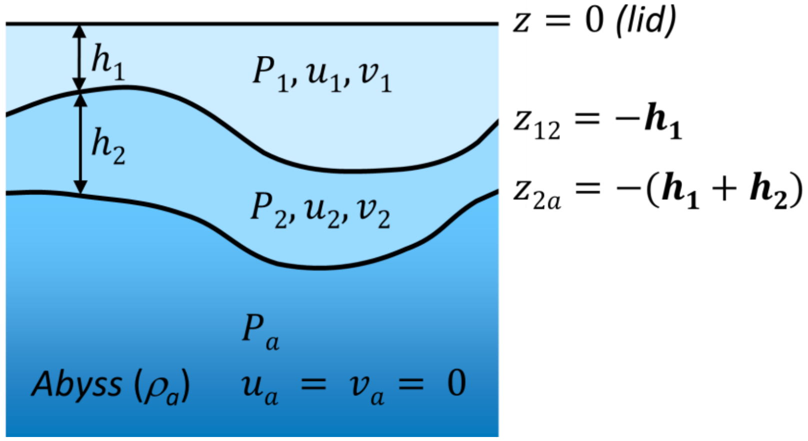
$$P_2 = g\Delta\rho(h_1 + h_2)$$

$$\frac{\partial u_2}{\partial t} - f v_2 = \frac{-1}{\rho_0}\frac{\partial P_2}{\partial x}$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -g'\frac{\partial}{\partial x}(h_1 + h_2)$$

Solution – Question 5

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Shallow water linear x-momentum equation in vector notation

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \boldsymbol{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Upper layer

$$\frac{\partial u_1}{\partial t} - f v_1 = -g' \frac{\partial}{\partial x}(2h_1 + h_2)$$

Lower layer

$$\frac{\partial u_2}{\partial t} - f v_2 = -g' \frac{\partial}{\partial x}(h_1 + h_2)$$

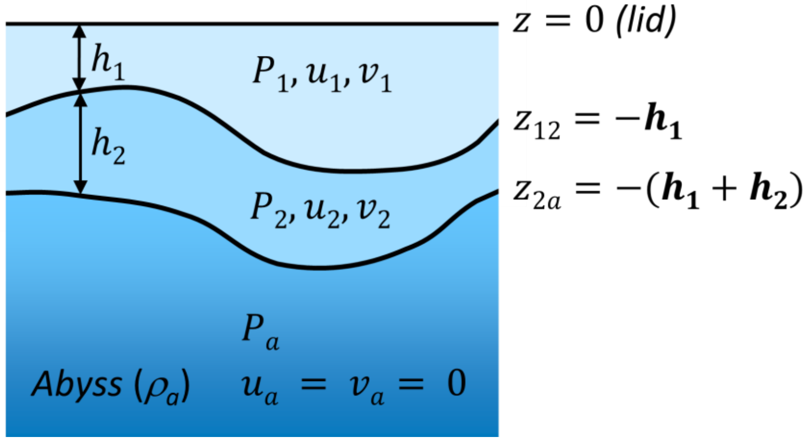
$$\frac{\partial \boldsymbol{u}}{\partial t} - f\boldsymbol{v} = -g' \frac{\partial}{\partial x}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\boldsymbol{h}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} - f\boldsymbol{v} = -g' \frac{\partial}{\partial x}\mathbb{C}\boldsymbol{h}$$

$$\mathbb{C} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

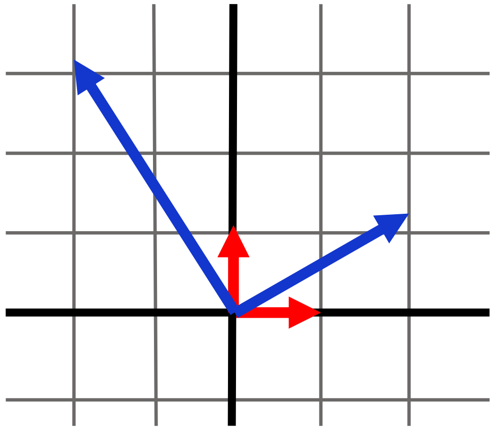
Solution – Question 6

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Eigenvalues and eigenvectors of $\mathbb{C}$

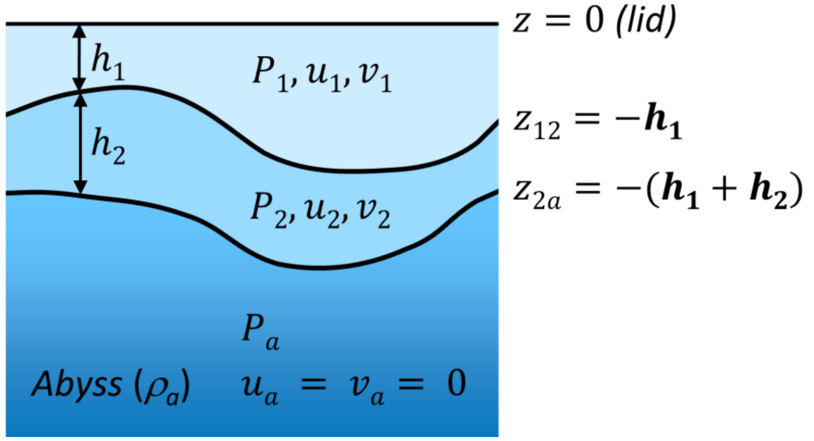
$$\mathbb{C} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$



$\mathbb{C}$ is real symmetric ($\mathbb{C} = \mathbb{C}^{\boldsymbol{T}}$)
⇨ It can thus be diagonalized,
↳ i.e. there exists a **basis** of eigenvectors **e** in which the matrix is diagonal: $\mathbb{C}\boldsymbol{e} = \lambda \boldsymbol{e}$

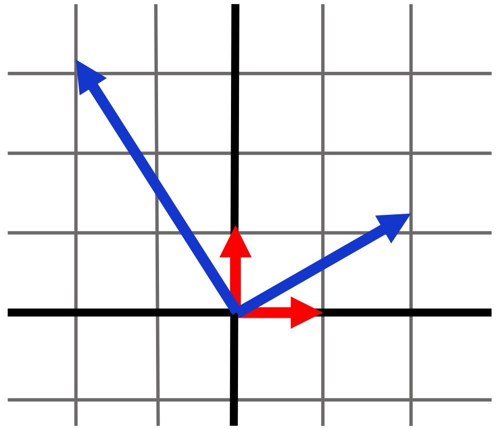
$$\boldsymbol{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad \mathbb{C}\boldsymbol{h} = \begin{pmatrix} 2h_1 + h_2 \\ h_1 + h_2 \end{pmatrix}$$

Solution – Question 6



Eigenvalues and eigenvectors of $\mathbb{C}$

$$\mathbb{C} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$



$det(\mathbb{C} - \lambda\mathbf{I})$ is $\mathbb{C}$'s ***characteristic polynomial***

$$= \left|\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = \left|\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}\right|$$

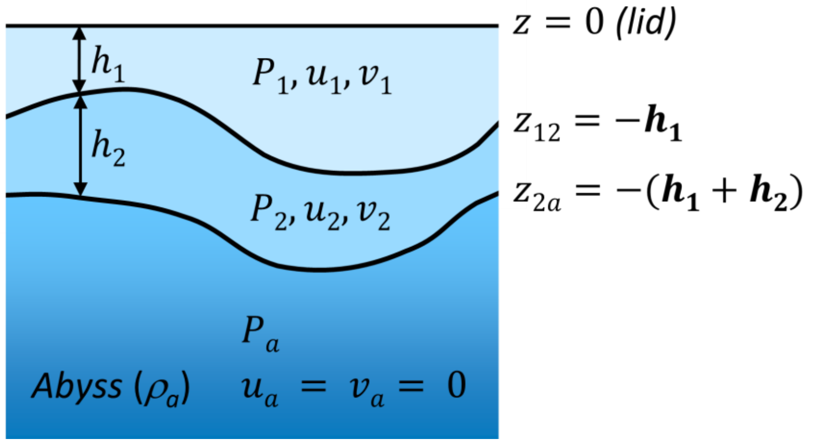
$$= \lambda^2 - 3\lambda + 1$$

Discriminant of the polynomial is $\Delta = 5$

Polynomial roots are $\lambda_1 = \dfrac{3+\sqrt{5}}{2}$ $\lambda_2 = \dfrac{3-\sqrt{5}}{2}$

Solution – Question 6

2 shallow water layers topped by a rigid lid overlying a motionless abyss



Eigenvalues and eigenvectors of $\mathbb{C}$

$$\mathbb{C} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \lambda = \frac{3 \pm \sqrt{5}}{2}$$

We look for non-zero eigenvectors $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ associated with each eigenvalue λ_1 and λ_2

$$\mathbb{C}\boldsymbol{e_1} = \lambda_1 \boldsymbol{e_1}$$

$$\Leftrightarrow (\mathbb{C} - \lambda_1 \boldsymbol{I})\boldsymbol{e_1} \overset{*}{=} 0$$

$det(\mathbb{C} - \lambda\mathbf{I}) = 0$

the system admits an infinity of solutions

$$\mathbb{C}\boldsymbol{e_2} = \lambda_2 \boldsymbol{e_2}$$

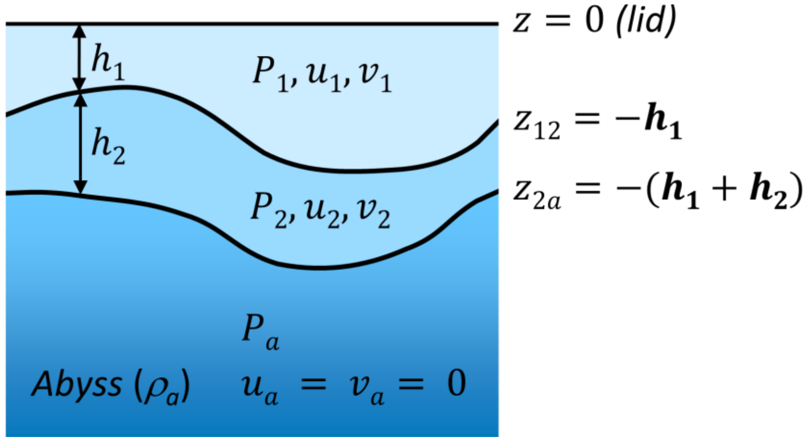
$$\Leftrightarrow (\mathbb{C} - \lambda_2 \boldsymbol{I})\boldsymbol{e_2} \overset{**}{=} 0$$

$$\boldsymbol{e_1} = \begin{pmatrix} 2 \\ \sqrt{5} - 1 \end{pmatrix}$$ is one solution of $*$

$$\boldsymbol{e_2} = \begin{pmatrix} -2 \\ \sqrt{5} + 1 \end{pmatrix}$$ is one solution of $**$

Solution – Question 7

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



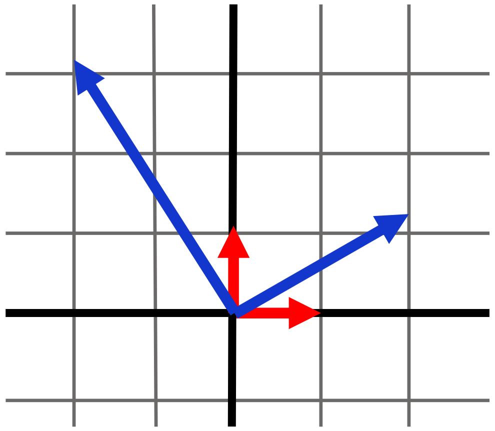
Variable transformation and independent equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - f\boldsymbol{v} = g' \frac{\partial}{\partial x} \mathbb{C}\boldsymbol{h}$$

$\mathbb{C} = \mathbf{PDP^{-1}}$ with $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

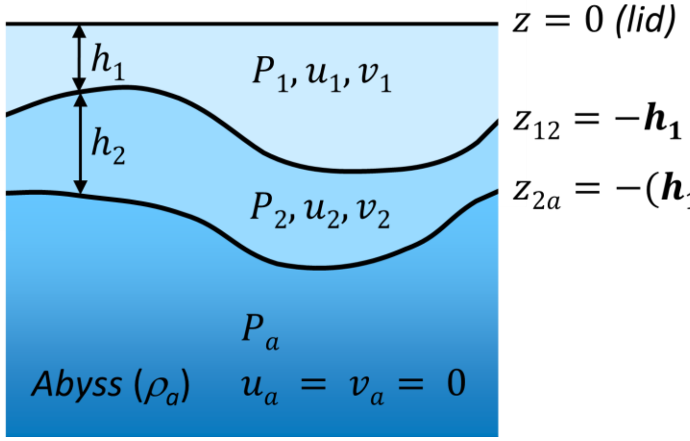
$\mathbf{P}$ is the passage matrix between the canonical base $\left\{\begin{pmatrix}1\\0\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and the new base $\{\boldsymbol{e_1}; \boldsymbol{e_2}\}$

$$\mathbf{P} = \begin{pmatrix} 2 & -2 \\ \sqrt{5} - 1 & \sqrt{5} + 1 \end{pmatrix}$$



Solution – Question 7

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



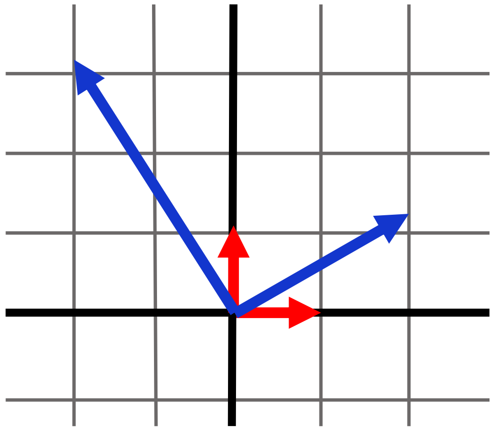
Variable transformation and independent equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - f\boldsymbol{v} = g' \frac{\partial}{\partial x} \mathbb{C}\boldsymbol{h}$$

$$\mathbb{C} = \mathbb{C}^T = \left(\mathbf{PDP}^{-1}\right)^T = \mathbf{P}^{-T}\mathbf{D}\mathbf{P}^T \quad \text{with} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

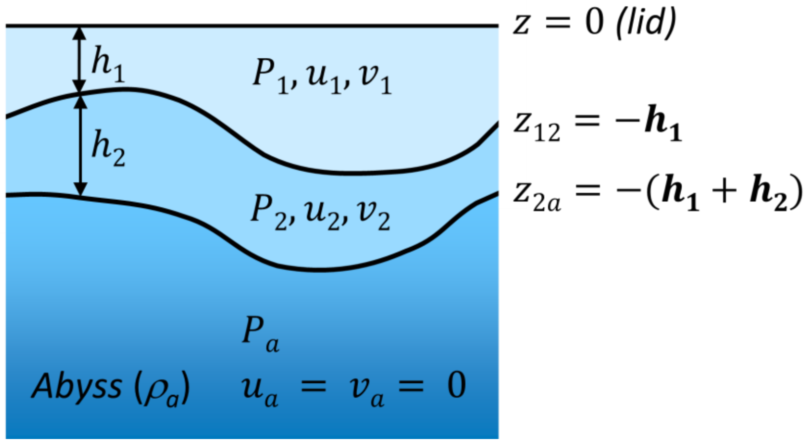
$\mathbf{P}$ is the passage matrix between the canonical base $\left\{\begin{pmatrix}1\\0\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and the new base $\{\boldsymbol{e_1}; \boldsymbol{e_2}\}$

$$\mathbf{P}^T = \begin{pmatrix} 2 & \sqrt{5}-1 \\ -2 & \sqrt{5}+1 \end{pmatrix}$$



Solution – Question 7

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Variable transformation and independent equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - f\boldsymbol{v} = g' \frac{\partial}{\partial x} \mathbb{C}^T \boldsymbol{h}$$

$$\mathbb{C} = \mathbb{C}^T = \left(\mathbf{PDP}^{-1}\right)^T = \mathbf{P}^{-T}\mathbf{D}\mathbf{P}^T \quad \text{with} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

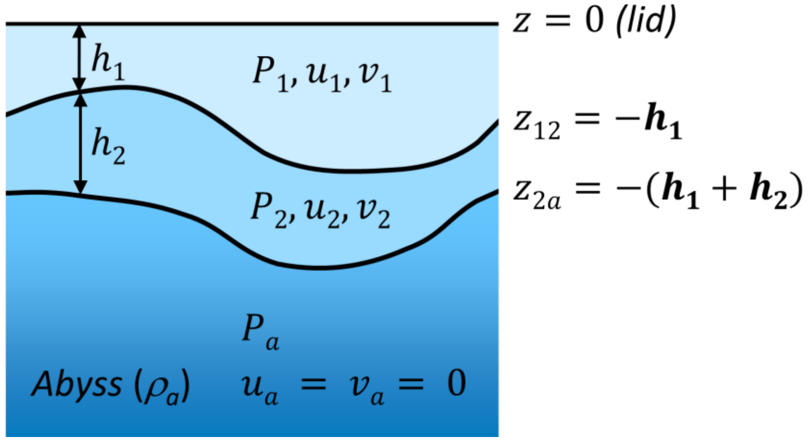
$\mathbf{P}$ is the passage matrix between the canonical base $\left\{\begin{pmatrix}1\\0\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and the new base $\{\boldsymbol{e_1}; \boldsymbol{e_2}\}$

$$\mathbb{C}\boldsymbol{h} = \mathbb{C}^T\boldsymbol{h} = \mathbf{P}^{-T}\mathbf{D}\mathbf{P}^T\boldsymbol{h} = \mathbf{P}^{-T}\mathbf{D}\widehat{\boldsymbol{h}}$$

with $\widehat{\boldsymbol{h}} = \mathbf{P}^{\mathrm{T}}\boldsymbol{h}$ coordinates in the new base

Solution – Question 7

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Variable transformation and independent equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - f\boldsymbol{v} = g' \frac{\partial}{\partial x} \mathbb{C}^T \boldsymbol{h}$$

$$\hat{\boldsymbol{u}} = \mathrm{P}^T \boldsymbol{u}$$

$$\hat{\boldsymbol{v}} = \mathrm{P}^T \boldsymbol{v} \quad \text{with} \quad \hat{\boldsymbol{h}} = \begin{pmatrix} \widehat{h_1} \\ \widehat{h_2} \end{pmatrix} \qquad \mathrm{P}^T = \begin{pmatrix} 2 & \sqrt{5} - 1 \\ -2 & \sqrt{5} + 1 \end{pmatrix}$$

$$\hat{\boldsymbol{h}} = \mathrm{P}^T \boldsymbol{h}$$

$$\mathbb{C}^T \boldsymbol{h} = \mathrm{P}^{-T} \mathrm{D} \hat{\boldsymbol{h}}$$

$$\mathrm{P}^T * \left(\frac{\partial \boldsymbol{u}}{\partial t} - f\boldsymbol{v} = g' \frac{\partial}{\partial x} \mathrm{P}^{-T} \mathrm{D} \hat{\boldsymbol{h}} \right)$$

Solution – Question 7

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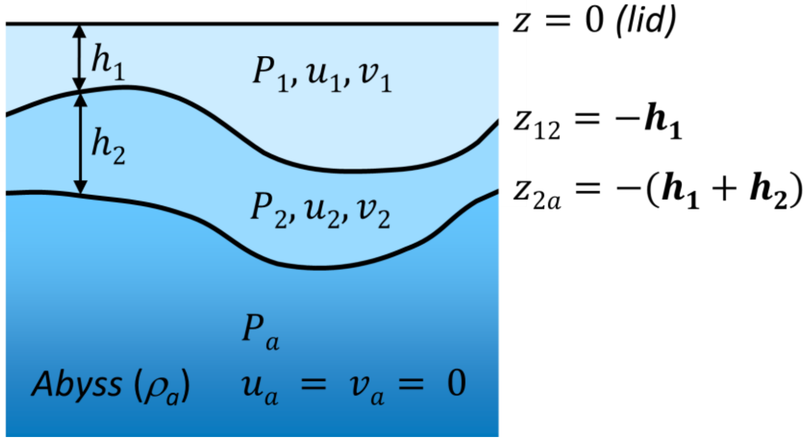

Variable transformation and independent equations

$$\frac{D\boldsymbol{u}}{Dt} - f\boldsymbol{v} = g'\frac{\partial}{\partial x}\mathbb{C}^T\boldsymbol{h}$$

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$$\hat{\boldsymbol{v}} = \mathbf{P}^T\boldsymbol{v} \quad \text{with} \quad \hat{\boldsymbol{h}} = \begin{pmatrix} \widehat{h_1} \\ \widehat{h_2} \end{pmatrix} \qquad \mathbf{P}^T = \begin{pmatrix} 2 & \sqrt{5}-1 \\ -2 & \sqrt{5}+1 \end{pmatrix}$$

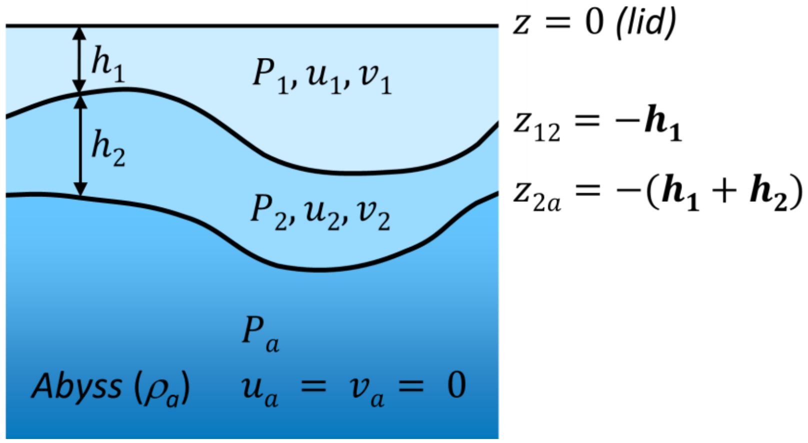
$$\hat{\boldsymbol{h}} = \mathbf{P}^T\boldsymbol{h}$$

$$\frac{\partial \mathbf{P}^T\boldsymbol{u}}{\partial t} - f\mathbf{P}^T\boldsymbol{v} = -g'\frac{\partial}{\partial x}\mathbf{P}^T\mathbf{P}^{-T}\mathbf{D}\hat{\boldsymbol{h}}$$

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} - f\hat{\boldsymbol{v}} = -g'\frac{\partial}{\partial x}\mathbf{D}\hat{\boldsymbol{h}} \qquad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Solution – Question 7

2 shallow water layers topped **by a rigid lid** overlying **a motionless abyss**



Variable transformation and independent equations

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$$\hat{\boldsymbol{u}} = \mathrm{P}^T \boldsymbol{u}$$
$$\hat{\boldsymbol{v}} = \mathrm{P}^T \boldsymbol{v}$$
$$\hat{\boldsymbol{h}} = \mathrm{P}^T \boldsymbol{h}$$

with $\hat{\boldsymbol{h}} = \begin{pmatrix} \widehat{h_1} \\ \widehat{h_2} \end{pmatrix} = \begin{pmatrix} 2h_1 + (\sqrt{5} - 1)h_2 \\ -2h_1 + (\sqrt{5} + 1)h_2 \end{pmatrix}$

$$\frac{\partial \widehat{u_1}}{\partial t} - f\widehat{v_1} = -g' \frac{\partial}{\partial x} \left(\frac{3 + \sqrt{5}}{2}\right) \widehat{h_1}$$
$$\frac{\partial \widehat{u_2}}{\partial t} - f\widehat{v_2} = -g' \frac{\partial}{\partial x} \left(\frac{3 - \sqrt{5}}{2}\right) \widehat{h_2}$$