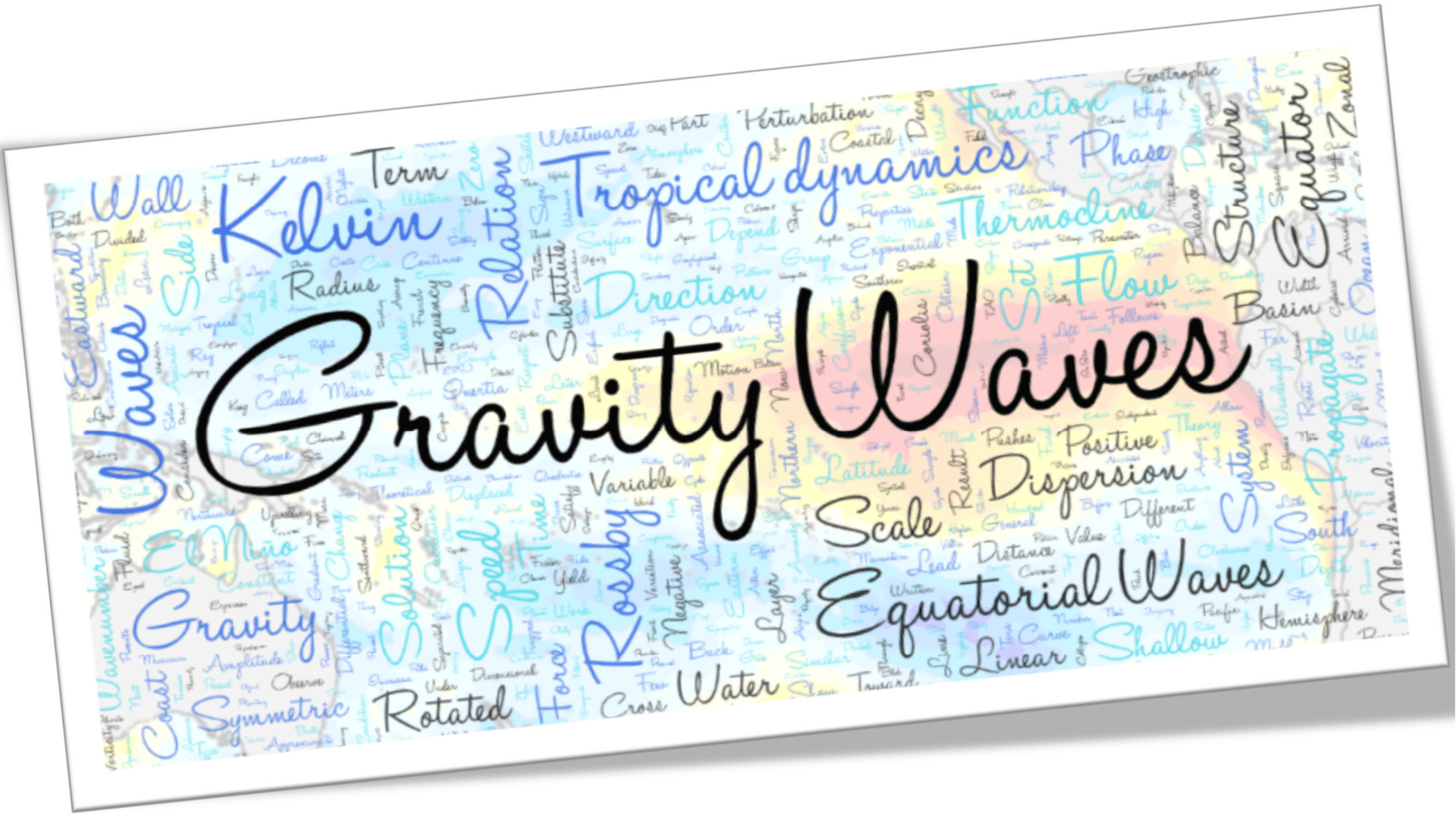
CHAPER 4
Gravity waves and tropical dynamics



CHAPTER 4
Gravity Waves and Tropical Dynamics

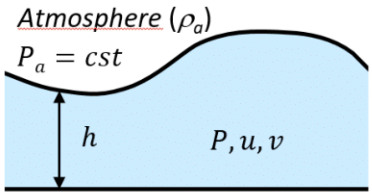
GFD4 Contents

This chapter focuses on gravity waves, which leads to the study of coastal Kelvin waves (see **#GFD4.2**) and extends the theory to equatorial Kelvin waves (see **#GFD4.3** and **#GFD4.4**). Then, we continue by discussing equatorial wave dynamics and its effects on tropical ocean variability (see **#GFD4.5**).

*QG theory filters out fast gravity waves (see **#GFD2.3d**), so we come back to the one-layer shallow-water equations derived in **#GFD1.2***

GFD4.1: Gravity Waves in a Rotating Fluid

4.1.a) Gravity waves in shallow water



⇨ Here is the **one-layer shallow water system** (x, y-momentum, and continuity equations, see **#GFD1.2f**):

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} \qquad \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Waves are the linear solutions of the equations

⇨ Let's start with something simple: a **one-dimensional non-rotating linear system**. The shallow water system can be simplified as follows:

- A **one-dimensional system** in the x-direction, so we ***cross out the y-direction terms*** (no v and no variations in y) and ***the y-momentum equation***.
- A **non-rotating system** means we ***cross-out the two Coriolis terms***.
- A **linear system**, so we can ***eliminate all the term which are quadratic in state variables*** (i.e. the advection terms).

↳ There is a little <u>subtlety</u> in the continuity equation, as we do not completely remove $h\frac{\partial u}{\partial x}$. We consider a **constant** average layer thickness H, such that $h = H + \eta$ and <u>linearize</u> this quadratic term by eliminating the product between the two state-variables (u and the variations in layer thickness η), so $h\frac{\partial u}{\partial x} \approx H\frac{\partial u}{\partial x}$.

2d-order Equation

⇨ This results in two equations: $\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \qquad \frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}$

↳ We then differentiate the x-momentum equation with respect to t and differentiate the continuity equation with respect to x, thus eliminating η. This leads to the following **second-order ordinary differential equation** for u:

$$\frac{\partial^2 u}{\partial t^2} = gH\frac{\partial^2 u}{\partial x^2}$$

Solution and Properties

Trigonometric function with imaginary exponential

⇨ The solution for this wave equation writes: $u = Re\,\tilde{u}\,e^{i(lx-\omega t)}$

▪ It is the real part of some amplitude coefficient $\tilde{u}$ times the classic imaginary exponential propagation part:

- l is the **zonal wavenumber** (2π divided by the x-wavelength),
- ω is the **angular frequency** (2π divided by the period).

This is a wave that propagates in the positive x-direction when $l > 0$.

Sine propagating in the $x > 0$ direction

▪ Taking a derivative of this trigonometric function yields the same function multiplied by some constant coefficients:

$$\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \frac{\partial}{x} \rightarrow il$$

Dispersion relation

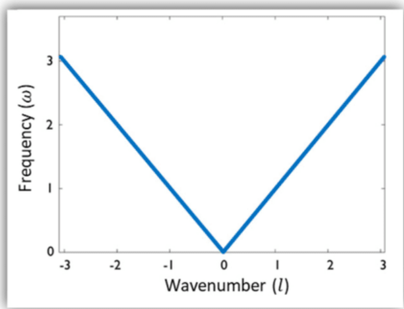
↳ Substituting the solution and its derivative into the wave equation results in **a relation between frequency ω and wavenumber *l*** (with two other geophysical parameters: gravity g and average layer thickness H): $\omega^2 = gHl^2$. This is the simple **dispersion relation** of a gravity wave, for which the **phase speed** is constant:

$$c = \frac{\omega}{l} = \pm\sqrt{gH}$$

All the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape along its propagation

▪ The phase speed does not depend on wavelength or frequency. Waves with different wavelengths (or structures made up of a collection of different wavelengths) travel at the same speed and will propagate without losing their shape. We say that these waves are **non-dispersive**.

▪ Their group speed $\frac{d\omega}{dl}$ remains the same as the phase speed because it is just a linear relationship between ω and l.



4.1.b) Adding rotation

⇨ The next step is to put the rotation back into the linear system. We **put the Coriolis terms back**. As the Coriolis force pushes perpendicular to the direction of movement, we have to go back to a ***two-dimensional situation*** with ***3 equations*** again. These are the single-layer linear shallow water equations on a flat bottom and an f-plane with linear perturbations in u, v and η:

f=f0

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

METHOD1: To solve this system, we can use the same method we just used, i.e. judiciously differentiate the equations in order to eliminate two of these three variables. This will lead to one high-order differential equation for one variable (u, v, or η). *You might have done this before…*

1) Derive the vorticity equation ($d(2)/dx - d(1)/dy$)
2) Derive the divergence equation ($d(1)/dx + d(2)/dy$)
3) Substitute the vorticity equation into the continuity equation
4) Substitute the divergence equation into the resulting equation
5) Differentiate with time and substitute with equation from step (3)

↳ With appropriate initial condition at $t = 0$, departures from geostrophic equilibrium follow:

$$\eta_{tt} - gH\nabla^2\eta + f^2\eta = 0$$

↳ Searching for plane-wave solutions ($\eta = \tilde{\eta}e^{i(lx+my-\omega t)}$) yields the dispersion relation:

$$\omega = \pm\sqrt{f^2 + gHk^2}$$

METHOD2: We can employ a more general (clever) method for finding wave solutions.

↳ It consists of substituting the plane-wave solutions $(u, v, \eta) = (\tilde{u}, \tilde{v}, \tilde{\eta})e^{i(lx+my-\omega t)}$ into the 3 equations separately.

▪ Solutions have the form of an amplitude coefficient times an imaginary exponential:

- l is the **zonal wavenumber** (2π divided by the x-wavelength),
- m is the **meridional wavenumber** (2π divided by the y-wavelength)
- ω is the **angular frequency** (2π divided by the period).

▪ The derivatives become coefficients: $\frac{\partial}{\partial x} \to il\times \quad \frac{\partial}{\partial y} \to im\times \quad \frac{\partial}{\partial t} \to -i\omega\times$

$$-i\omega\tilde{u} - f\tilde{v} = -igl\tilde{\eta}$$
$$-i\omega\tilde{v} + f\tilde{u} = -igm\tilde{\eta}$$
$$-i\omega\tilde{\eta} + H(il\tilde{u} + im\tilde{v}) = 0$$

↳ Substituting the solution and its derivatives into the linear system results in a set of three algebraic equations, in which the three unknowns are the coefficients of amplitude ($\tilde{u}, \tilde{v}$, and $\tilde{\eta}$).

↳ The parameters are the wave properties (l, m, and ω) and the geophysical constants (f, g, and H).

Solutions and algebraic system

4.1.c) Inertia-gravity (Poincaré) waves

⇨ We can write this set of equations in matrix form, resulting in an algebraic system:

$$\begin{pmatrix} -i\omega & -f & igl \\ f & -i\omega & igm \\ ilH & imH & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

▪ This equation is trivially satisfied if there is no wave-like perturbation ($\tilde{u} = \tilde{v} = \tilde{\eta} = 0$).

▪ The condition for the system to be satisfied and for the wave to have some amplitude is that ***the determinant of the matrix must be zero***.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \bullet \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \bullet \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \bullet \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

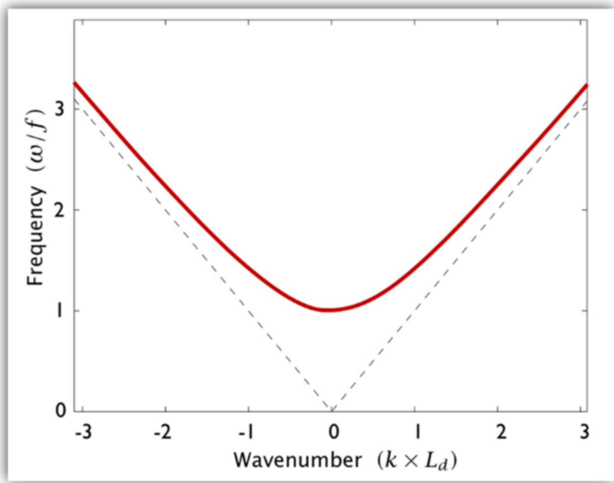
↳ It results in the dispersion relation for gravity waves in a rotating fluid:

$$\omega\left[\omega^2 - f^2 - gH(l^2 + m^2)\right] = 0$$

k is the horizontal wave number: $k^2 = l^2 + m^2$

⇨ The **solutions** are either:

- a steady geostrophic flow ($\omega = 0$, no oscillation in time ≡ a fixed stationary wave)
- a propagating wave that satisfies: $\omega = \pm\sqrt{f^2 + gHk^2}$



➢ If we set $f = 0$ we recover the dispersion relation for gravity waves in a non-rotating fluid (see **#GFD 4.1a**), i.e. non-dispersive waves with a constant phase speed $c = \pm\sqrt{gH}$.

➢ The additional f^2 under the square root means that the relationship between ω and l is **not linear anymore**.

↳ It is shown in the figure to the left: frequency as a function of wavenumber. When $k > 0$, the wave propagates in the positive x-direction, and when $k < 0$, it propagates in the opposite direction.

▪ Dashed lines are **non-dispersive gravity waves without rotation** (see **#GFD4.1a**).

▪ The red curve shows the system with rotation, i.e. adding f^2 under the square root in the dispersion relation. These are ***inertia-gravity or Poincaré waves***.

➢ For short-waves (large values of the wavenumber l) rotation does not make much difference to the way the waves propagate. They behave like ordinary, non-dispersive gravity waves.

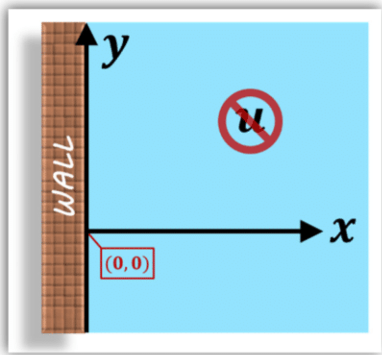
➢ For larger scales (wavelength much longer than the Rossby radius $\sqrt{f^2/gH}$), the **curve flattens out**, so the frequency has a **lower limit** of f, and the waves become **very dispersive**. At very small wavenumbers, the wave starts to behave rather oddly. As the horizontal scale of the wave becomes larger, the phase speed becomes faster. The slope of a line joining the origin to the curve gets steeper (see **#GFD3.1d**). But the group speed (the tangent to the curve, see **#GFD3.1d**) is equal to the phase speed for short waves and then for larger scales, it disappears. So, there is **no transmission of information from one position to another**, even though the **oscillations that are separated in space are perfectly coherent**. This is not really a wave anymore. It is **coherent oscillations in space separated by some distance**. In fact, it is just motion in **inertial circles**. This is why the waves are called inertia-gravity waves.

↳ It is a bit of a negative result. For large-scales, we have waves that basically collapse to inertial motion. ***We are left wondering if there is a way in which we can have large-scale propagating geophysical waves in a rotating planet***. The answer is yes, and they are called Kelvin waves (see **#GFD4.2** and **#GFD4.3**).

GFD4.2: Boundary Kelvin Waves

4.2.a) Adding a wall

⇨ We need to introduce a constraint to the equations to add a lateral boundary to the problem.



$$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0$$

We put a north-south wall on the western side of the Ocean and the flow cannot cross the wall, i.e. no flow perpendicular to the wall

⇨ In the x-direction, there is no flow through the wall (at $x = 0$), which suggests that we look for a solution with $\boldsymbol{u = 0}$ **everywhere**.

↳ This results in **geostrophic balance** (equilibrium between pressure $\mathbf{F_P}$ and Coriolis force $\mathbf{F_C}$) in the x-direction.

↳ In the y-direction, we recover the equations for non-rotating shallow-water gravity waves (see **#GFD4.1a**). So, in the y-direction, we have non-dispersive gravity waves propagating northwards or southwards with a fixed phase speed, independent of horizontal scales ($|c| = \sqrt{gH}$):

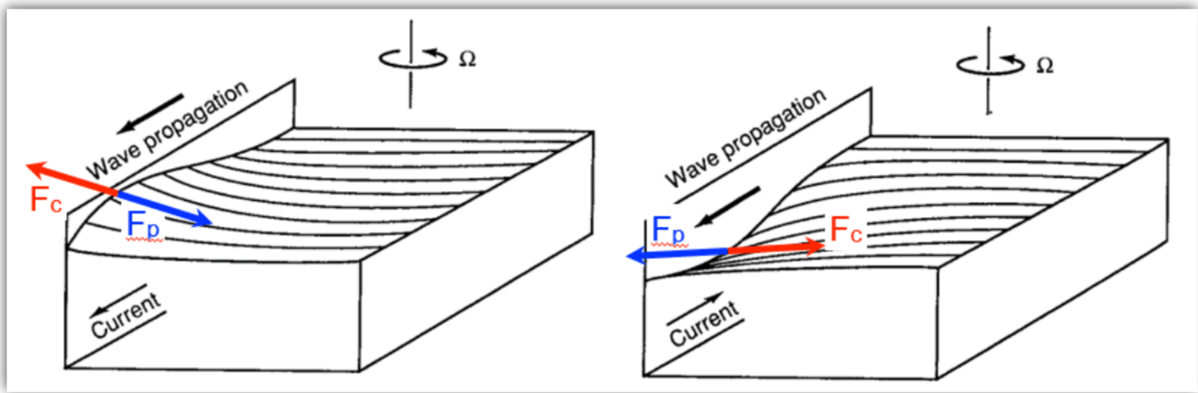
$$fv = g\frac{\partial \eta}{\partial x}$$

Diagnostic equation: ***Geostrophic balance***

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial v}{\partial y} = 0$$

Prognostic equations: ***Non-dispersive waves***



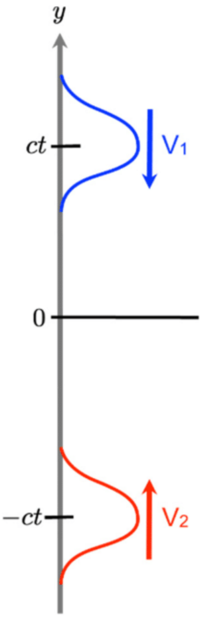
▪ If the fluid is heaped up against the wall, the pressure force will be pushing out into the fluid. The pressure gradient force and the Coriolis force will balance and in the northern hemisphere ($f > 0$) we will have southward flow.

▪ If there is a dip against the wall pressure, gradient force is pushing towards the wall and the Coriolis force balances this, so in the northern hemisphere ($f > 0$) the flow is northwards.

⤷ We will have **oscillations** between **northward** and **southward** flow alternating with the crests and troughs of the wave, and the whole thing is propagating like a gravity wave along the wall. These waves are ***coastal Kelvin waves***.

⇨ **On the diagram**, it is mentioned that the wave is **propagating southwards**. We have not proved that yet and to do so we need to make a closer examination of the geostrophic balance equation (see **#GFD4.2b**).

4.2.b) Geostrophic balance



⇨ Since the wave is ***non-dispersive***, all signals must travel at the same speed $c = \sqrt{gH}$. The solution for v at $y = 0$ and time t must be **the superposition of two independent waves traveling in opposite directions:**

- A wave coming from the north $V_1(x, y + ct)$
- A wave coming from the south $V_2(x, y - ct)$

All the wavelengths propagate at the same speed. A wave pattern will propagate at this speed

⤷ Anything at $y = 0$ and time t must consist of the sum of everything that was at a distance $c \times t$ either to the north or to the south. ***Anything else has either gone too far*** or ***has not arrived yet*** because there is only one speed that these waves can propagate at.

⇨ The corresponding surface displacement is $\eta = \sqrt{H/g}(-V_1 + V_2)$.

⤷ This can be shown by substitution into the y-momentum equation:

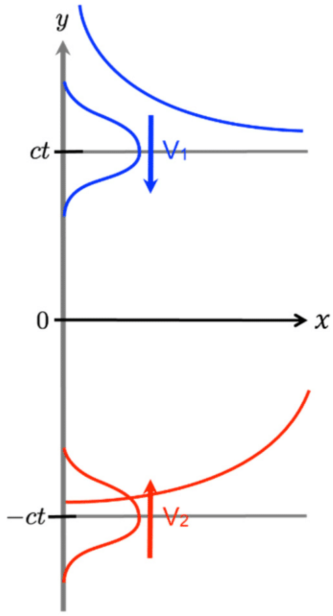
$$\frac{\partial}{\partial t}(V_1 + V_2) = -\sqrt{gH}\frac{\partial}{\partial y}(-V_1 + V_2) \qquad \rightarrow \frac{\partial V_1}{\partial t} = c\frac{\partial V_1}{\partial y}$$

$$\frac{\partial}{\partial t}(-V_1 + V_2) = -\sqrt{gH}\frac{\partial}{\partial y}(V_1 + V_2) \qquad \frac{\partial V_2}{\partial t} = -c\frac{\partial V_2}{\partial y}$$

⇨ To obtain the x-dependence of these functions, ***we use the diagnostic equation***, geostrophic balance, which gives:

$$\frac{\partial V_1}{\partial x} = -\frac{f}{\sqrt{gH}}V_1 \qquad \frac{\partial V_2}{\partial x} = \frac{f}{\sqrt{gH}}V_2$$

⤷These relations have ***exponential solutions*** with x: $V_1(x, ct)e^{-x/L_R}$ and $V_2 = V_2(x, ct)e^{x/L_R}$ respectively, with a scale distance of the Rossby radius $L_R = c/f$.



$x \geq 0$
f> 0

With $x \geq 0$ and $f > 0$:

- $\boldsymbol{V_1}$ has a **decaying** exponential solution in x with boundary layer width L_R.
- $\boldsymbol{V_2}$ **grows exponentially** away from the wall, and so fails to satisfy the condition of boundedness at infinity. This solution thus must be eliminated (for physical reasons).

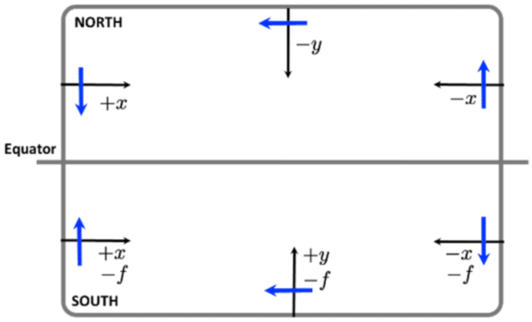
⤷ We thus retain the solution $\boldsymbol{V_1}$ that implies that **coastal Kelvin waves must propagate southwards** (negative in y direction) along a **wall on the western side** of the basin (x positive offshore) **in the northern hemisphere** ($f > 0$).

The scale of the costal Kelvin wave is the Rossby radius $L_R = c/f$ (see **#GFD1.2a**)

4.2.c) Properties of Kelvin waves

⇨ So far, we have considered Kelvin waves on the **western side** of an ocean basin in the northern hemisphere:
- our x-coordinate was positive towards the ocean basin center ($x \geq 0$)
- The planetary vorticity was positive ($f > 0$).

↳ The only admissible solution has a zonal structure decaying exponentially offshore e^{-x/L_R}

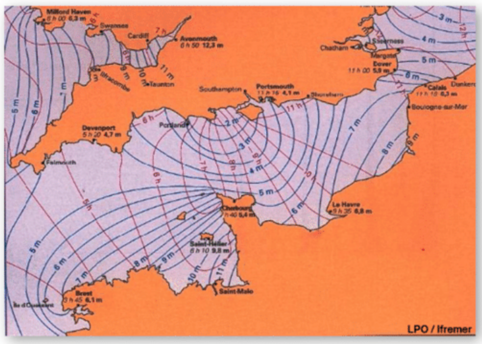


⇨ On the **eastern side** of the ocean basin, the x-coordinate is negative towards the center of the basin. Following the same logic, with V_1 and V_2, this change of sign leads to the conclusion that along a wall on the eastern side of the basin the wave must propagate northwards.

↳ In the **northern hemisphere** a Kelvin wave **keeps the coast to its right** as it is pushed against it by the Coriolis force.

⇨ As f-plane dynamics are isotropic in x and y, if we have geostrophic balance in the north-south direction, Kelvin waves propagating in the zonal direction, along the northern wall of the basin (negative y-coordinate) will propagate westwards. See *Vallis* (2017, #3.7) for the rotated equation system.

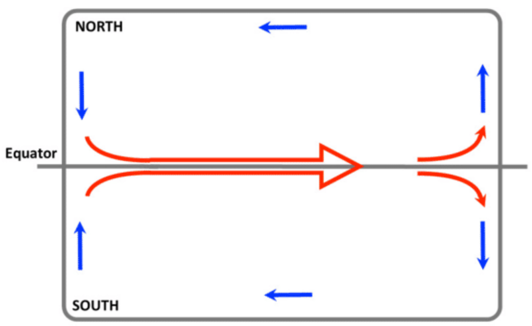
↳ **In the northern hemisphere Kelvin waves lean against the coast with the coast on their right as they propagate.**



As an illustration, this picture shows the English Channel where the tides come from the Atlantic Ocean through the channel. They can be described as coastal Kelvin waves, leaning against the French coast. The amplitude of the tidal variation is much higher on the French side than on the English side: up to 11 meters near St Malo, and only about 2 or 3 meters near Southampton. This explains why a tidal power plant was built on the French side.

⇨ In the **southern hemisphere** f changes sign, so all these considerations are reversed, and Kelvin waves propagate with the coast to their left.

↳ Kelvin waves propagate around the basin anti-clockwise in the northern hemisphere and clockwise in the southern hemisphere. We are left wondering what happens when **coastal Kelvin waves meet at the equator**, along the Brazilian coasts for instance. In fact, they can carry on along the equator as **equatorial Kelvin waves**, propagating eastwards along the equatorial waveguide. Then, at the eastern boundary (African coast), the equatorial Kelvin waves will continue poleward in each hemisphere.



📖 ***Imagine*** that you put a wall along the equator. It would be a southern boundary in the northern hemisphere and a northern boundary in the southern hemisphere. In both hemispheres, ***Kelvin waves can lean on this wall and propagate eastward***. Suddenly, the wall collapses, and the Kelvin waves in each hemisphere can just ***lean against each other*** as they travel eastwards along the equator. ***This can only happen on the equator where the sign of f changes***. See **#GFD4.3** for theoretical details.

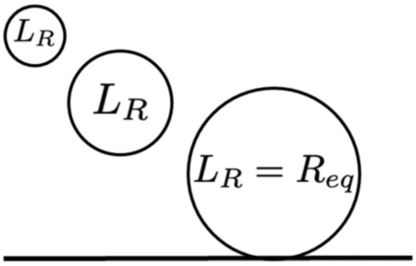
GFD4.3: Equatorial Scaling and Kelvin Wave Solution

4.3.a) Scales of motion near the Equator

⇨ Let's focus on the **equatorial region**. At the equator:

- The latitude is zero ($\phi = 0$)
- The planetary vorticity is also zero ($f = 0$).
- The variation of the Coriolis parameter with distance y in the north-south direction is maximum at the equator: $\beta = \frac{\partial f}{\partial y} = \frac{2\Omega}{a} cos\phi = 2.28 \times 10^{-11}\, m^{-1}. s^{-1}$.
- We can use the β approximation. Unlike in the extra-equatorial regions where $f = f_0 + \beta y$, at the equator $f_0 = 0$ and thus the β-approximation is reduced to $f = \beta y$.

⇨ **The Rossby radius** is the typical scale of motion: $L_R = \frac{NH}{f} = \frac{\sqrt{g'H}}{f} = \frac{c}{f}$. In the mid-latitudes, this works fine. Close to the equator, it does not work as a useful scale because $f = 0$. We thus have to consider another way of imagining what the relevant scale is at the equator.



⇨ On the left is a sketch showing **circles** that describe the **Rossby radius** as a function of latitude. As f decreases approaching the equator, the Rossby radius gets larger.

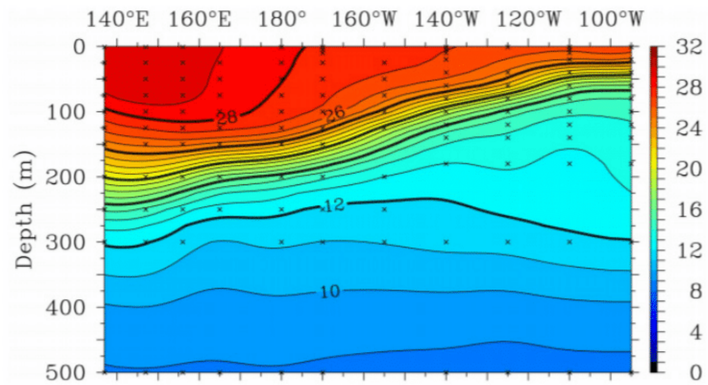
↳ There is a critical latitude $y = R_{eq}$, where the Rossby circle first touches the equator and the center of the circle is exactly the Rossby radius away from the equator. This distance R_{eq} can be defined as the **equatorial radius**.

↳ Using the β-approximation at $y = R_{eq}$ $(f = \beta R_{eq})$ and $R_{eq} = L_R = \frac{c}{\beta R_{eq}}$, it follows that:

$$R_{eq} = \sqrt{\frac{c}{\beta}}.$$

⇨ To **evaluate the amplitude** of this scale of motion at the equator, we have to quantify the phase speed of the gravity waves: $c = \sqrt{g'H}$.

↳ The diagram on the right presents the temperature as a function of depth and longitude in the equatorial Pacific, from in-situ TAO data. The region of strong vertical gradients is called the thermocline. In a conceptual one-layer shallow water model, it separates the active layer from the resting abyss layer.



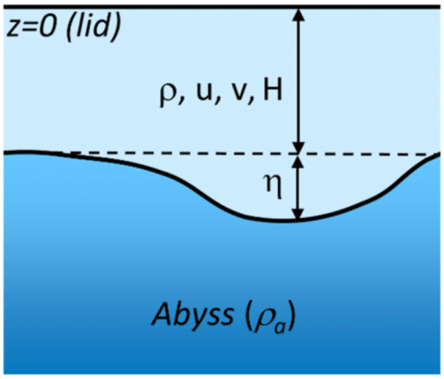
- The change in density across the thermocline is $\sim 2kg/m^3$ for an average density $\sim 1000kg/m^3$. This gives a **reduced gravity** of the order of $g' = g\frac{\Delta\rho}{\rho} \sim 0.02m. s^{-2}$.
- The typical **depth of the thermocline** in the equatorial Pacific changes quite a lot with longitude. It ranges from a couple of hundred meters in the west to a few tens of meters in the eastern basin. We approximate $H \sim 100m$.

↳ Internal gravity waves propagating on the thermocline have a **phase speed**

$$c = \sqrt{g'H} \sim \sqrt{2} = 1.4\, m. s^{-1}$$

- The equatorial **radius of deformation** is $R_{eq} = \sqrt{\frac{c}{\beta}} \sim 250\text{km} \sim 2.2^0$
- The time T_{eq} for a wave to travel distance R_{eq} is $T_{eq} = \frac{1}{\sqrt{\beta c}} \sim 2$ days

4.3.b) Linear equatorial shallow water model



⇨ Let's analyze the **shallow water equations** for a conceptual equatorial model consisting of one active layer with a rigid lid overlying a motionless abyss.

• We use the $\boldsymbol{\beta}$ **approximation** at the equator: $f = \beta y$. The shallow water equations (see **#GFD1.2g**) are:

$$\frac{\partial u}{\partial t} - \beta y v = -g' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \beta y u = -g' \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

↳ This slight change, going from an $\boldsymbol{f}$**-plane** with a constant coefficient f (see **#GFD4.1b**) to an equatorial β-plane in which there is a y tin the Coriolis term will yield some slightly different results.

4.3.c) The equatorial Kelvin wave solution

⇨ We start with **a special case** in which there is no flow across the equator. This is the **Kelvin wave framework**, just like when we had a wall (see **#GFD4.2a**). The flow is along the equator (u) and <u>$\boldsymbol{v = 0}$ everywhere</u>.

⇨ The equations simplify to:

$$\cancel{\frac{\partial v}{\partial t}} + \boxed{\beta y u = -g' \frac{\partial \eta}{\partial y}}$$

Cross-equatorial Geostrophic balance

$$\frac{\partial u}{\partial t} - \cancel{\beta y v} = -g' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}\right) = 0$$

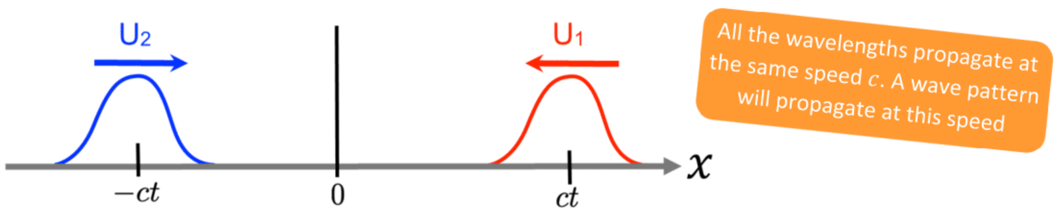
Non-dispersive waves $c = \sqrt{g'H}$

↳ We obtain a set of equations similar to the one we had for the coastal Kelvin waves (see **#GFD4.2a**), i.e. <u>2 prognostic equations</u> consistent with **non-dispersive waves** propagating along the equatorial wave-guide with a phase speed $c = \sqrt{g'H}$ (✋) and a <u>diagnostic equation</u> for **cross-equatorial geostrophic balance**, that will determine the meridional structure of the waves.

⇨ Using the same logic as before (see **#GFD4.2b**), the solution for u at $x = 0$ and time t can consist only of **the superposition of two independent waves traveling in opposite directions:**

- A wave propagating ***westwards*** $U_1(x + ct, y)$
- A wave propagating ***eastwards*** $U_2(x - ct, y)$

↳ Anything at $x = 0$ and time t must consist of the sum of what was at a distance $c \times t$ either to the east or to the west. Anything else has either gone too far or has not arrived yet because there is only one speed (c) at which these waves can propagate.



⇨ The solution for η can be written in terms of ***U₁*** and ***U₂***, as $\eta = \sqrt{\frac{H}{g'}}(-U_1 + U_2)$

This can be verified by substituting the solution into the prognostic equations.

⇨ As in **#GFD4.2b**, the **cross-equatorial geostrophic balance** informs us about the **meridional structure of the wave** and we will consequently discard one solution.

↳ To proceed, we substitute the solutions for $u(x,y,t)$ and $\eta(x,y,t)$ into the diagnostic equation. The following expressions must be satisfied separately:

$$\beta y U_1 = c\frac{\partial U_1}{\partial y} \qquad \beta y U_2 = -c\frac{\partial U_2}{\partial y}$$

⇨ These relations have exponential solution with y: $U_1 \sim e^{\frac{\beta}{2c}y^2}$ and $U_2 \sim e^{-\frac{\beta}{2c}y^2}$ respectively.

↳ Only the solution $\boldsymbol{U_2}$, i.e. the ***wave propagating eastward***, is ***exponentially decaying*** in y^2.

✋ Note the difference with coastal waves that depended on non-zero f, and thus, y. Now we have a y^2 dependence (because of the extra y in the equation set, see **#GFD4.3b**). The decay works both to the north and south with the same propagation direction.

⇨ The scale distance of these waves is $\sqrt{\beta/2c} = \sqrt{1/2R_{eq}^2}$ (see **#GFD4.3a**) and it is symmetric about the equator. Kelvin waves decay away from the equator regardless of whether y is negative or positive. Equatorial Kelvin waves are ***equatorially trapped***.

4.3.d) Equatorial Kelvin wave properties

⇨ The **equatorial Kelvin wave** solutions for the three variables (u, v, η) can be written as a function of ψ, a dimensionless waveform that propagates in the x-direction:

$$u = c\,\psi(x - ct)\, e^{-y^2/2R_{eq}^2}$$
$$\boxed{v = 0}$$
$$\eta = H\,\psi(x - ct)\, e^{-y^2/2R_{eq}^2}$$

⇨ **Kelvin waves are a special solution for equatorial waves** for which $v = 0$.

They are:
- non-dispersive waves
- trapped at the equator
- propagate eastwards at a phase speed $c = \sqrt{g'H}$

GFD4.4: Equatorial Waves – General Solution

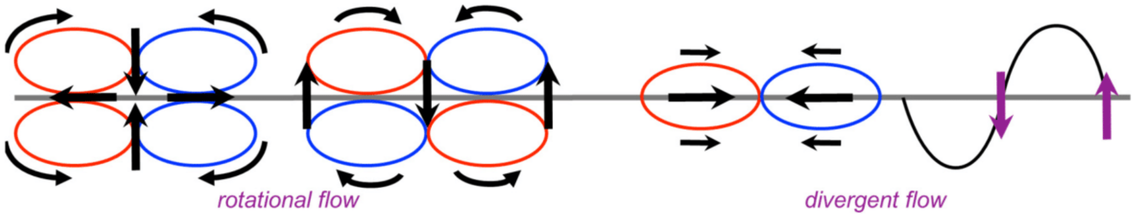
4.4.a) The general solution

⇨ To derive the general solutions for equatorial waves, we substitute **wave-like solutions** into the **equatorial shallow water equations** (see **#GFD4.3b**):

$$\frac{\partial u}{\partial t} - \beta y v = -g'\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + \beta y u = -g'\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Expected wave structures

⇨ We first consider **different types of wave structures** that we might encounter. Below, colored circles represent the geostrophic stream function (η, the depth of the thermocline in the Ocean or surface pressure in the Atmosphere). Positive η anomalies are in red, negative in blue, associated with anti-cyclonic and cyclonic circulation respectively in the northern hemisphere.



• The first example (left sketch) is symmetric about the equator, i.e. positive/negative anomalies in η north and south of the equator. The rotational flow around a positive anomaly is clockwise in the northern hemisphere and anti-clockwise in the southern hemisphere. It is the other way around for the negative anomalies.

↳ η is symmetric and so is the zonal current (u). But the meridional current (v) is anti-symmetric about the equator, so v is out of phase by a quarter of a wavelength – in **quadrature**.

• The second scenario (middle sketch) shows anti-symmetric thermocline perturbations associated with symmetric v. Extrema in v are **displaced by a quarter of the wavelength**.

• Not all the solutions have <u>rotational</u> type flow. Let's picture a <u>divergent</u> type flow. The third sketch exemplifies the equatorial Kelvin wave solution. $v = 0$ and there is a convergence/divergence between equatorial positive and negative η anomalies. It is associated with downward/upward movement of the stream function. This configuration is consistent with eastward propagation.

Wave-like solutions

⇨ We thus look for **wave-like solutions** that ***propagate along the equator*** (x-direction, positive or negative). For v, we have added this $\frac{\pi}{2}$ as a phase displacement for the solution, so:

$$u = \tilde{u}(y)e^{i(lx-wt)}$$
$$v = \tilde{v}(y)e^{i\left(lx-wt+\frac{\pi}{2}\right)}$$
$$\eta = \tilde{\eta}(y)e^{i(lx-wt)}$$

l is the zonal wavenumber

☝ The amplitude coefficient depends on y. This is going to complicate the resolution.

Harmonic (wave) equation

↳ We substitute these solutions into the **equatorial shallow water equations** and (after a laborious process, see details on the next pages) we obtain a ***harmonic (second-order differential) equation*** for the amplitude $\tilde{v}(y)$:

$$\frac{d^2\tilde{v}}{dy^2} + \frac{\beta^2}{c^2}(Y^2 - y^2)\tilde{v} = 0 \qquad Y^2 = \left(\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega}\right)\frac{c^2}{\beta^2}$$

$c = \sqrt{g'H}$ is a constant

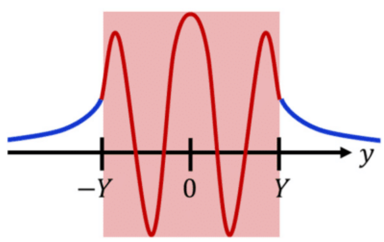
☝ It is not an algebraic linear equation because of the y-dependence in of $\tilde{v}$.

⇨ In this equation, there is a coefficient $\frac{\beta^2}{c^2}(Y^2 - y^2)$ in front of $\tilde{v}$. It can be positive or negative depending on the sign of $Y^2 - y^2$.

▪ **<u>In the vicinity of the equator</u>**, $y^2 < Y^2$, the coefficient is positive and the spatial structure of the solution will be oscillating in y.

▪ **<u>Outside the Y-bound</u>** ($y^2 > Y^2$), the coefficient is negative and the equation admits exponentially decaying solutions.

↳ This portrays a picture of the zonally-propagating wave where near the equator the spatial structure is an oscillation and at a certain distance, it just decays away to zero. The **wave is trapped at the equator** and Y is the **width of the equatorial waveguide**. As the expression of Y depends on various properties of the wave ($\omega, \beta, c = \sqrt{g'H}, l$), different waves will have different widths but it basically scales the wave structure in a similar way to the equatorial radius (R_{eq}) for the Kelvin waves.



Derivation of the equatorial wave equation for v

$$\left\{\begin{aligned}\frac{\partial u}{\partial t}-\beta y v&=-g'\frac{\partial \eta}{\partial x}\\ \frac{\partial v}{\partial t}+\beta y u&=-g'\frac{\partial \eta}{\partial y}\\ \frac{\partial \eta}{\partial t}+H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)&=0\end{aligned}\right.$$

$$u=\tilde{u}(y)e^{i(kx-\omega t)} \qquad v=\tilde{v}(y)e^{i(kx-\omega t\pm\pi/2)}$$

$$\eta=\tilde{\eta}(y)e^{i(kx-\omega t)} \qquad =\tilde{v}(y)e^{i(kx-\omega t)}e^{\pm i\pi/2}$$

$$=\pm i\tilde{v}(y)e^{i(kx-\omega t)}$$

v in quadrature with u,
+ or - makes no difference, we choose +

$$\left\{\begin{aligned}-i\omega\tilde{u}-i\beta y\tilde{v}+ig'k\tilde{\eta}&=0\\ \omega\tilde{v}+\beta y\tilde{u}+g'\frac{\partial\tilde{\eta}}{\partial y}&=0\\ -i\omega\tilde{\eta}+H\left(ik\tilde{u}+i\frac{\partial\tilde{v}}{\partial y}\right)&=0\end{aligned}\right.$$

We want to eliminate u and η to get an equation for v.

We drop tildes and prime on g, and we use subscript notation for derivatives. The linear system can be written:

$$\left\{\begin{aligned}\omega u+\beta y v-gk\eta&=0 \qquad (1)\\ \omega v+\beta y u+g\frac{\partial \eta}{\partial y}&=0 \qquad (2)\\ -\omega\eta+Hku+H\frac{\partial v}{\partial y}&=0 \qquad (3)\end{aligned}\right.$$

$$\left\{\begin{aligned}\partial/\partial y(1)+k\times(2)&\rightarrow \omega u_y+\beta v+\beta y v_y+\omega k v+\beta y k u=0 \qquad (A)\\ \omega\times(2)+g\times\partial/\partial y(3)&\rightarrow \omega^2 v+\beta y\omega u+gHku_y+gHv_{yy}=0 \qquad (B)\\ \omega\times(1)-gk\times(3)&\rightarrow \omega^2 u+\beta y\omega v-gk^2Hu-gkHv_y=0 \qquad (C)\end{aligned}\right.$$

⇨ $gHk\times(A)-\omega\times(B)\rightarrow$

$$gHk(\beta v+\beta y v_y+\omega k v)+gHk^2\beta y u-\omega^3 v-\beta y\omega^2 u-gH\omega v_{yy}=0$$

$$-gH\omega v_{yy}+gHk\beta y v_y+(gHk\beta+gH\omega k^2-\omega^3)v+(gHk^2\beta y-\beta y\omega^2)u=0 \qquad (D)$$

⇨ $(D)+\beta y\times(C)\rightarrow$

$$-gH\omega v_{yy}+gHk\beta y v_y+(gHk\beta+gH\omega k^2-\omega^3)v+\beta^2y^2\omega v-\beta y gHk v_y=0$$

⇨ $\div -gH\omega \rightarrow \dfrac{d^2\tilde{v}}{dy^2}+\left[\dfrac{\omega^2}{g'H}-k^2-\dfrac{k\beta}{\omega}-\dfrac{\beta^2}{g'H}y^2\right]\tilde{v}=0$

or $\dfrac{d^2\tilde{v}}{dy^2}+\dfrac{\beta^2}{c^2}(Y^2-y^2)\tilde{v}=0$ where $\left\{\begin{array}{l} c \text{ is the gravity wave speed)} \\ Y^2=\dfrac{g'H}{\beta^2}\left[\dfrac{\omega^2}{g'H}-k^2-\dfrac{k\beta}{\omega}\right]\end{array}\right.$

4.4.b) Meridional structure

⇨ The **general solutions** of the $\tilde{v}$ harmonic equation have the form of a discrete set of meridional structures:

$$\tilde{v} \propto H_n(y')e^{-\frac{y'^2}{2}} \qquad (y' = y/R_{eq})$$

$$R_{eq} = \sqrt{c/\beta}$$

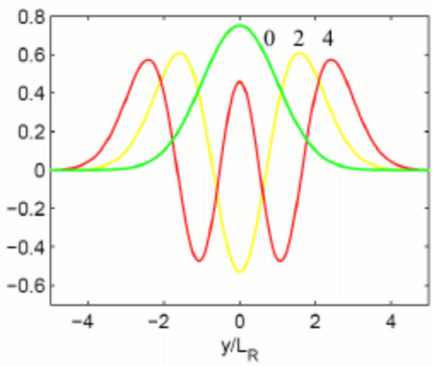
↳ $\tilde{v}$ is the product of a ***Hermite polynomial*** (H_n) and a ***decaying exponential***. This is similar to what we had for the meridional structure of equatorially-trapped Kelvin waves (see **#GFD4.3c**), except that now it is multiplied by the Hermit polynomial. The product of these two functions is called a **parabolic cylinder functions**.

⇨ The **Hermit polynomials** are defined as:

$$H_0(y') = 1$$
$$H_1(y') = 2y'$$
$$H_2(y') = 4y'^2 - 2$$
$$H_3(y') = 8y'^3 - 12y'$$
$$H_4(y') = 16y'^4 - 48y'^2 - 12 \quad ...$$

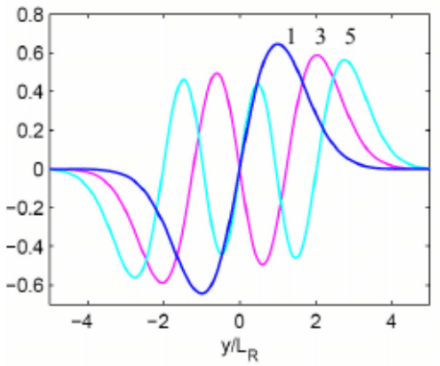
↳ They have mathematically-useful properties such as H_n can be written in term of the previous and the next polynomial and there is a simple expression of the derivative:

$$y'H_n = nH_{n-1} + \frac{1}{2}H_{n+1} \qquad \frac{dH_n}{dy'} = 2nH_{n-1}$$



• **For even numbers n**, $H_n(y')$ is symmetric about the equator and so is $\tilde{v}$. The symmetric meridional structures are illustrated on the left for $n = 0, 2, 4$. For $n = 0$, $\tilde{v} \propto e^{-y'^2/2}$, for $n = 2$ there is a single wiggle in the middle, for $n = 4$ there are two wiggles in the middle, etc…

↳ These structures are associated with a ***cross-equatorial flow and create thermocline displacements asymmetric with respect to the equator*** (i.e. a maximum in the thermocline displacement on one side of the equator, and a minimum on the other side).



• **For odd numbers n** ($n = 1, 3, 5, ...$), the meridional structures of the equatorial wave are anti-symmetric in v and changes sign across the equator. As n increases, the number of wiggles increases too.

↳ These structures are associated with ***no cross-equatorial flow (convergence/divergence) and symmetric thermocline displacements***.

✋ The ***Kelvin wave solution*** is not in this set of solutions, because it is associated with $v = 0$. You have to consider $n = -1$ and in that case, u and η will be symmetric.

4.4.c) The dispersion relations

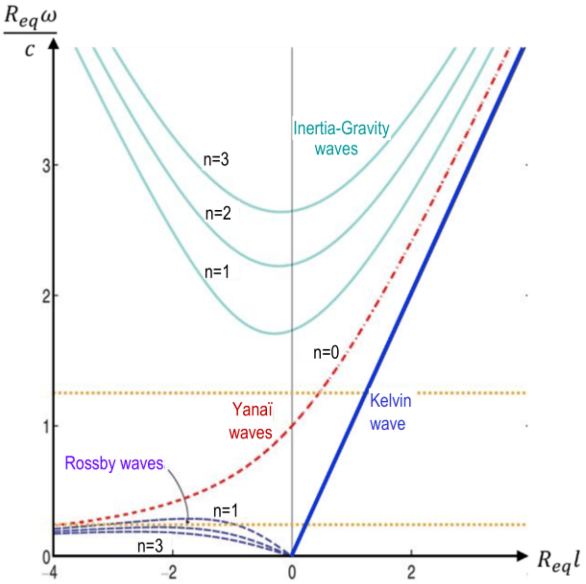
⇨ To derive the propagation properties, we substitute the general parabolic cylinder function solutions into the differential equation (see **details** on the following page). This leads to a ***set of dispersion relations***:

$$\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega} = (2n+1)\frac{\beta}{c} = \frac{(2n+1)}{{R_{eq}}^2}$$

Here, $c = \sqrt{g'H}$ is a constant
$\frac{\omega}{l}$ is the phase speed of the wave

↳ These are the ***theoretical dispersion relations*** for shallow water modes on the equatorial β-plane. This is a **family of dispersion relations** $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{l}, \boldsymbol{n})$ for distinct tropical waves (including equatorial Kelvin waves) associated with discreet values of n (index for the Hermite polynomials).

⇨ As the dispersion relations are cubic in ω, there are **3 mathematical roots** for each value of $n \geq 1$. Positive roots are represented below: ω as a function of the zonal wavenumber l. For $l > 0$, equatorial waves propagate eastwards, while for $l < 0$, they propagate westwards.



• For high frequencies ($\omega \gg 1$), we can neglect $\frac{\beta l}{\omega}$ to obtain $\omega^2 = (2n+1)\beta c + c^2 l^2$. This is a dispersion relation for ***equatorial inertia-gravity waves***. They propagate in either direction and are similar to inertia-gravity waves in mid-latitudes (see **#GFD4.1c**). The dispersion curve has a minimum frequency that depends on n. It is slightly off-centered (not completely symmetric) because there is the β-effect in the equations.

• For low frequencies, we can neglect $\frac{\omega^2}{c^2}$ to obtain $\omega = -\frac{\beta l}{l^2 + (2n+1)R_{eq}^{-2}}$. These are ***equatorial Rossby waves*** similar in principle to their counterparts in mid-latitudes (see **#GFD3.1f**) that critically depend on the β-effect. They propagate westwards, but their group speed can be eastwards for high wavenumbers.
✋ There is a mathematical solution for a negative ω.

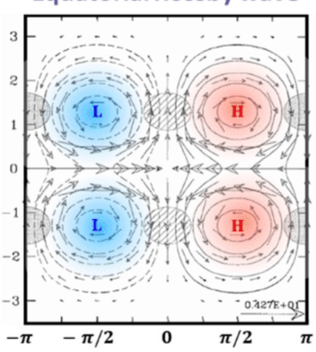
• Between inertia-gravity and Rossby wave solutions, there is a hybrid solution for $n = 0$. At low frequencies, this wave propagates westwards and behaves like a Rossby wave. And at higher frequencies, it propagates eastwards and behaves like a gravity wave. This wave is called a ***mixed Rossby gravity wave*** or Yanaï wave.

• The blue straight line corresponds to non-dispersive waves that propagate eastwards along the equator. These are the ***equatorial Kelvin waves*** (see **#GFD4.3**) which behaves like a gravity wave in absence of rotation (see **#GFD4.1a**).

4.4.d) Waves properties

⇨ Let's have a look at the two-dimensional structure of some of these tropical waves.

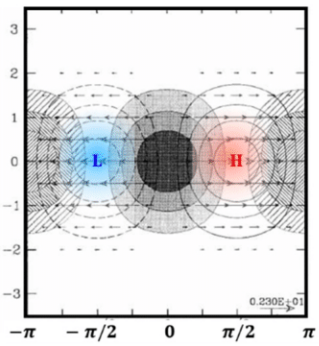
$\boldsymbol{n = 1,\ l^* = 1}$
Equatorial Rossby wave



• First is the **Rossby wave structure** for $n = 1$. Contours shows the north-south **symmetric $\boldsymbol{\eta}$** (the stream function, or dips in the thermocline or high-pressure areas in the atmosphere). The geostrophic flow circulates around η perturbations and arrows depict the **anti-symmetric meridional flow**. This wave propagates towards the west.

• Putting $n = -1$ into the dispersion relation yields an eastward-propagating non-dispersive Kelvin wave with $v = 0$, associated with alternate **convergent/divergent** zonal flow.

$\boldsymbol{n = -1, l^* = 1}$
Equatorial Kelvin wave



Derivation of the dispersion relations

$$\frac{d^2v}{dy^2} + \frac{1}{R_{eq}^4}(Y^2 - y^2)v = 0$$

$$R_{eq} = \sqrt{\frac{c}{\beta}}, \quad Y^2 = \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right) R_{eq}^4 \quad \{= (2n+1)R_{eq}^2\}$$

$$y' = y/R_{eq}, \quad Y' = Y/R_{eq} \rightarrow \quad \frac{1}{R_{eq}^2}\frac{d^2v}{dy'^2} + \frac{1}{R_{eq}^4}(Y'^2 - y'^2)vR_{eq}^2 = 0$$

dropping primes $\frac{d^2v}{dy^2} + (Y^2 - y^2)v = 0$ solution $v = H_n e^{-y^2/2}$

should lead to non-dimensional dispersion relation $Y^2 = 2n + 1$

using $\frac{dH_n}{dy} = 2nH_{n-1}$ and $yH_n = nH_{n-1} + \frac{H_{n+1}}{2}$

$$\frac{dv}{dy} = \frac{dH_n}{dy}e^{-y^2/2} - yH_ne^{-y^2/2} = \left[\frac{dH_n}{dy} - yH_n\right]e^{-y^2/2}$$

$$\frac{dv}{dy} = \left[2nH_{n-1} - \left(nH_{n-1} + \frac{H_{n+1}}{2}\right)\right]e^{-y^2/2} = \left[nH_{n-1} - \frac{H_{n+1}}{2}\right]e^{-y^2/2} = [yH_n - H_{n+1}]\,e^{-y^2/2}$$

$$\frac{d^2v}{dy^2} = \left[H_n + y\frac{dH_n}{dy} - \frac{dH_{n+1}}{dy} - y\,(yH_n - H_{n+1})\right]e^{-y^2/2}$$

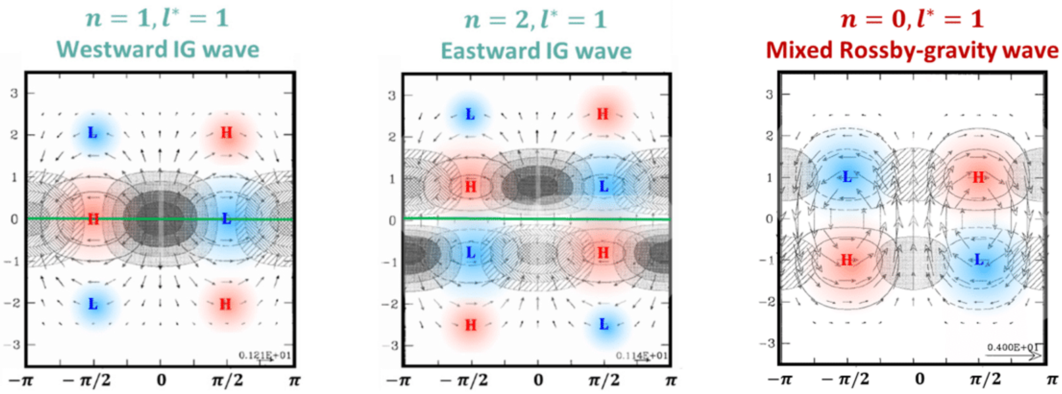
$$= [H_n + 2ynH_{n-1} - 2(n+1)H_n - y^2H_n + yH_{n+1}]\,e^{-y^2/2}$$

$$= [H_n + y\,(2nH_{n-1} - yH_n + H_{n+1}) - 2(n+1)H_n]\,e^{-y^2/2}$$

$$= [-H_n - 2nH_n + y^2H_n]\,e^{-y^2/2}$$

so $[y^2 - (2n+1)]\,H_ne^{-y^2/2} + (Y^2 - y^2)H_ne^{-y^2/2} = 0$

thus $Y^2 = 2n + 1$



• **Inertia-gravity waves** have divergent structures and they can propagate either eastwards or westward. Eastward/westward structures look fairly similar but they are not exactly the same because there is a slight asymmetry between the two directions.

• **Mixed Rossby gravity waves** have anti-symmetric structures in η and a mixture of rotational and divergent flow. They can propagate either eastwards or westwards.

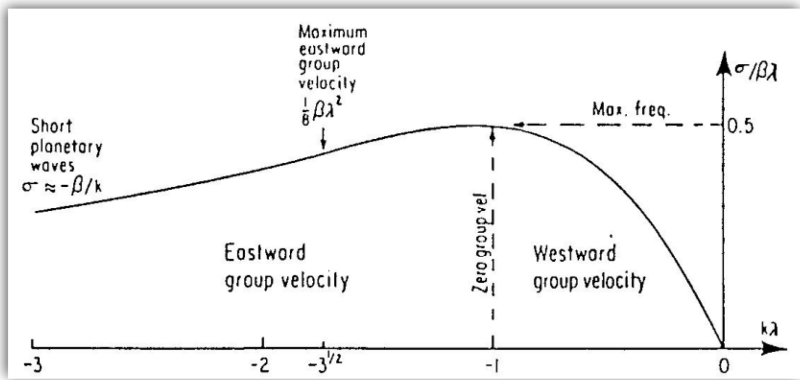
GFD4.5: Equatorial Waves – Special Cases and Examples

4.5.a) Equatorial Rossby waves

⇨ At low frequencies (long-wave approximation), close to the origin point (where $\omega = 0$), $\omega \ll f$. In the theoretical dispersion relationships for shallow water modes on the equatorial β plane ($\omega^2/c^2 - l^2 - \beta l/\omega = (2n+1)R_{eq}^{-2}$, see **#GFD4.4c**) we can **neglect the first term in ω^2** and it rearranges to:

$$\omega = -\frac{\beta l}{l^2 + (2n+1)R_{eq}^{-2}}$$

↳ This is the dispersion relation for long equatorial Rossby waves. It is very similar to mid-latitude barotropic Rossby waves (see **#GFD3.1**). We now have $(2n+1)R_{eq}^{-2}$ in the denominator where we had L_R^2 (the Rossby radius) for the mid-latitude Rossby waves in a one-layer model (see **#GFD3.1f**).



▪ For shorter waves the phase speed remains westward, but the group speed becomes eastward. These short-waves are of little importance because they are very dispersive. They are hard to observe because they are slow and dissipate.

▪ As in **#GFD3.1f**, for small wavenumber ($l \ll 1$, **long-waves**) equatorial Rossby waves are **almost non-dispersive** ($\omega = -\frac{cl}{2n+1}$) and can be detected on equatorial Rossby rays.

Non-dispersive waves: all the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape

▪ The dispersion relation is fairly straight at the origin and then curves round to a maximum frequency. The latter corresponds to zero group speed.

4.5.b) Equatorial Rossby rays

⇨ It is important to recall that as the wave propagates its dispersion relation changes. This is because the wave may change latitude, and f enters into the dispersion relation.

↳ The propagation of Rossby waves can be traced along the equatorial Rossby rays by computing the ratio of the group speeds (as in **#GFD3.2c** or **#GFD3.3c**). This provides the trajectory $\frac{dx}{dy}$. We consider here that f is "slowly varying". For long Rossby waves, the direction of the group speed is given by:

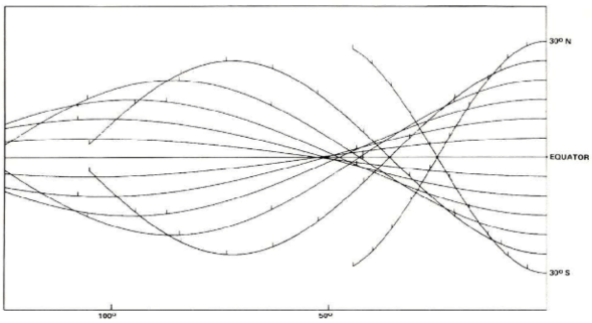
$$\frac{dx}{dy} = \frac{\partial\omega/\partial n}{\partial\omega/\partial l} = -\frac{2\omega}{\beta}\left(-\frac{\beta l}{\omega} - \frac{\beta^2 y^2}{c^2}\right)^{1/2}$$

⇨ The integral yields an equation for the latitude: $y = -\left(\frac{c^2 l}{\omega\beta}\right)^{\frac{1}{2}} cos\left(\frac{2\omega}{c}x + \theta_0\right)$

↳ Rossby waves of constant frequency and zonal wavenumber will **change their meridional wavenumber** and thus their direction of propagation.

• They end up oscillating about the equator by **refraction**: just like if the equator had a high-refractive-index and waves were attracted towards regions of high-refractive-index. It is another way to show that equatorial Rossby waves are **equatorially trapped**.

• This behavior is modified by the presence of a mean flow (current, winds).



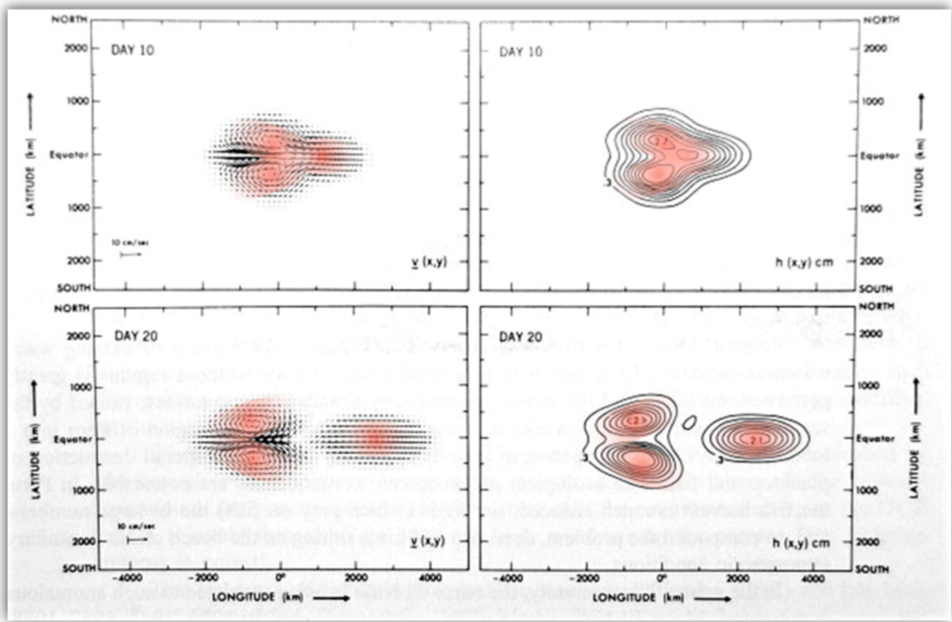
4.5.c) Oceanic adjustment

⇨ Here is a ***practical example*** illustrating how the Ocean will adjust to an initial thermocline perturbation. Below are snapshots from an ocean shallow water model simulation.

• In this experiment, at $t = 0$, a bell-shaped perturbation to the thermocline is allowed to dissipate. Initially, the thermocline has been artificially pushed down (downwelling) in a symmetric round-blob like structure, resembling a Gaussian or cosine squared.

• 10 days later, the induced flow field (a) and thermocline displacement (b) emerge. We observe a symmetric single bulge Kelvin wave ($n = -1$) propagating eastwards, while a symmetric dipole associated with a double bulge Rossby wave ($n = 1$) propagates westwards.

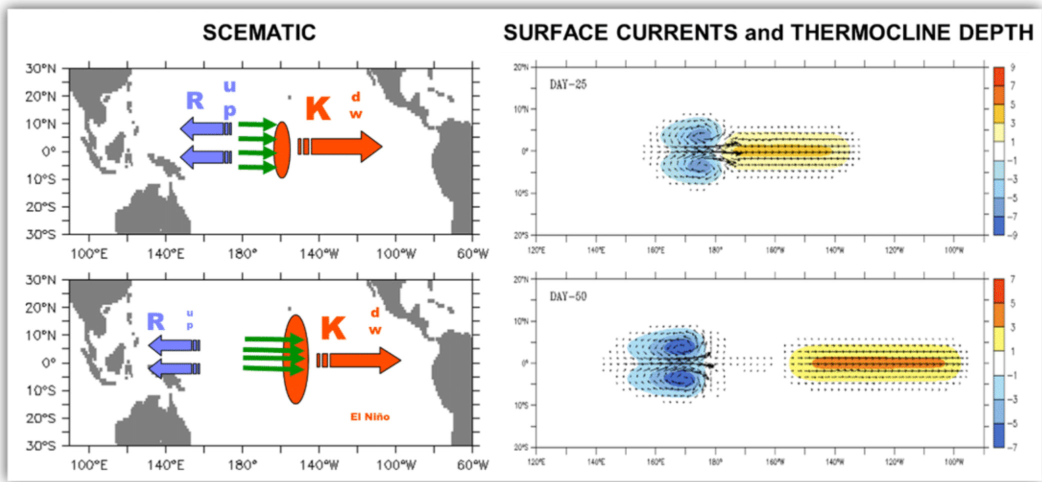
• At $t = 20$ days, the two structures are completely separated. The equatorial Kelvin wave propagate faster than the equatorial Rossby wave. This is consistent with the slope of the dispersion relation for long-wave approximation (☝ see **#GFD4.4c**).



4.5.d) ENSO theories: the delayed oscillator

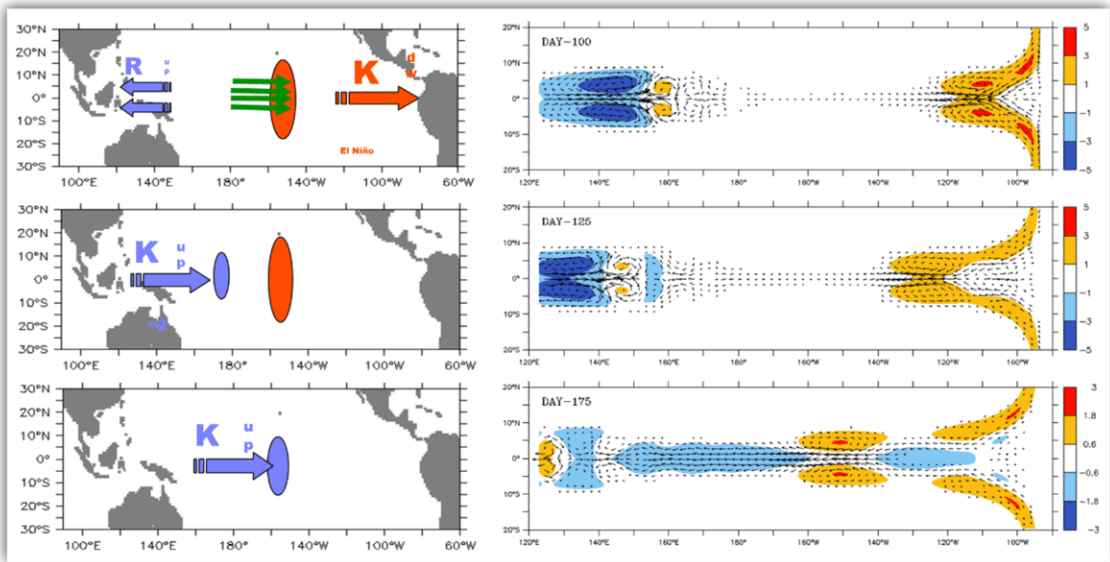
⇨ This was put forward as a candidate for the explanation for the **life-cycle of El Niño**.

↳ **El Niño** is a perturbation to the depth of the thermocline and the sea surface temperature in the eastern equatorial Pacific Ocean. **El Niño** starts very often with an **abrupt change in the surface wind-stress** forcing, a westerly wind anomaly in the equatorial western basin. The thermocline is pushed down (***downwelling***) ahead of the wind anomaly and pulled up (***upwelling***) behind it. The thermocline perturbations propagate eastwards as a ***downwelling Kelvin wave*** and westwards as an ***upwelling Rossby wave***. The combined effects of the two waves will tend to **flatten out the thermocline**, resulting in a **warming** of the eastern Pacific temperatures.



▪ A ~couple of months later, the ***downwelling Kelvin wave*** arrives at the South-American coast. Part of its energy is transmitted along the coast as coastal Kelvin waves, but a significant part reflects into a **downwelling Rossby wave**. This amplifies the deepening of the thermocline in the eastern equatorial basin and increases the sea surface warming.

▪ When the ***wind-forced upwelling Rossby wave*** arrives at the other coast, it reflects and transforms itself into an eastward-propagating ***upwelling Kelvin wave***. (In its third life, it will also reflect into an upwelling Rossby wave.) Along their propagation, these ***upwelling waves*** will raise the thermocline depth and thus cancel the original wind-forced perturbation.



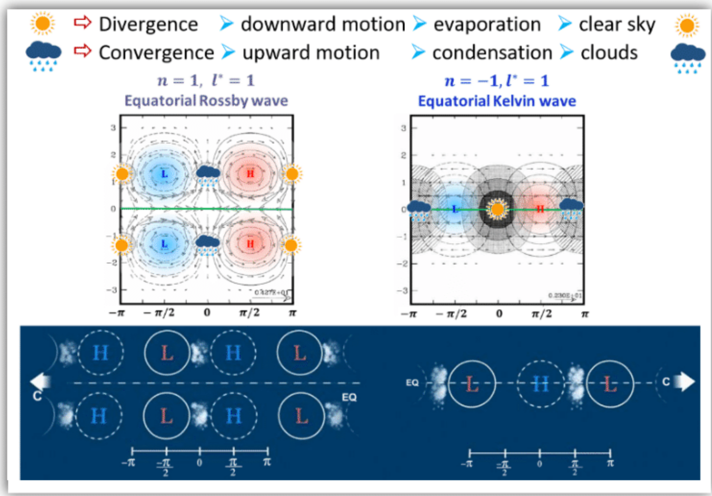
↳ This theory is called the __**delayed oscillator theory**__: **The initial El Niño perturbation sows the seeds of its own destruction** a few months later after these waves have crossed the Pacific Ocean and come back again.

⇨ It is __**one of the theories**__ to explain the **El Niño cycle** but __it is not the only one__.

4.5.e) Tropical convection in the atmosphere

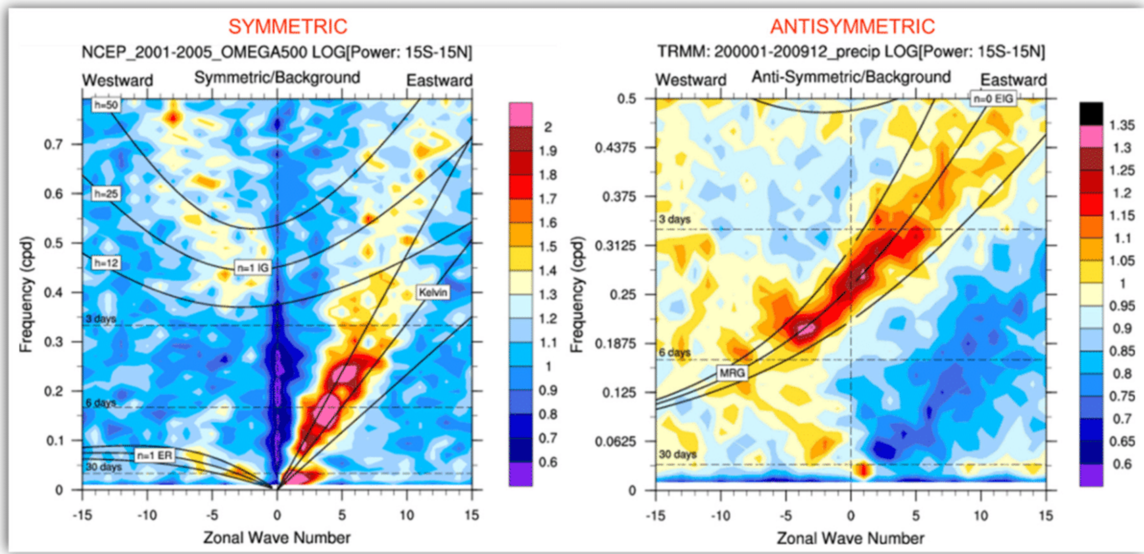
⇨ Here is an example for the **atmosphere**. The equatorial theory developed in this chapter was presented more from an oceanographic perspective, but it works just as well in the atmosphere.

↳ The main difference is that the **equatorial radius is much wider**: the geophysical parameters are such that the equatorial zone in the atmosphere is not just a few hundred kilometers (see **#GFD4.3a**) but is ~ the tropical band, namely 20°S-20°N (the width of a Hadley cell).



⇨ The figure below is the ***Wheeler-Kiladis diagram*** (from Wheeler and Kiladis, 1999). Observed atmospheric tropical variability is plotted as a function of its zonal and temporal scale. The red noise background is first filtered-out (this removes substantial part of the variance that does not have much structure) and the variable is separated into its meridionally-symmetric and anti-symmetric components with respect to the equator.

↳ For instance, the symmetric component of vertical velocity (left panel of the figure below) shows variability patterns that line-up nicely against the theoretical dispersion curves: Kelvin ($n = -1$), Rossby ($n = 1$), and Inertia-gravity waves ($n = 1$). The anti-symmetric component of the precipitation (right panel of the figure below) is consistent with the mixed Rossby gravity wave ($n = 0$).



✋ The **equivalent of $\boldsymbol{H}$** (previously the thermocline depth), used to compute the theoretical curves is some height in the atmosphere, but it is not the whole depth of the troposphere. It is a bit more complicated because H is modified by convection. But if you pick the right value, you can find a curve that lines up with the observed variability.

In the next chapter, we will discuss scale interactions. So far, we have primarily focused on linear dynamics. We will now go full **nonlinear** and investigate **scale interactions** and **turbulence**.