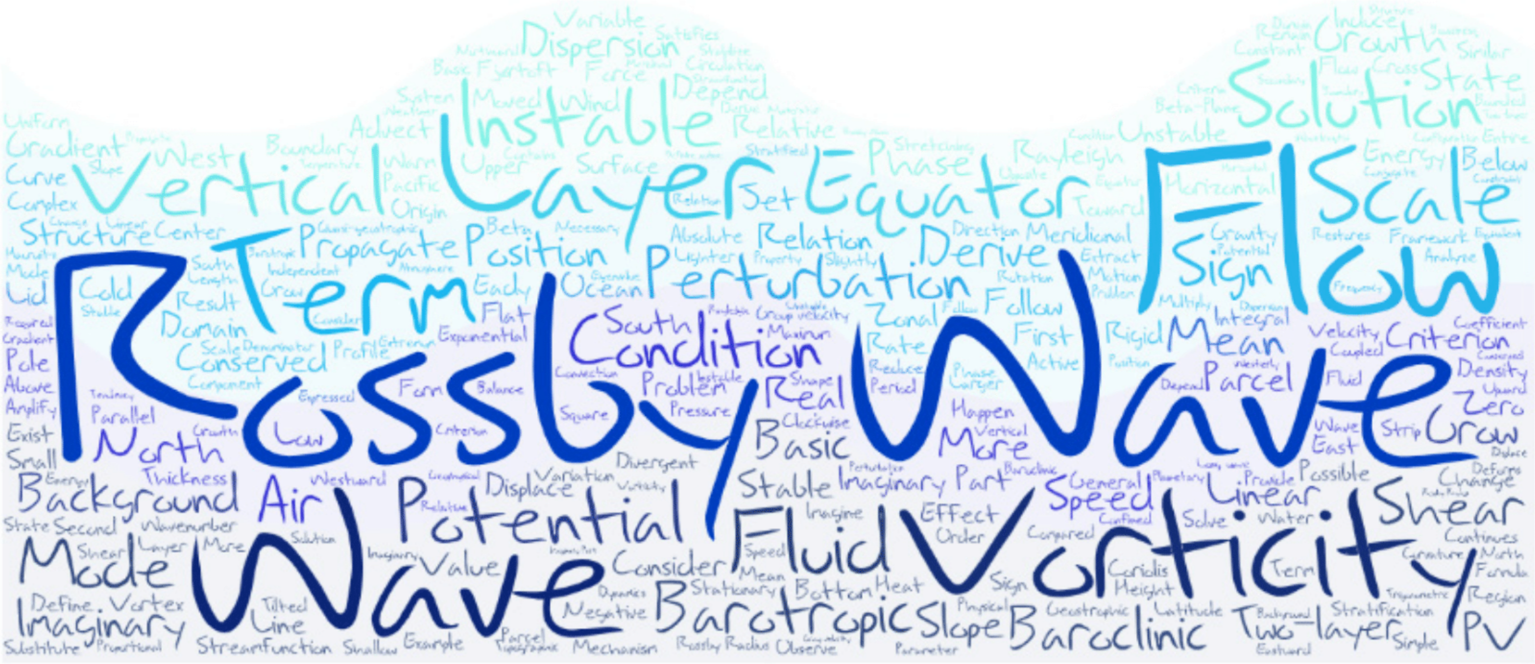
CHAPER 3
Rossby waves and instability



CHAPTER 3
Rossby waves and instability

GFD3 Contents

In this chapter, we stay in the **quasi-geostrophic framework** (see **#GFD2**) and focus on **Rossby waves**. We start with a general idea of what happens to a parcel of air or water if it is displaced on the planet where there is a variation in the Coriolis parameter, i.e. what are the consequences of **conserving the potential vorticity**.

We will derive the **dispersion relation** for Rossby waves by looking for **trigonometric/wave-like solutions**. We will overview different cases:

1) **Barotropic Rossby waves** (see **#GFD3.1**) and **topographic Rossby waves** (see **#GFD3.1g**),
2) **Baroclinic Rossby wave**, in a multi-layer model (see **#GFD3.2a**) and then in a continuously stratified fluid (see **#GFD3.2b**). We will decompose the variability in the vertical, i.e. extract independent **vertical modes**.

We will study the wave solution propagating through a non-uniform **background flow with shear**. Waves are solutions with trigonometric variations and imaginary exponentials, so the time variation is an oscillation and there is a propagation. What if the exponential becomes real?

3) It results in a perturbation that grows in time exponentially and becomes **unstable**. We will review the conditions required for this to happen in a barotropic (see **#GFD3.3**) and then baroclinic (see **#GFD3.4**) frameworks.

GFD3.1: Barotropic Rossby waves

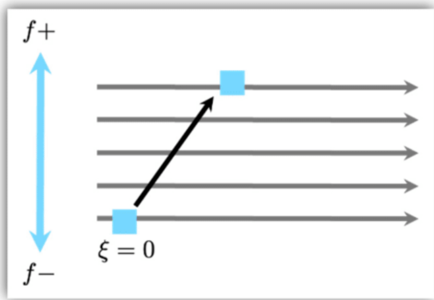
3.1.a) Parcel displacement in a vorticity gradient

General idea

⇨ Let's consider as parcel of fluid:

▪ In a non-divergent barotropic framework, i.e. the ***absolute vorticity is conserved*** (see **#GFD1.3b**) following the parcel: $q = f + \xi$.

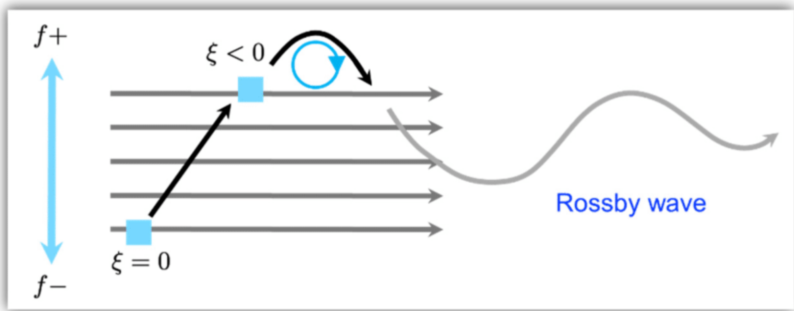
▪ On a planet with some curvature, i.e. the planetary vorticity (Coriolis parameter) f varies with latitude (larger f to the north - smaller f to the south).



⇨ At the origin, this parcel of fluid has no relative vorticity ($\xi = 0$). Imagine, for some reason, there is a perturbation that displaces the parcel (a little bit) to the north, where f gets larger.

⇨ In accord with the conservation of absolute vorticity ($f + \xi$), the relative vorticity of the flow will compensate for this increase in f and ξ must become negative ($\xi < 0$). Negative relative vorticity is associated with a **clockwise curvature of the flow**.

↳ So, the flow curves back down towards the south and the parcel will return to its latitude of origin. This is a *stable situation*, i.e. the solution oscillates such that the force that restores it to its position of origin is somehow proportional to the distance from the origin position.



⇨ You can imagine it overshooting and going back down south in which case it will come back north and it will produce a wave, a Rossby wave. A wave for which the restoring force is not just the Coriolis force, but the variation of the Coriolis force with the latitude.

☝ We need **variable f** for this to happen, so this **cannot work on a f-plane**.

3.1.b) The conservation of vorticity

⇨ We will derive the **dispersion relation for Rossby waves**.

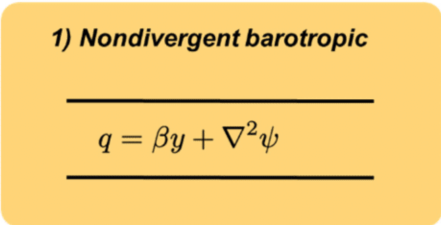
⇨ We remain in the **general framework of quasi-geostrophy** in which the potential vorticity is conserved following the flow, i.e. $\frac{Dq}{Dt} = 0$, with the material tendency given by the local tendency plus the advection terms.

↳ Here, the advection terms depend on:

- a background **zonal** flow $\boldsymbol{U}$
- and the small perturbation flow $(\boldsymbol{u'}, \boldsymbol{v'})$ associated with the wave.

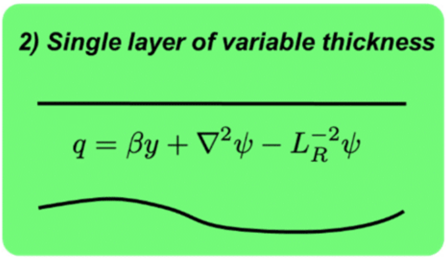
This gives: $\frac{D}{Dt} = \frac{\partial}{\partial t} + (U + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$

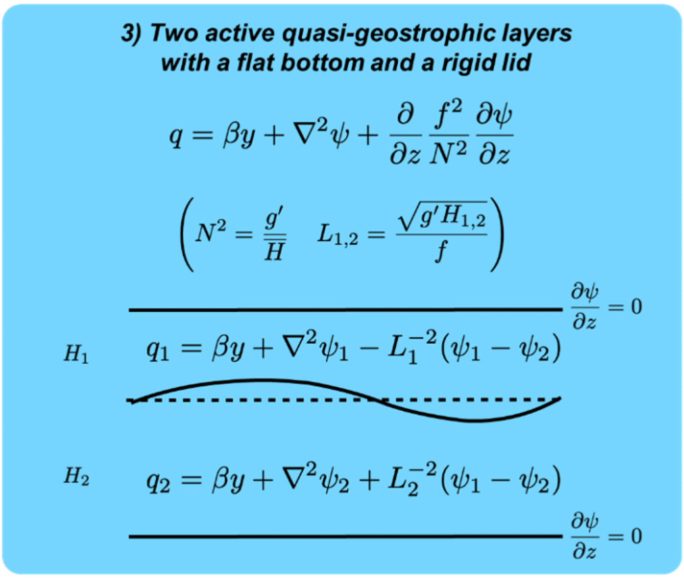
⇨ ***q* depends on the type of flow we consider**. We will study the Rossby wave dispersion relation in **three different contexts**:



1) ***Non-divergent barotropic flow*** (see **#GFD3.1c**) = One active layer bounded above and below by a rigid lid and a flat bottom. The flow is uniform in the vertical, i.e. it is barotropic. In this case, the potential vorticity is the absolute vorticity (see **#GFD2.3h**): $q = f + \xi = \beta y + \nabla^2\psi$. The stream function ($\psi$) is the ***perturbation of the stream function associated with the wave*** and the ***stream function of the background flow U***.

2) Then we will study the effect of variable layer thickness *on* a barotropic flow (see **#GFD3.1f**). In this case, you can generate vorticity by divergence and there is a vortex stretching term in the potential vorticity formulation (see **#GFD2.3h**). It is often called ***equivalent barotropic***, as there is only one active layer, the layer below is a motionless abyss.





3) We will finally consider the full baroclinic framework (see **#GFD3.2**) bounded above and below by a rigid lid and a flat bottom. We impose no vertical velocity at these boundaries, i.e. $\frac{\partial\psi}{\partial z} = 0$. In this framework, we consider that the fluid is Boussinesq, so the vortex stretching term in the continuously-stratified fluid potential vorticity formula is slightly simplified (see **#GFD2.3i**).

↳ If you **discretize** the vertical derivative and do a finite difference, you can easily derive the potential vorticity expressions for a discretized two-layer framework (see **#GFD3.2a**). You obtain simple differences between the stream functions. In the upper layer, it reduces to the inverse square of the Rossby radius times the difference between the stream function in the two layers. In the lower layer, a distinct Rossby radius (the thicknesses can be different in each layer) times by the difference between the stream function in the two layers.

↳ The formulae for the potential vorticity are coupled: q_1 depends on Ψ_2 and q_2 depends on Ψ_1.

Note that q is conserved with the flow. $\psi = \psi_B + \psi$. With $\psi_B = -Uy$, $\nabla^2\psi_B = 0$. ψ_B is thus crossed out from the PV equation

3.1.c) CASE 1: Non-divergent barotropic case

Equation

$$q = \beta y + \nabla^2 \psi$$

⇨ We **develop** the substantial derivative in the potential vorticity conservation equation (see **#GFD2.3g**), using the characteristics of the background flow:

$$u = U + u' = U - \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = v' = \frac{\partial \psi}{\partial x}$$

We obtain: $\frac{\partial}{\partial t}(\beta y + \nabla^2\psi) + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}(\beta y + \nabla^2\psi) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\beta y + \nabla^2\psi) = 0$

We can cross out some of these terms:

- As βy does not vary with time or x.
- We **linearize the equation** and consider that perturbations are small compared to the mean flow. Terms with a square of perturbation are neglected.

$$\frac{\partial}{\partial t}(\beta y + \nabla^2\psi) + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}(\beta y + \nabla^2\psi) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\beta y + \nabla^2\psi) = 0$$

↳ The linear equation in perturbation ψ can be written: $\left(\frac{\partial}{\partial t} + U\right)\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0$

Solutions

⇨ We are going to look for **wave-like solutions**, **plane-wave solutions**:

$$\psi = Re\, \tilde{\psi} e^{i(lx + my - \omega t)}$$

▪ They have the form of an amplitude coefficient times an imaginary exponential:

- l is the **zonal wavenumber** (2π divided by the x-wavelength),
- m is the **meridional wavenumber** (2π divided by the y-wavelength),
- ω is the **angular frequency** (2π divided by the period).

Properties

▪ Taking a derivative of this trigonometric function yields the same function multiplied by some constant coefficients:

$$\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \frac{\partial}{x} \rightarrow il \qquad \nabla^2 \rightarrow -(l^2 + m^2)$$

↳ Substituting the solution and its derivatives into the linear potential vorticity equation gives:

$$-i\omega\left(-(l^2 + m^2)\right) + il\left(-(l^2 + m^2)\right)U + \beta il = 0$$

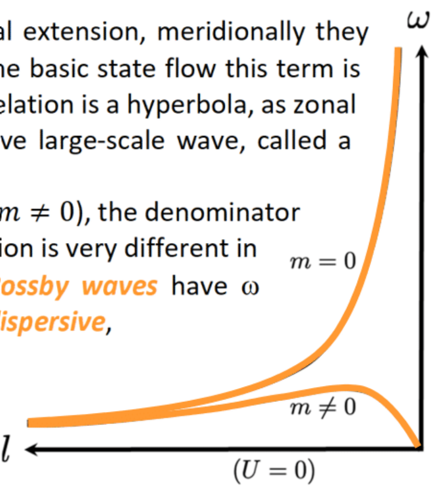
Dispersion relation

⇨ It results in a relation between ω, l and m (with 2 other geophysical parameters U and β). This is the **dispersion relation for barotropic Rossby waves**: $\omega = Ul - \frac{\beta l}{l^2 + m^2}$

- The phase speed $c = \frac{\omega}{l}$ is equal to $U - \frac{\beta}{l^2 + m^2}$. $l^2 + m^2$ and β are always positive. So, the **Rossby waves always propagate westwards**, opposite to the background eastward flow U.

- With $m = 0$ (i.e. the waves have an infinite meridional extension, meridionally they cover the entire planet), ω is proportional to $-\beta/l$. Relative to the basic state flow this term is negative, so we plot it on the negative quadrant. The dispersion relation is a hyperbola, as zonal scales get bigger, frequencies get higher. This is a very dispersive large-scale wave, called a **Rossby Haurwitz wave**.

- As soon as you set a meridional scale to your structure ($m \neq 0$), the denominator does not disappear. When $l = 0$ then $\omega = 0$. The dispersion relation is very different in this case. For the meridionally-confined structures, the ***long Rossby waves*** have ω almost proportional to l, which means that they are almost ***non-dispersive***, until a certain point. The maximum ω is found for $l = m$, and then for the shorter waves (for larger l), they become very dispersive.

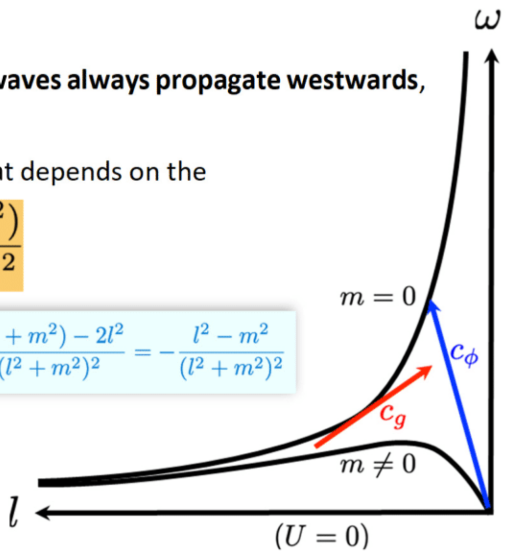


Non-dispersive waves: all the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape

Wave propagation speed

3.1.d) Rossby wave dispersion

• The **phase speed**: $c = \frac{\omega}{l} = U - \frac{\beta}{l^2 + m^2}$



↳ With $l^2 + m^2$ and β always positive, the **Rossby waves always propagate westwards**, opposite to the background eastward flow U.

• The **group speed** on the other hand has a sign that depends on the sign of its numerator: $c_g = \frac{\partial \omega}{\partial l} = U + \frac{\beta(l^2 - m^2)}{(l^2 + m^2)^2}$

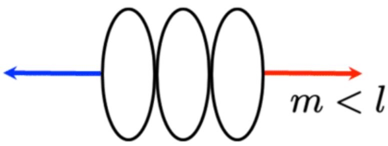
$$\frac{\partial \omega}{\partial l} = U - \beta \frac{\partial}{\partial l}(l(l^2+m^2)^{-1}) = \frac{1}{(l^2+m^2)} + l(-(l^2+m^2)^{-1} \times 2l) = \frac{(l^2+m^2) - 2l^2}{(l^2+m^2)^2} = -\frac{l^2 - m^2}{(l^2+m^2)^2}$$

⇨ On this graph, the phase speed is the **arrow that points from the origin towards the curve** (whichever curve is used), while the **group speed is the tangent to the curve**.

↪ Relative to the background current, the direction of propagation of the energy of the wave **depends on the shape of the wave** (ratio of zonal to meridional scales):

➢ If $l = m$, the group speed is zero.

➢ If $l > m$, i.e. waves which have a larger meridional scale than their zonal scale, then the ratio term in the group speed formula is always positive. Relative to the eastward background flow (U), the phase speed of these waves (in blue here) will be to the west, while their group speed will be to the east.



➢ If $l < m$, the waves are elongated in the zonal direction and the ratio term is negative. The group speed and the phase speed are both to the west (relative to the eastward background flow (U),). These waves are more non-dispersive and are easier to observe because they will not lose their shape as they propagate westwards.

⇨ From the dispersion relation, it comes that:
- **Rossby waves are dispersive**. **Longer waves go faster**.
- **Waves closer to the equator go faster** (β is maximum at the equator, zero at the poles)

Westward propagation illustration

3.1.e) Rossby wave propagation mechanism

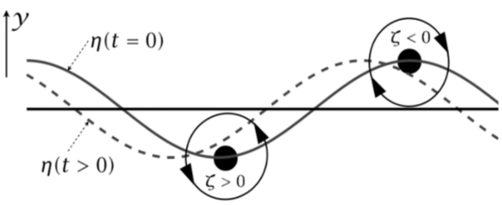
⇨ Why do Rossby waves propagate to the west?

• **Remember the parcel** which was displaced from its position of origin. To the north, it has acquired negative relative vorticity resulting in a clockwise circulation. To the south, positive relative vorticity has been induced, i.e. an anti-clockwise circulation.

• Imagine now a **streamline of potential vorticity** that follows the parcel. It has been moved to the north or to the south, portraying a wave.

⇨ **How would the stream line be displaced by this secondary circulation?**

It will be pushed away from the origin on the west, and towards it on the east. So, at a later time, the streamline will follow the dashed curve, effectively moving it to the left on the diagram. The Rossby wave is thus propagating to the west.



↳ ***The secondary circulation induced by the constraint of conserving the vorticity produces the westward propagation.***

3.1.f) CASE 2: Divergent case (variable layer thickness)

$$q = \beta y + \nabla^2\psi - L_R^{-2}\psi$$

⇨ In the case of a variable layer thickness, the same advection operator is applied to a different definition of potential vorticity. The latter contains a **vortex stretching term** (see **#GFD2.3h**), which is $-L_R^{-2}\psi$.

☝ In the PV conservation equation, the stream function is the sum of contributions from:
- the stream function associated with the perturbation ψ
- and the **background flow stream function** $\psi_B = -Uy$

Equation

↳ $\nabla^2\psi_B = 0$, but for the divergent case, the contribution of the background flow U remains in the ***vortex stretching term***, such that:

$$\left(\frac{\partial}{\partial t} + (U+u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\left(\beta y + \nabla^2\psi - L_R^{-2}(\psi - Uy)\right) = 0$$

⇨ As in the non-divergent case (**#GFD3.1c**):
- $U + u' = U - \frac{\partial\psi}{\partial y}$ and $v' = \frac{\partial\psi}{\partial x}$
- Terms associated with **time and x variation** of βy and $L_R^{-2}Uy$ can be crossed out.
- We **linearize the equation**, so terms with a square of perturbation are neglected.

↳ The resulting PV conservation equation can be sorted into **two terms**:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2\psi - L_R^{-2}\psi\right) + \left(\beta + L_R^{-2}U\right)\frac{\partial\psi}{\partial x} = 0$$

▪ The first LHS term is the material tendency of the perturbation relative vorticity and vortex stretching term, i.e. ***the mean flow advecting the perturbation***.

▪ The second term is the other way round, i.e. ***the perturbation flow affecting the mean***. This is the perturbation meridional flow ($v' = \frac{\partial\psi}{\partial x}$) advecting the potential vorticity associated with the background flow plus β.

Solutions and dispersion relation

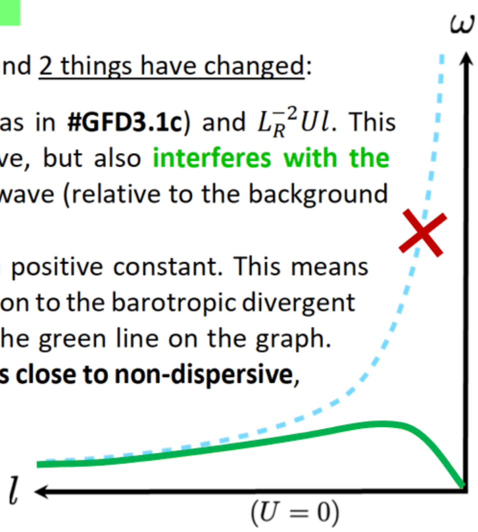
⇨ As in **#GFD3.1c**, we derive the **dispersion relation** by substituting ***plane-wave solutions*** ($\tilde{\psi} e^{i(lx+my-\omega t)}$), and their derivative properties into the PV conservation equation. It follows that:

$$\omega = Ul - l\frac{\beta + L_R^{-2}U}{l^2 + m^2 + L_R^{-2}}$$

↳ The second term is more complicated than before and 2 things have changed:

• ***On the numerator***, there are now two terms: βl (as in **#GFD3.1c**) and $L_R^{-2}Ul$. This means that the **background flow** not only displaces the wave, but also **interferes with the properties of the wave**. In particular, the ***phase speed*** of the wave (relative to the background flow) will be altered by the background flow.

• ***On the denominator***, there is an extra term L_R^{-2}, a positive constant. This means that the very dispersive **Rossby Haurwitz waves** are not a solution to the barotropic divergent framework (see dotted line). The solution always resembles the green line on the graph. The phase speed is bounded and **long-Rossby waves are always close to non-dispersive,** with group speed to the west (even for $m = 0$).



3.1.g) Topographic Rossby waves

⇨ We focus now on a slightly different case in which potential vorticity can be changed by externally constraining the layer thickness. From **#GFD2.3h**, the quasi-geostrophic potential vorticity can be written as:

$$q = f_0 + \beta y + \xi - \frac{f_0}{H}\delta h$$

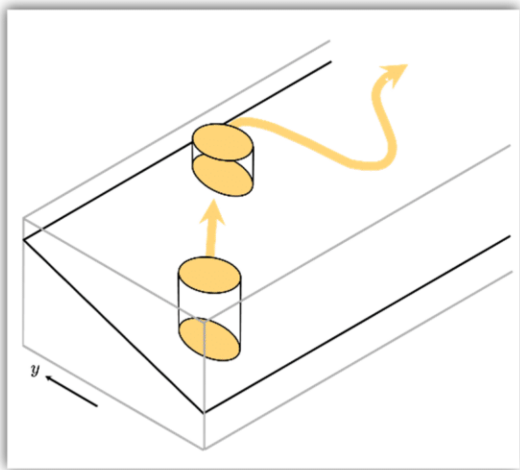
Note that, in the example below, we can disregard the changes in the planetary vorticity (βy).

⇨ Consider an ocean getting shallower towards the north (see illustration on the right).

↳ In this example, the thickness of the Ocean is proportional to the latitude y, $h_{OC} = \alpha y$. Since the Ocean gets thinner to the north, α is negative.

This topographic effect will add a constrain on δh, so the quasi-geostrophic potential vorticity is:

$$q = f_0 + \beta y + \xi - \frac{f_0}{H}(\alpha y + \eta)$$



⇨ Imagine that a column of fluid is displaced northward.

↳ Because of the change in the topography, the column is squashed (it gets shallower).

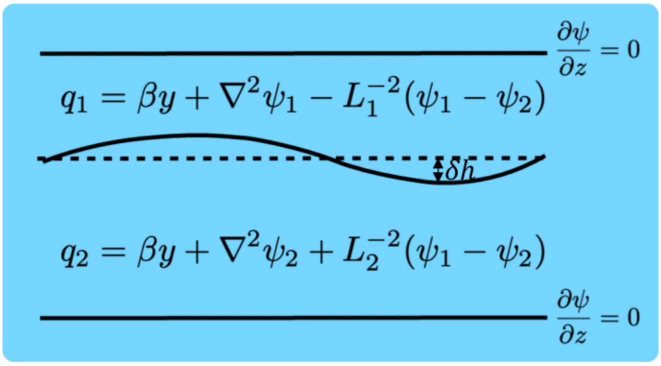
↳ Its potential vorticity has to be conserved, which triggers negative relative vorticity. This induces a clockwise flow, similar to **#GFD3.1a**. Just like before, the displacement can generate a Rossby wave, which lives on this slope.

↳ Mathematically, the external constraint term (topographic effect) $-\frac{f_0}{H}\alpha y$ is positive and identical to the β-effect. *In the northern hemisphere, an ocean floor that is shallowing to the north will have the same effect as β. In the southern hemisphere the ocean floor must shallow to the south.*

⇨ These waves are called topographic Rossby waves.

GFD3.2: Baroclinic Rossby waves

3.2.a) CASE 3: Two active layers



⇨ In this framework, we have two active layers in which the quasi-geostrophic potential vorticity is conserved (see **#GFD2.3h**):

$$q_1 = \frac{f + \xi_1}{h_1} \approx \frac{1}{H_1}\left(f + \xi_1 - \frac{f}{H_1}\delta h\right)$$
$$q_2 = \frac{f + \xi_2}{h_2} \approx \frac{1}{H_2}\left(f + \xi_2 + \frac{f}{H_2}\delta h\right)$$

⇨ We retrieve **geostrophic stream functions** for each layer (see **#GFD1.2e**):

$$\boldsymbol{f}_0 \times \boldsymbol{u}_1 = -\frac{1}{\rho_0}\nabla P_1 = -g\nabla(h_1 + h_2) \quad \text{and} \quad \boldsymbol{f}_0 \times \boldsymbol{u}_2 = -\frac{1}{\rho_0}\nabla P_2 = -g\nabla(h_1 + h_2) - g'\nabla h_2$$

$$\psi_1 = \frac{g}{f_0}(h_1 + h_2) \quad \text{and} \quad \psi_2 = \frac{g}{f_0}(h_1 + h_2) + \frac{g'}{f_0}h_2$$

↳ The interface displacements (from the rigid lid) are $\delta h = -h_2 = \frac{f_0}{g'}(\psi_1 - \psi_2)$. Therefore, the vortex stretching term is a coupled term defined in terms of the difference between the two stream functions.

As in **#GFD3.1f**, the conservation equation can be expanded by using the advection operators defined in terms of the stream function. For simplicity, the background flow (U) has been disregarded in this example. Simplification of time and x-invariant terms and linearization yields:

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 - L_1^{-2}(\psi_1-\psi_2)\right] + \beta\frac{\partial\psi_1}{\partial x} = 0$$
$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 + L_2^{-2}(\psi_1-\psi_2)\right] + \beta\frac{\partial\psi_2}{\partial x} = 0$$

$$L_1^{-2} = \frac{f_0^2}{g' H_1} \quad L_2^{-2} = \frac{f_0^2}{g' H_2}$$

As in **#GFD1.2f**, the two equations for the potential vorticity are coupled (the top layer equation depends on ψ_1 and ψ_2, so does the bottom layer equation).

The problem now is to **decouple these equations**. Similar to the shallow water equations (in **#GFD1.2**), we have to define new variables that are a linear combination of ψ_1 and ψ_2 which will provide <u>two independent equations</u>. For this example, we do not need to compute the eigenvalues and eigenvectors of the coupling matrix. The variable transformations are more straightforward.

1) We can first define a **barotropic mode**, noted $\bar{\psi}$, as the weighted sum of the stream functions by the layer thicknesses.

$$\bar{\psi} = \frac{H_1\psi_1 + H_2\psi_2}{H_1+H_2}$$

It can also be expressed in terms of the Rossby radius:

$$\bar{\psi} = \frac{L_2^{-2}\psi_1 + L_1^{-2}\psi_2}{L_1^{-2}+L_2^{-2}} \qquad L_R^{-2} = L_1^{-2} + L_2^{-2}$$

2) We can define a **baroclinic mode** $\hat{\psi}$ which is just the difference between the two layers:

$$\hat{\psi} = \psi_1 - \psi_2$$

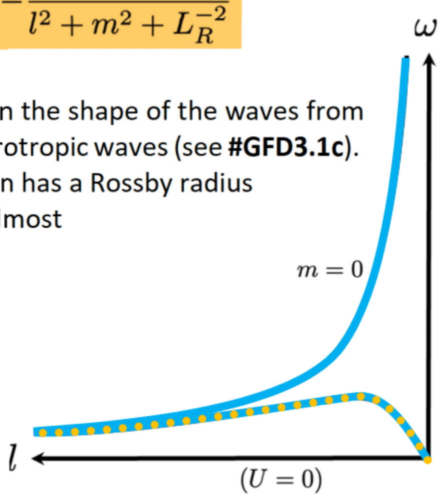
If we manipulate the set of equations so the variables are $\bar{\psi}$ and $\hat{\psi}$, we obtain two independent equations, one for the barotropic mode and one for the baroclinic mode:

$$\frac{\partial}{\partial t}\nabla^2\bar{\psi} + \beta\frac{\partial\bar{\psi}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\left[(\nabla^2 - L_R^{-2})\hat{\psi}\right] + \beta\frac{\partial\hat{\psi}}{\partial x} = 0$$

Naturally, the **barotropic mode** equation resembles the barotropic potential vorticity equation (see **#GFD3.1c**), while the **baroclinic mode** equation includes an extra stretching term.

These two modes are associated with distinct dispersion relations:

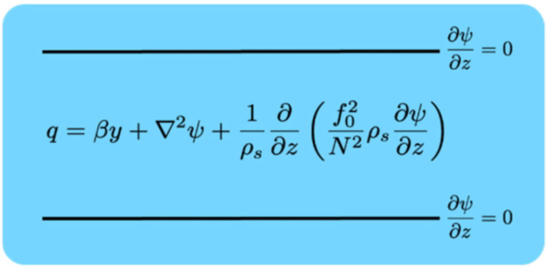
$$\omega = -\frac{\beta l}{l^2+m^2} \quad \text{and} \quad \omega = -\frac{\beta l}{l^2+m^2+L_R^{-2}}$$



• The **barotropic mode** dispersion relation will depend on the shape of the waves from the extreme **Rossby Haurwitz wave** to the non-dispersive long barotropic waves (see **#GFD3.1c**).

• The **baroclinic mode** is slower and its dispersion relation has a Rossby radius term in the denominator $L_R^{-2} = L_1^{-2} + L_2^{-2}$ and always yields almost non-dispersive long Rossby waves.

3.2.b) CASE 4: Extension to the vertical continuum



⇨ Let us now consider the dynamics of linear waves in stratified quasi-geostrophic flow on a β-plane, in a domain confined between two **rigid surfaces** at $z = 0$ and $z = -H$, with a **resting basic state**.

↳ The **quasi-geostrophic potential vorticity** for a continuously stratified fluid is conserved following with the flow (see **#GFD2.3i**).

⇨ The interior flow is governed by the quasi-geostrophic potential vorticity conservation (see **#GFD2.3i**):

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

↳ As in **#GFD3.1f**, the conservation equation can be expanded and **linearized**, leading to:

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho \frac{{f_0}^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] + \beta \frac{\partial \psi}{\partial x} = 0$$

↳ If the boundaries are flat, rigid, and slippery surfaces, then $w = 0$ at the boundaries. Also, if there is no surface buoyancy gradient, the linearized equation is:

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z}\right) = 0 \text{ at } z = 0 \text{ and } z = -H$$

Vertical structure

⇨ As in the single-layer case (**#GFD3.1c**), we seek solutions of the form of ***plane-wave solutions***, $\psi = Re\, \tilde{\psi}(z) e^{i(lx+my-\omega t)}$, where $\tilde{\psi}(z)$ **determines the vertical structures of the waves**.

↳ We indeed have to account for the fact that **the wave amplitude might vary in the vertical**. For the two-layer case, we had two modes because we had two layers. For the vertically continuous case, ***we have functions of z***.

⇨ **Substituting** the solution and its derivative into the linear potential vorticity equation does not yield a simple algebraic expression. It results in a **differential equation** for the wave coefficient $\tilde{\psi}(z)$:

$$\omega\left[-(l^2 + m^2)\tilde{\psi}(z) + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho \frac{{f_0}^2}{N^2}\frac{\partial \tilde{\psi}(z)}{\partial z}\right)\right] - \beta l \tilde{\psi}(z) = 0$$

Vertical separation

↳ To solve this equation, we make a separable dependence assumption, implying that **horizontal and vertical structures of the waves can be separated**, $\tilde{\psi}(z)$ satisfies:

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho \frac{{f_0}^2}{N^2}\frac{\partial \tilde{\psi}(z)}{\partial z}\right) = -\Gamma \tilde{\psi}(z) \quad (*)$$ (Γ is the separation constant)

⇨ Then the equation of motion becomes:

$$-\omega[(l^2 + m^2) + \Gamma]\tilde{\psi} - \beta l \tilde{\psi} = 0$$

↳ And the dispersion relation follows:

$$\omega = -\frac{\beta l}{(l^2 + m^2) + \Gamma}$$

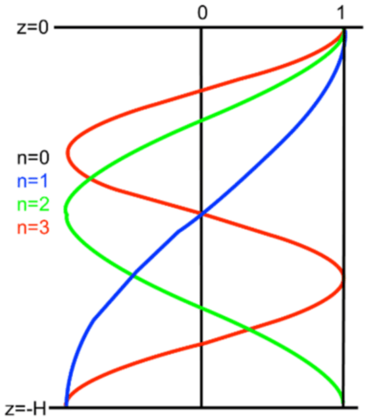
⇨ Equation $(*)$ constitutes an ***eigenvalue problem*** for the vertical structure, with boundary conditions $\frac{\partial \psi}{\partial z} = 0$ at $z = 0$ and $z = -H$. The resulting eigenvalues Γ are proportional to the inverse of the squares of the deformation radii for the problem and the eigenfunctions are the vertical structure functions.

What do these vertical structures look like?

A simple example:

⇨ For simplification, we consider waves propagating in a **Boussinesq fluid**, with a **constant stratification**. The eigenproblem for the vertical structure (with previous boundary conditions) is:

$$\frac{{f_0}^2}{N^2}\frac{\partial^2\tilde{\psi}(z)}{\partial z^2} = -\Gamma\tilde{\psi}(z)\ (**)$$



↳ There is a sequence of solutions to this equation, namely:

$$\tilde{\psi}_n(z) = \cos(n\pi z/H),\quad n = 1,2\ldots,\quad k_v = n\pi/H$$

▪ The first solution ($n = 1$, blue line) is half a wave in the vertical, the second solution ($n = 2$, green line) is a full-wave in the vertical. The third mode is one and a half waves in the vertical, etc... These constitute n baroclinic modes.

↳ The structure of the baroclinic modes becomes more ***complex*** as the vertical wavenumber ***n increases***.

▪ Each mode has a different eigenvalue:

$$\Gamma_n = (n\pi)^2\left(\frac{f_0}{NH}\right)^2,\quad n = 1,2\ldots$$

▪ This equation can be used to define the deformation radii for this problem, namely:

$$L_n \equiv \frac{1}{\sqrt{\Gamma_n}} = \frac{NH}{n\pi f_0} = \frac{N}{f_0 k_v}$$

▪ The phase speed of the Rossby waves is given by ω/l.

▪ The dispersion relation is different for each mode:

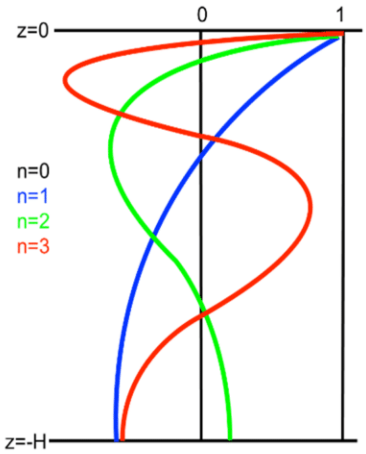
$$\omega = -\frac{\beta l}{(l^2+m^2)+\frac{f^2}{N^2}k_v^2}$$

For each different vertical structure, we have a different Rossby wave with different properties.
↳ For each mode, Rossby waves have a different phase speed.

A more realistic stratification:

⇨ In a **Boussinesq fluid**, the eigenproblem for the vertical structure is more complex as the stratification depends on the vertical ($N^2(z)$):

$$\frac{\partial}{\partial z}\left(\frac{{f_0}^2}{N^2}\frac{\partial\tilde{\psi}(z)}{\partial z}\right) = -\Gamma\tilde{\psi}(z)\qquad (***)$$



↳ For a stable stratification, this differential equation with its boundary conditions is reduced to a *Strum-Liouville* system which can be solved numerically.

▪ The structure of the baroclinic modes which depends on the structure of the stratification, becomes increasingly ***complex*** as the vertical wavenumber ***n increases***.

▪ The ***variability of the vertical structure*** is confined in the ***thermocline layer*** where the stratification is maximum.

In addition to these baroclinic modes, the **barotropic mode** with $n = 0$, that is $\tilde{\psi}(z) = 1$, is also a solution of $(*)$ for any density profile (black line).

↳ The dynamics of the barotropic mode is independent of height and independent of the stratification of the basic state, and so these Rossby waves are identical to the Rossby waves in a homogeneous fluid contained between two flat rigid surfaces (see **#GFD3.1c**).

3.2.c) Vertically propagating Rossby waves

⇨ Rossby waves propagate horizontally, as the restoring force is in the horizontal.

↳ But they have a **vertical component to their propagation** as well. For instance, Rossby waves can be out of phase in different layers, and they can thus effectively propagate with a vertical component to their propagation.

⇨ The **vertical wavenumber** for each mode $k_{vn} = n\pi/H$ can also be expressed in terms of c_n, the gravity wave phase speed for the n^{th}-mode, as $k_{vn} = N/c_n$. ✋ c_n is not the Rossby wave speed!

⇨ The **dispersion relation for very long Rossby wave** ($l^2 + m^2 \ll k_v^2 f_0^2/N^2$) can be approximated, as:

$$\omega = -\frac{\beta l}{(l^2+m^2) + \frac{f_0^2}{N^2}k_v^2} \approx -\frac{\beta l N^2}{f_0^2 k_v^2} = \frac{\beta l c_n^2}{f_0^2}$$

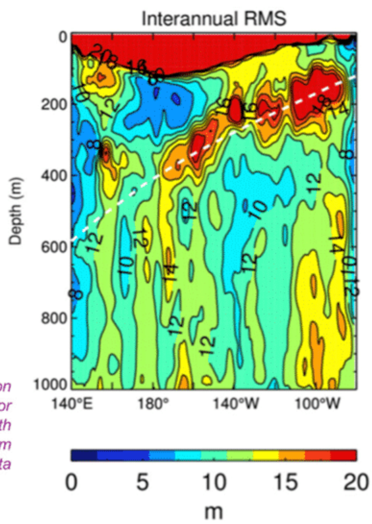
- The horizontal group speed is: $\dfrac{\partial \omega}{\partial l} = -\dfrac{\beta N^2}{f_0^2 k_v^2} = -\dfrac{\beta c_n^2}{f_0^2}$
- The vertical group speed is: $\dfrac{\partial \omega}{\partial k_v} = \dfrac{2\beta l N^2}{f_0^2 k_v^3} = \dfrac{2\beta l c_n^3}{f_0^2 N}$

↳ We can trace the signal path associated with the **vertical propagation** in the $x-z$ plane by calculating the **ratio between the two group speeds**:

$$\frac{dz}{dx} = \frac{c_g^z}{c_g^x} = -\frac{2lc_n}{N} = \frac{2f_0^2\omega}{\beta N c_n}$$

This provides the **slope** at which the energy propagates, allowing us to **trace the direction of propagation** of perturbations.

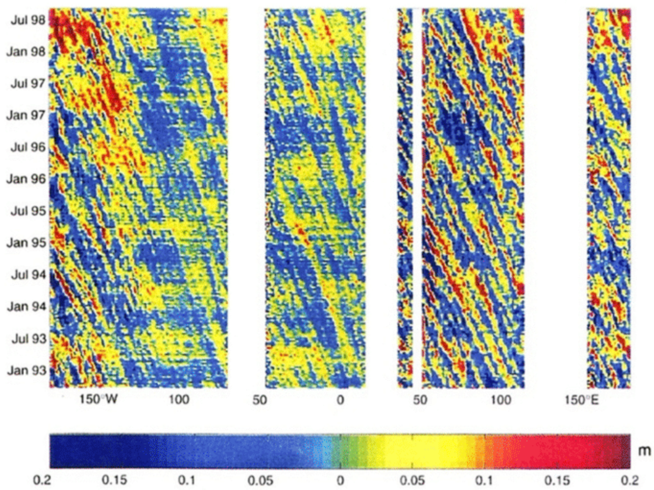
On the right is a figure by Vergara et al. (2017) highlighting a ray, showing the propagation of a perturbation to the thermocline depth. It gives observational evidence of vertically propagating Rossby waves.



Propagation pathway for isotherm depth variability from ARGO data

3.2.d) Observations

Below is a (quite old) global longitude-time representation of the Sea Level anomalies (perturbations) at 25°S from Topex/Poseidon altimetry data from 1993 to 1998. The longitude in degrees covers the 3 tropical Oceans: Pacific, Atlantic, and Indian.



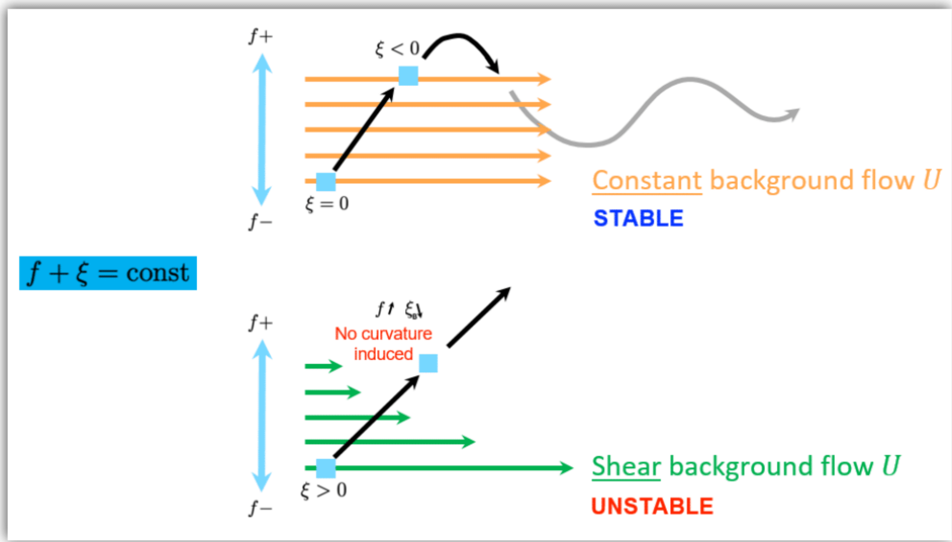
The **diagonal stripes** are the signature of westward propagation. It takes about **five years to cross the Pacific basin**. This cannot be the signature of an external/barotropic Rossby wave, because that would go too fast to be picked up by this altimeter time resolution (dt=10days). It could be the adjustment of the sea level to a perturbation on the thermocline, i.e. the trace of a (slower) baroclinic Rossby wave traversing the Pacific in a few years. However, it is not entirely sure whether this is exactly what it is or whether it has to do with non-linear phenomena like eddies.

GFD3.3: Barotropic Instability

3.3.a) Growing Rossby waves?

⇨ What happens when we allow **the basic state flow** to become more interesting/complicated?

• Up to now, we have assumed that the basic state flow is just **uniform** westerlies ($U = cst$). Our parcel was displaced around an equilibrium controlled by a horizontal restoring force and it created a Rossby wave (see **#GFD3.1a**).



General idea

⇨ What happens if the basic flow resembles a **sharp jet** in the westerlies, with strong **meridional shear**?

• At the origin, the relative vorticity (ξ) is not zero anymore. Imagine dropping a wheel into the flow, it will spin anti-clockwise (because of the shear), i.e. with positive relative vorticity ($\xi > 0$).

↳ - When the parcel is moved north, the planetary vorticity gets bigger (f ↗).

- According to the conservation of potential vorticity (absolute vorticity $f + \xi$), we would expect the relative vorticity to get smaller (ξ ↘).

- but the background flow was chosen such that its shear is smaller there than in the south, so no secondary circulation develops (no curvature is induced). The particle just goes north, as if it is allowed to just take off.

⇨ This is the beginning of the consideration of instability.

3.3.b) Perturbations on a parallel shear flow

$$q = \beta y + \nabla^2 \psi$$

⇨ Let's consider a **barotropic non-divergent flow** on a β-plane ($f = f_0 + \beta y$), with a **geostrophic parallel shear flow in the background**:

$$u = \bar{u}(y) = \frac{1}{\rho f}\frac{d\bar{p}}{dy}$$

⇨ On top of the background flow, we have perturbations:

$$u = \bar{u}(y) + u' = \bar{u}(y) - \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = v' = \frac{\partial \psi}{\partial x}$$

↳ The **momentum and continuity equations** can be written:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

⇨ **Linearization** yields:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} - fv' = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$
$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} \qquad\qquad + fu' = -\frac{1}{\rho}\frac{\partial p'}{\partial y}$$

⇨ The **vorticity equation** is derived by taking the curl of the momentum equations ($\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$) and simplifying. It follows: $\frac{\partial \xi}{\partial t} + \bar{u}\frac{\partial \xi}{\partial x} - v'\frac{\partial^2 \bar{u}}{\partial y^2} + v'\frac{\partial f}{\partial y} = 0$

↳ In terms of the stream function, **the linear vorticity equation** on a β-plane writes:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0$$

↳ We obtain a vorticity equation in which:
- The first term is the advection of perturbation vorticity by the background flow,
- The second term is the advection by the perturbation flow of the absolute vorticity associated with the basic state.

Dispersion relation

⇨ The difference compared to the uniform background flow case (**#GFD3.1c**) is that we have meridional variations (shear) in the basic state flow.

⇨ From the **linear vorticity equation in a parallel shear flow** (see **#GFD3.3b**), we derive a dispersion relation for the Barotropic Rossby waves by introducing solutions of ***plane-wave form***: $\psi = Re\,\tilde{\psi}e^{i(lx+my-\omega t)}$ (similar to **#GFD3.3b**):

$$\omega = Ul - \frac{(\beta - U_{yy})l}{l^2 + m^2}$$

↳ Note the presence of the ***relative vorticity of background flow (U_{yy}) in the inner term***.

3.3.c) Stationary Rossby Waves

⇨ Atmospheric scientists are interested in stationary Rossby waves, i.e. Rossby waves which stay in place, with $\omega = 0$. The relationship between the properties of the wave (l^2 and m^2) and the background flow follows:

$$U(l^2 + m^2) = (\beta - U_{yy})$$

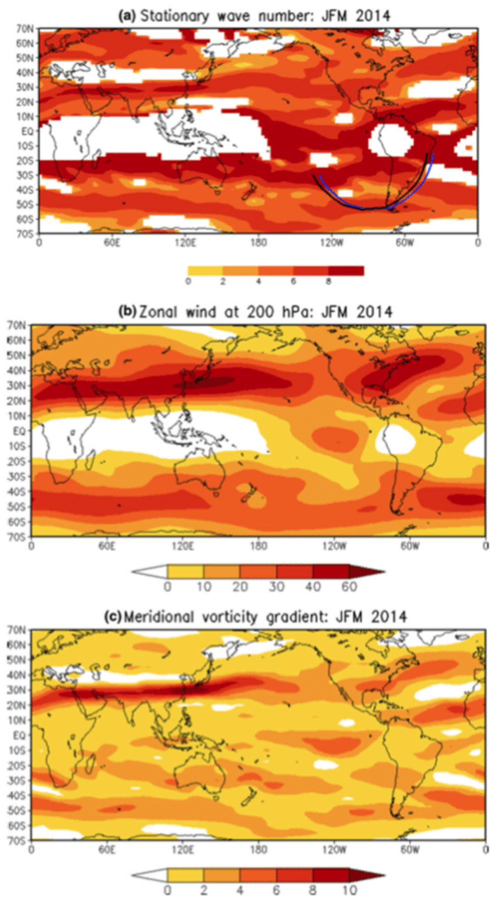
↳ There is a relation between the horizontal wavenumber k ($k^2 = l^2 + m^2$) and the background flow:

$$k = \sqrt{(\beta - U_{yy})/U}$$

The wavelength is $2\pi\sqrt{\frac{U}{\beta}}$

⇨ For the stationary wave to exist, the wavenumber has to be real, i.e. $(\beta - U_{yy})/U$ has to be positive. This means that $\beta - U_{yy}$ must have the same sign as U (*which usually means both must be positive).*

↳ **Fig.c** (from Coelho et al., 2016) shows the meridional gradient term (U_{yy}), which is positive almost everywhere. This means that for stationary Rossby waves to exists, we must have easterlies. **Fig.b** shows the associated zonal wind component (U) and **Fig.a** represents the stationary wavenumber k. White areas denote regions in which $\beta - U_{yy}$ and U have different signs and k is



complex/imaginary. In these regions, stationary or very low-frequency Rossby waves cannot exist. Contrastingly, there is a large area in the eastern Pacific where stationary Rossby waves can exist, this is the pacific waveguide.

⇨ Ray paths of the stationary Rossby waves can also be calculated as before (see **#GFD3.2c**) from the ratio of the zonal group speed ($\frac{\partial\omega}{\partial l}$) and meridional group speed ($\frac{\partial\omega}{\partial m}$):

$$\mathbf{c}_g = \left(U + \frac{\beta_*(l^2-m^2)}{k^4}, -\frac{2\beta_* lm}{k^4}\right) \qquad (\beta_* = \beta - U_{yy},\ \ k^2 = l^2+m^2)$$

↳ Plotting the different components of the group speed provides the theoretical direction of the stationary Rossby wave, as illustrated in **Fig.a**.

3.3.d) Growing solutions

⇨ Whenever we consider waves, the other side of the coin is instability. Let's get on to this idea of a solution that can grow. A gravity wave is stable, while a thunderstorm is unstable. A Rossby wave is stable and barotropic instability leads to rapid development. In **#GFD3.3a**, we considered a parcel of fluid that can take off in the meridional direction.

We are seeking ***unstable solutions*** for the **linear vorticity equation in a parallel shear flow** derived in **#GFD3.3b**:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial\psi}{\partial x} = 0$$

⇨ In the linear vorticity equation, the coefficients of the x-derivatives are not themselves functions of x. Thus, we may seek **solutions** that are harmonic functions (sines and cosines) in the x-direction, but the ***y-dependence must remain arbitrary at this stage*** and we seek solution such that:

$$\psi(x,y,t) = \phi(y)e^{i(lx-\omega t)}$$

↳ We **substitute** this solution into the vorticity equation (see **details of the calculous** on the following page) and it is very similar to what we did with the vertical dependence in **#GFD3.2b**. We obtain a differential equation for $\phi(y)$, namely:

Rayleigh equation

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta - d^2\overline{u}/dy^2}{\overline{u}(y)-c}\phi = 0$$

↳ This is the **linear vorticity equation for disturbances to parallel shear flow**, in which $c = \omega/l$, known as **Rayleigh's equation**. **We are not going to solve this equation**! We are going to analyze it for the possibility of growth.

⇨ The wave part of the solution is trigonometric with imaginary exponentials. But if what is inside the exponential has an ***imaginary part*** then you would get a real exponential.

- If ω is purely real then $c = \omega/l$ is the phase speed of the wave.
- If ω has a positive imaginary component (ω_i) then the wave will grow exponentially and will thus be ***unstable***: $\omega = \omega_r + i\omega_i,\ \ \omega^* = \omega_r - i\omega_i$ is the complex conjugate.

↳ Supposing that l is real, the phase speed $c = \omega/l$ can be complex too:

$$c = c_r + ic_i,\ \ c^* = c_r - ic_i$$

✋ l could be complex but it would not add anything. It would just be more mathematics.

We are interested in whether it is possible for c to have an imaginary part. Because if it does that means ω has an imaginary part, which means there is a possibility of instability. We are going to analyze **Rayleigh's equation** for the possibility of c having an imaginary part.

⇨ ⇨ If we add channel boundary conditions ($\phi = 0$ at $y = 0, L$), in general, we get a set of solutions for ϕ associated with complex conjugate pairs of values for c (or ω). The imaginary part of the solution is associated with growth or decay. The growth rate is the imaginary part of ω.

Details for the derivation of the Rayleigh's equation

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0 \qquad\qquad \psi = \phi(y)e^{i(lx-\omega t)}$$

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = \frac{\partial}{\partial x}\left(\phi\, il\, e^{<>}\right) + \frac{\partial}{\partial y}\left(\phi_y e^{<>}\right) = \left(-\phi l^2 + \phi_{yy}\right)e^{<>}$$

$$\psi_x = \phi\, il\, e^{<>}$$

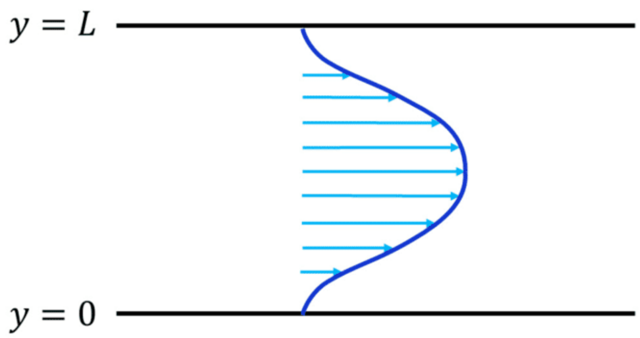
$$-\cancel{i}\omega(-\phi\, l^2 + \phi_{yy}) + \cancel{i}l\,\overline{u}(-\phi\, l^2 + \phi_{yy}) + (\beta - \overline{u}_{yy})\phi\,\cancel{i}l = 0$$

$$-\frac{\omega}{l}(\phi_{yy} - \phi\, l^2) + \overline{u}(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$-\left(\frac{\omega}{l} - \overline{u}\right)(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$(\overline{u} - c)(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$\phi_{yy} - l^2\phi + \left(\frac{\beta - \overline{u}_{yy}}{\overline{u} - c}\right)\phi = 0$$

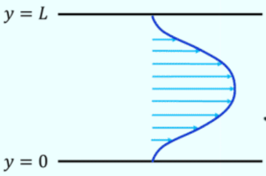


3.3.e) Conditions for growth: the Rayleigh criterion

⇨ ⇨ If we add channel boundary conditions ($\phi = 0$ at $y = 0, L$), in general, we get a set of solutions for ϕ associated with complex conjugate pairs of values for c (or ω). The imaginary part of the solution is associated with growth or decay. The growth rate is the imaginary part of ω.

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta - d^2\overline{u}/dy^2}{\overline{u}(y) - c}\phi = 0$$

⇨ We **multiply** Rayleigh's equation by **the complex conjugates of ϕ** and **integrate the equation across the domain from 0 to L**. The two first terms are integrated by parts.



$$\frac{d}{dy}(\phi\phi_y) = \phi_y^2 + \phi\phi_{yy} \implies d(\phi\phi_y) = \phi_y^2\,dy + \phi\phi_{yy}\,dy$$

$$\int \phi(\phi_{yy} - l^2\phi)\,dy = \int (\phi\phi_{yy} - l^2\phi^2)\,dy = \int_0^L d(\phi\phi_y) - \int_0^L (\phi_y)^2\,dy - \int_0^L l^2\phi^2\,dy$$

$$= [\phi\phi_y]_0^L - \int_0^L |\phi_y|^2 + l^2\phi^2\,dy$$

It gives: $$-\int_0^L \left(\left|\frac{d\phi}{dy}\right|^2 + l^2|\phi|^2 \right) dy + \int_0^L \frac{\beta - d^2\overline{u}/dy^2}{\overline{u} - c}|\phi|^2 dy = 0$$

↳ ***The equation equates to zero if its real*** (see **#GFD3.3f**) ***and imaginary parts are both zero.***

↳ The first LHS term is positive definite and real. Therefore, if there is anything **imaginary** in this integral, it must be in the second term.

⇨ To get rid of the imaginary part in the denominator, we multiply top and bottom by $(\overline{u} - c)^*$. We get $\overline{u} - c$ on the outside of the integral. We do not care about $\overline{u}$ or the real part of c because they are both real. We only care about the imaginary part of c (c_i). This procedure allows us to isolate the imaginary part of the integral:

$$c_i \int_0^L \left(\beta - \frac{d^2\overline{u}}{dy^2}\right) \frac{|\phi|^2}{|\overline{u} - c|^2} dy = 0$$

↳ This quantity has to be equal to zero because it is the only imaginary bit of the whole equation.

↳ Either - $c_i = 0$, i.e. there is no imaginary part for the phase speed and the flow is **stable**
or - $c_i \neq 0$, in which case **the integral must be zero**.

⇨ How can this integral be zero? The ratio is real and positive, which means that $\beta - \overline{u}_{yy}$ is either zero everywhere or ***at the very least*** **$\beta - \overline{u}_{yy}$ must change sign somewhere in the domain** between (0 and L).

⇨ This term, which can be written $\frac{d}{dy}(f_0 + \beta y - \overline{u}_y)$, i.e. the meridional **gradient of the absolute vorticity must change sign** somewhere in the domain.

$\beta - \overline{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
↳ If the Rayleigh criterion is satisfied, we might have an instability.

⇨ This is a necessary condition for the integral to be zero, which is
- a necessary condition for the phase speed to have an imaginary part, which is
- a necessary condition for **instability**
↳ This is the **Rayleigh criterion for instability**

⇨ The condition is to have an extremum (maximum or minimum) in the **absolute vorticity**, i.e. its gradient changes sign somewhere in the domain, i.e. the velocity profile has an inflection point.

⇨ This Rayleigh condition is a **necessary condition** for **barotropic instability**.

3.3.f) More conditions for growth: the Fjørtoft criterion

⇨ There is another necessary condition called the **Fjørtoft condition**. The Rayleigh condition dealt with the imaginary part of the linear vorticity equation in a parallel shear flow. But the **real part** must also be satisfied, so:

$$-\int_0^L \left(\left|\frac{d\phi}{dy}\right|^2 + l^2|\phi|^2\right)dy + \int_0^L (\bar{u}-c_r)\left(\beta - \frac{d^2\bar{u}}{dy^2}\right)\frac{|\phi|^2}{|\bar{u}-c|}dy = 0$$

The first LHS term is negative, which means that:

$$\int_0^L (\bar{u}-c_r)\left(\beta - \frac{d^2\bar{u}}{dy^2}\right)\frac{|\phi|^2}{|\bar{u}-c|}dy > 0$$

⇨ This inequality $\int A(u-c) > 0$ is similar to the imaginary part of the equation we dealt with deriving the **Rayleigh condition**, i.e. $\int A = 0$

Fjørtoft criterion

Fjørtoft logic:

- It consists of decomposing $(u-c)$ into two terms: $(u-u_0)+(u_0-c)$:

$$\int A(u-u_0) = \int A(u-c) + \int A(c-u_0) > 0$$

- With $(c-u_0)$ being a constant and with $\int A = 0$, the last term cancels.

↳ It follows that $\int A(u-u_0) > 0$ **must be true for any value of u_0**

- With $\dfrac{|\phi|^2}{|\bar{u}-c|^2} > 0$, this means ***at the very least***

$(\bar{u}-u_0)(\beta-\bar{u}_{yy})$ **must be positive somewhere in the domain for any value of u_0**.

$\forall u_0$, $(\bar{u}-u_0)(\beta-\bar{u}_{yy})$ must be positive somewhere in the domain.
↳ If the Fjørtoft criterion is satisfied, we might have an instability.

⇨ ***Large values of u_0***:

For a very large positive value of u_0, $(u-u_0) < 0$ and $(\beta-\bar{u}_{yy})$ must be < 0 somewhere.
For a very large negative value of u_0, $(u-u_0) > 0$ and $(\beta-\bar{u}_{yy})$ must be > 0 somewhere.

↳ This a weaker version of saying that $(\beta-\bar{u}_{yy})$ must change sign. This is the **Rayleigh criterion**, saying that the gradient of the absolute vorticity has to change sign in the domain.

↳ So large values of u_0 add nothing to the Rayleigh criterion.

⇨ ***Medium values of u_0***:

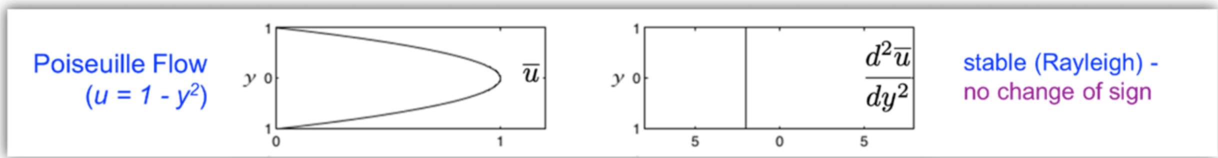
It is most useful to choose **u_0 to be the value of $U(y)$ at which $(\beta-\bar{u}_{yy})$ vanishes**. This leads to the **Fjørtoft criterion**. Moderate values of u_0, such as $(u-u_0)$ also changes sign in the domain somewhere, adds an extra criterion which is more of a constraint than just the **Rayleigh criterion** (see example in **#GFD3.3g**). The **Fjørtoft criterion** is satisfied if **the magnitude of the absolute vorticity has an extremum inside the domain**, and not at the boundary or at infinity – the velocity profile must have an inflection point inside the flow. **Fjørtoft** necessary condition is **thus a stronger condition** than the **Rayleigh criterion**.

☝ Both **Rayleigh** and **Fjørtoft** criteria are just ***necessary conditions***. They are not sufficient conditions. This means that, when analyzing a potential vorticity map, if one of these conditions is satisfied, it does not mean that the flow is unstable, it means that ***it is possible for the flow to be unstable***.

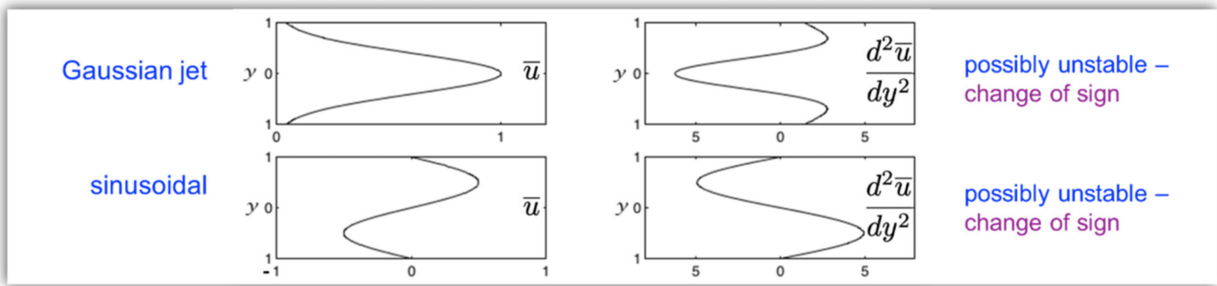
On the other hand, the non-satisfaction of a necessary condition is a sufficient condition, which means that ***if the Rayleigh or the Fjørtoft condition is not satisfied then the flow is stable***.

3.3.g) Stable and unstable profiles

⇨ In the examples below, parallel shear-flow could induce instability. The left column displays the zonal component of the flow, while the right column shows the associated second derivative.



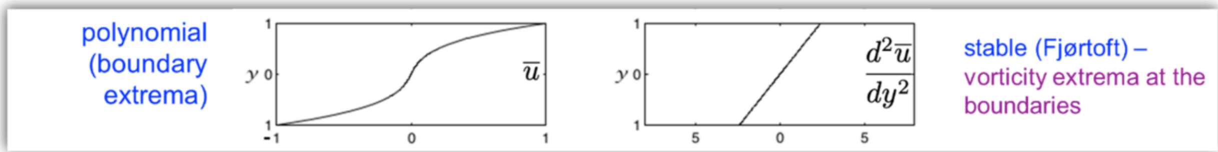
1) A Poiseuille flow corresponds to the quadratic form of a viscous fluid flowing in a pipe. The derivative of the vorticity is a constant that does not change sign in the domain. By **Rayleigh's criterion**, **it must therefore be stable**.



2) A Gaussian jet has an exponential form with extrema. It is therefore potentially **unstable**.

3) A sinusoidal profile has another sinusoidal function as its derivative. Likewise, this flow is potentially **unstable**.

The ***β-effect can be either stabilizing or destabilizing***. If the β-effect were present and large enough to have $\left(\beta - \bar{u}_{yy}\right)$ one-signed, it would stabilize the Gaussian or sinusoidal jets.



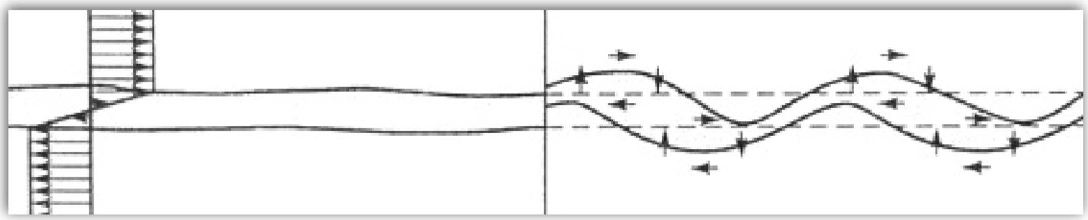
4) This third-order polynomial profile is **stable** by **Fjørtoft's criterion** (note that the vorticity extrema are at the boundaries).

- By the **Rayleigh** criterion, it could be unstable because the basic flow vorticity has extremes.

- The **Fjørtoft's** criterion dictates that $\bar{u}$ has to have the same sign as $-\bar{u}_{yy}$ somewhere in the domain. Here, they have opposite signs everywhere. It thus fails **Fjørtoft's** criterion. Fjørtoft's u_0 constant could shift $\bar{u}$ but the sign requirement must be true for all values of u_0. In this case, it fails for $u_0 = 0$. The polynomial profile is thus **stable**.

3.3.h) Physical mechanism

How does the flow become barotropically unstable? Here is an example of a background flow:

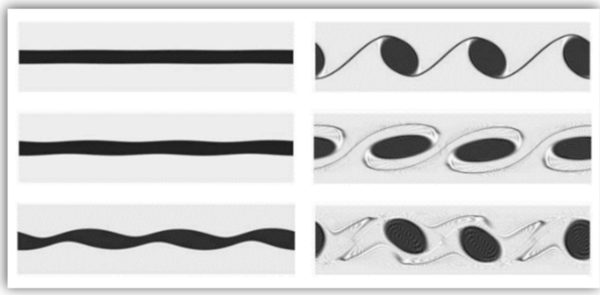


▪ To the **south**, we have uniform **easterlies**, and to the **north**, uniform **westerlies**. In these two regions, there is no background vorticity.

▪ In between, there is a **transition zone** with a strip of parallel shear flow, i.e. a strip of background negative vorticity (clockwise) – an extremum. The flow is potentially unstable.

↳ Consider a small perturbation, such as a streamline is moved slightly to the north. It exports its vorticity into a region where there is none. At the same time, on the other side of the vorticity strip, but just out of phase, the same thing happens.

The secondary circulation is going to displace the vorticity contours, so that it deforms the vorticity strip and the situation amplifies and the deformation continues.



GFD3.4: Baroclinic Instability

3.4.a) Baroclinic instability

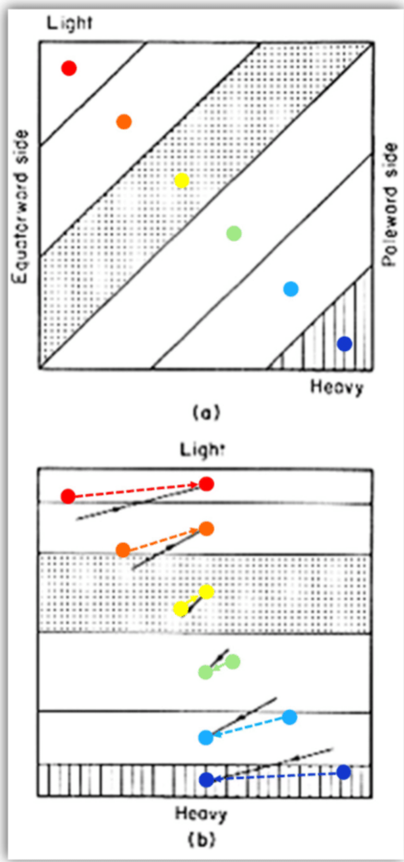
⇨ It is similar but for a baroclinic flow, in which there is a **vertical dependence in the background flow**. ***How would a perturbation grow on this sort of flow?***

↳ Let's start by thinking about energy. Instabilities are growing perturbations, where do they get their energy from?

- We have seen in **#GFD3.3** that barotropic instabilities take their energy out of **the horizontal shear** of the background flow.
- Baroclinic instability also takes its energy from some property of the background flow.

Let's consider the ***configuration shown in the figure below***. Upward is at the top and northward to the right.

We define a very ***idealized geophysical situation*** on a rotating planet, in which there are tilted layers of different densities. To the north and at low levels we have heavy dense fluid (cold water in the ocean). Towards the equator and at upper levels the water gets warmer and lighter. In between, there are tilted homogeneous density layers. In each of these layers of different density, the ***colored dot is placed at the center of gravity*** of the layer. Note that if it is in the atmosphere, we need to take into consideration the compressibility of air and we must consider potential temperature.



If we take all these layers and flatten them out, where would the center of gravity go? Imagine filling the same amount of space and laying out each layer horizontally. The densest layer is spread horizontally at the bottom, while the lightest layer becomes the surface layer of our ocean. The center of gravity of the denser layers has moved downwards and for the lighter layers it has moved upwards as the fluid rearranges. ***The center of gravity of the whole fluid would go down*** because heavy layers have more influence on the total center of gravity than the lighter layers.

↳ **By rearranging the fluid, we have moved the center of gravity downwards**. This means **we have liberated potential energy** to supply kinetic energy. Release of instability can be considered as a transfer of energy from a basic state to a flow.

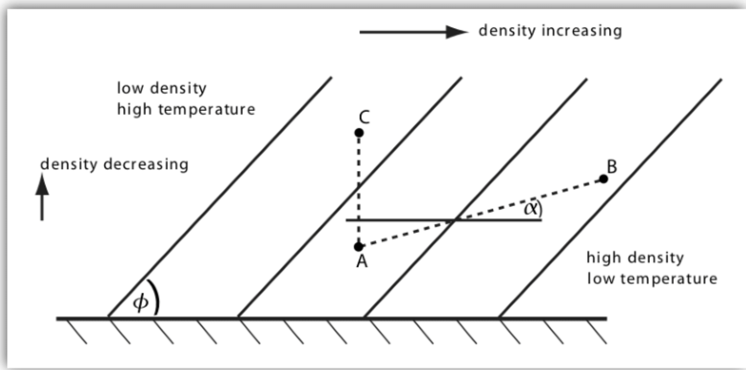
3.4.b) Sloping convection

Sloping convection is another way of thinking about baroclinic instability.
Is the following structure stable to perturbations?

- We consider a fluid that is **barotropically stable**, i.e. there is no reversal of the barotropic vorticity gradient (Rayleigh criterion, see **#GFD3.3e**), a parcel of fluid is not going to take off.
- The fluid is also **statically stable**, i.e. we do not have vertical gravity instability (cold air is at the bottom and warm air is at the top).
- Density contours are tilted, so that (potentially) cold air remains in the down/north region, while potentially warm air is up and south. In a rotating system, we can imagine a steady basic state with inclined density contours (we need rotation to balance the pressure gradient forces).

Displacement A-C: A parcel of fluid A is displaced vertically into position C. As it moves into this lighter layer, the parcel will be heavier than its surroundings and it is going to drop back down. This is static ***stability***.

Displacement A-B: A parcel of fluid A is displaced northwards into position B, along a slope. It moves into denser air and it is lighter than its surroundings. It can thus keep on going up and north. This a potential ***instability***/baroclinic instability called ***Sloping convection***. The energy stored in the density structure is released.



3.4.c) Optimal scales for growth

⇨ At what kind of <u>scales</u> does this happen? One of the things that is important to understand is that the process of baroclinic instability depends on some sort of communication between different levels, and there are certain scales on which that happens.

⇨ Let's go back to the definition of the quasi-geostrophic potential vorticity equation (f-plane Boussinesq, see **#GFD2.3i**) and do a basic scale analysis:

- The <u>relative vorticity</u> ($\nabla^2\psi$) is of the order of $\sim\psi/L^2$, with L the typical length scale at which we have vorticity gradients.
- The <u>vortex stretching term</u> is of the order of $\sim \dfrac{f^2\Psi}{N^2H^2} = \dfrac{\Psi}{{L_R}^2}$

↳ This means that these two terms are of comparable magnitude when L is comparable to L_R. On length scales comparable to the Rossby radius, both of these terms will be important and this is what we need to amplify perturbations:

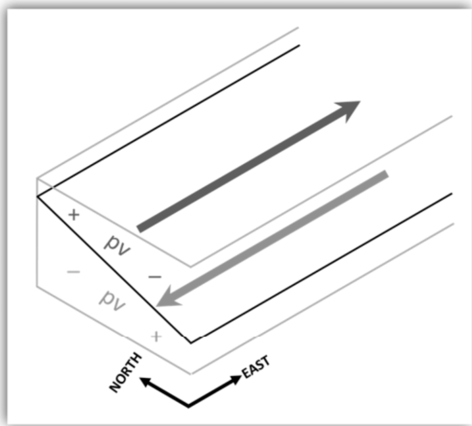
- if $L \gg L_R$, the relative vorticity term will be small and vertical coupling will dominate. There will not be much difference between the top and bottom layers. At these scales, the fluid will essentially be **barotropic**.
- if $L \ll L_R$, then the relative vorticity will dominate and there will not be any coupling between the layers. The fluid would behave just like uncoupled/**independent layers**.
- **On the particular length scale of the Rossby radius**, there is some interplay between these two terms and this will allow **the liberation of potential energy stored in the background state** (horizontal variations of density or vertical shear of the wind).

⇨ If we are right that the Rossby radius scale is the scale on which perturbations can grow, then we should see these scales naturally in a geophysical fluid, just like ***Darwinian selection***. We observe the scales that amplify. If the mechanism of amplification depends on it being a certain scale then this is the scale that should be seen on weather maps or diagnostics of ocean variability.

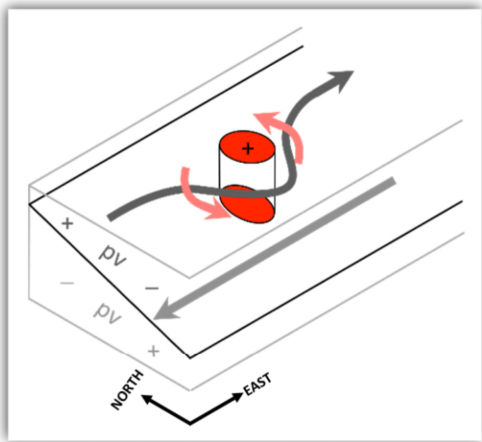
↳ This is the scale we indeed observe: when looking at weather maps (see **#GFDintro**), we see cyclones and anticyclones. Altimetric sea surface height show ocean eddies - all on the Rossby radius scales $\left(L_R = \frac{NH}{f}\right)$.

3.4.d) Physical mechanism

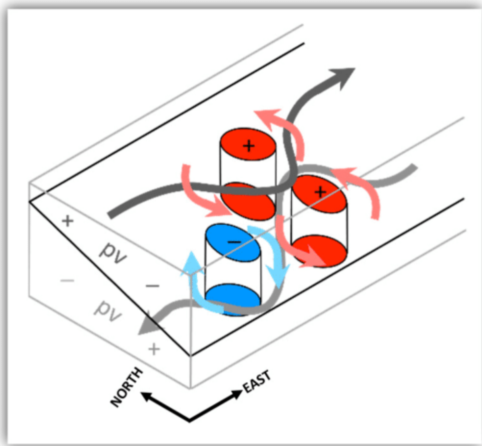
⇨ ***How does it work?*** Here is a schematic to explain the physical mechanism.



1) Consider a ***two-layered shear flow in thermal wind balance***. There are two layers with a slope between them. In the upper layer, the current is flowing eastwards, while in the lower layer, it is flowing westwards. The slope means that the layer thickness varies from north to south and accordingly the potential vorticity for the upper layer increases towards the north. In the lower layer, it is the other way around.



2) We introduce a positive potential vorticity **perturbation** (PV+) into the top layer with an associated cyclonic flow that diverts the upper-layer eastward jet.



3) Positive vorticity is associated with positive layer thickness that will **squeeze the layer below** and **drive a circulation in the same way**. In the lower layer, west of the upper-layer perturbation, this circulation will advect PV- southward, and east of the upper-layer perturbation, it will advect PV+ northwards creating **a perturbation dipole in the lower layer**. This will also divert the lower-layer westward jet.

4) In the center of the dipole, there is a southward component in the lower-layer flow, which in turn will **impact the upper layer dynamics**. This induces southward advection of more positive potential vorticity in the layer above, **amplifying the original perturbation**, which will grow.

↳ If there is the right phase relation between perturbations, they can mutually amplify and grow. In this configuration, there is **a slope of the dynamical perturbation towards the west with height** and which is consistent with the extraction of energy from the basic state sloping density surfaces to produce a circulation anomaly which can grow exponentially.

✋ At the same time, due to the upper-level potential vorticity gradient and the gradient of f, the entire structure propagates westwards (relative to the mean flow) as a Rossby wave.

3.4.e) Modal solutions

⇨ Here is an overview of the **theoretical framework** in order to analyze under which ***conditions baroclinic instability is possible***.

↳ Here is a reduced version of the linear perturbation PV equation, in which q' is a perturbation potential vorticity and Q is the potential vorticity associated with the background state.

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial Q}{\partial y} = 0$$

⇨ We seek wave-like solutions in x and the amplitude coefficients as a function of y (as in **#GFD3.3d**) and also of z because of the presence of a vertical component in the variations of the background flow, $\psi' = \tilde{\psi}(y,z)e^{i(lx-\omega t)}$. We substitute this solution into the linear vorticity equation:

$$(U-c)(\tilde{\psi}_{yy} + \frac{\partial}{\partial z}\frac{{f_0}^2}{N^2}\tilde{\psi}_z - l^2\tilde{\psi}) + Q_y\tilde{\psi} = 0$$

With the boundary conditions (equivalent to $w = 0$) at top and bottom: $(U-c)\tilde{\psi}_z - U_z\tilde{\psi} = 0$

↳ This is the equivalent of our Rayleigh equation in the barotropic case (see **#GFD3.3d**) but this equation contains both horizontal and vertical derivatives.

3.4.f) Conditions for growth

⇨ In order to analyze the conditions for growth, as in **#GFD3.3e**, we multiply the equation by the complex conjugates of $\tilde{\psi}$ and integrate the equation across the whole domain, i.e. in the north-south direction (from 0 to L) and also in height (from 0 to H). It leads to:

$$\int_0^L\int_0^H |\Psi_y|^2 + f_0^2/N^2|\tilde{\psi}_z|^2 + l^2|\tilde{\psi}|^2 dzdy - \int_0^L \left\{ \int_0^H \frac{Q_y}{U-c}|\tilde{\psi}|^2 dz + \left[\frac{f_0^2/N^2U_z|\tilde{\psi}|^2}{U-c}\right]_0^H \right\} dy = 0$$

↳ The first LHS term is ***positive definite and real***. Therefore, if there is anything imaginary in this integral, it must be in the second term.

⇨ We analyze this term for the ***possibility of it having an imaginary part*** for the phase speed. To get rid of the imaginary part in the denominator, we multiply top and bottom by $(\bar{u}-c)^*$, and isolate the imaginary part:

$$-c_i\int_0^L \left\{ \int_0^H \frac{Q_y}{|U-c|^2}|\tilde{\psi}|^2 dz + \left[\frac{f_0^2/N^2U_z|\tilde{\psi}|^2}{|U-c|^2}\right]_0^H \right\} dy = 0$$

↳ If $c_i \neq 0$ then the integral must be zero. Instead of just having one criterion, the Rayleigh criterion (see **#GFD3.3e**), we need to think about all the circumstances in which this integral could be zero:

- Disregarding the second term (no vertical shear of the background flow, $U_z = 0$), there is the same condition as for the barotropic case, i.e. the basic state potential vorticity gradient (Q_y) could change sign somewhere in the domain.
- Disregarding the first term (no horizontal gradient of background PV, $Q_y = 0$), the vertical shear (U_z) has to have the same sign at the top ($z = H$) and bottom ($z = 0$).

Then there is an interplay between the two terms. If these two terms have the opposite sign, it means:

- The gradient of potential vorticity (Q_y) has to have the opposite sign to the vertical shear (U_z) at the top level ($z = H$), or
- Q_y has the same sign as a vertical shear (U_z) at the bottom level ($z = 0$).

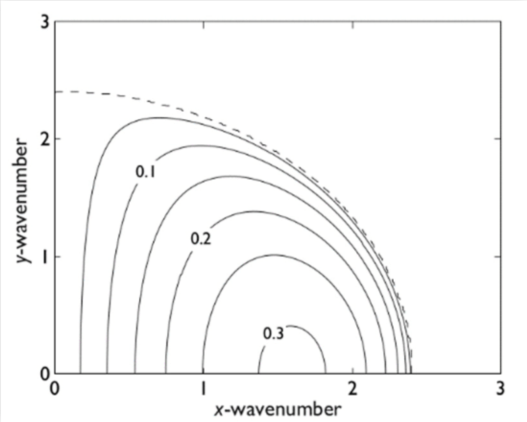
⇨ There are **4 possibilities**, called the **Charney-Stern-Pedlosky criteria**. These are necessary conditions but not sufficient conditions: ***if at least one of these four criteria is satisfied then we might have an instability***. If this the case, then waves can grow either in the interior of the fluid if we have a PV extremum for instance or on the boundaries if we have temperature gradients on the boundaries.

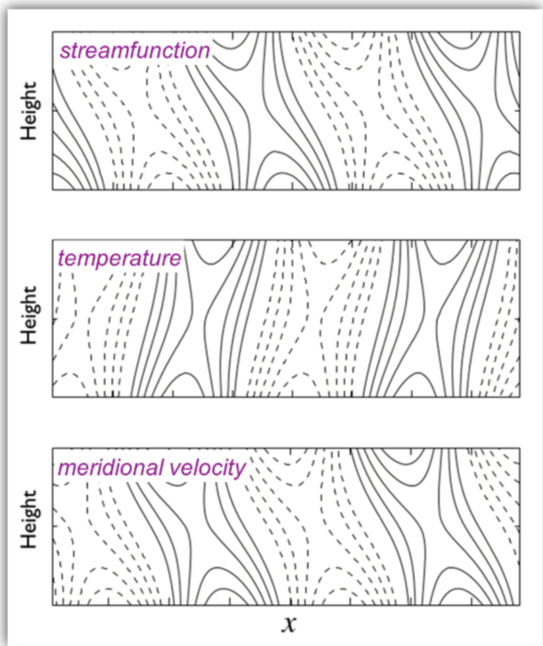
3.4.g) The Eady problem

The analytical solution can be derived for a simple configuration. It is the Eady problem, in which:
• The motion is on an f-plane ($\beta = 0$).
• The fluid is uniformly stratified (N^2 is constant).
• The basic state has uniform shear: $U(z) = Uz/H$.
• The motion is contained between two rigid, flat horizontal surfaces.

↳ Constant vertical shear implies that the ***basic state PV is zero*** (Q = 0), which makes the Eady problem a special case that can be ***solved analytically***. ***Solutions have modes that grow on the boundaries***.

The non-dimensionalized **growth rate** as a function of the zonal and meridional wavenumbers (non-dimensionalized by the Rossby radius: figure on the right) shows stable conditions for short-waves and for any given zonal wavenumber the most unstable wavenumber is that with the gravest meridional scale. This figure also highlights the scale of the maximum exponential growth, close to the Rossby radius.





▪ The maximum growth rate is
$$\sim 0.31\frac{U}{L_R}$$

▪ Wavenumber and wavelength at which the instability is the greatest are:

$$k_m = \frac{1.6}{L_R} \qquad \lambda_m = \frac{2\pi}{k_m} = \frac{2\pi}{1.6}L_R$$

▪ The structures of these modes for the most unstable Eady mode are tilted with height towards the west.

More details can be found in Vallis (2017)

3.4.h) What we learn from the Eady problem

• The maximum growth rate is $0.31U/L_R$ and there is a length scale associated with the maximum instability, close to the Rossby radius scale (a factor of 3.9).
• There is a short-wave cutoff – short-waves are not unstable.
• The circulation (meridional current, stream-function) must slope westwards with height in westerly shear to extract energy from the basic state.

✋ In the Eady problem, the instability relies on an interaction between waves at the upper and lower boundaries. If either boundary is removed, the instability dies.

↳ To get a qualitative sense of the nature of the instability, we choose some typical parameters:

⇨ ***For the ocean***, we choose:

$$H \sim 1km, \qquad U \sim 0.1m.s^{-1}, \quad N \sim 10^{-2}s^{-1}$$

We then obtain:

$$\text{Rossby radius } L_R = \frac{NH}{f} \approx \frac{10^{-2} \times 1000}{10^{-4}} \approx 100km$$

$$\text{Instability scales } \sim 3.9 \times L_R \approx 400km$$

$$\text{Growth rate } \sim 0.3\frac{U}{L_R} \approx \frac{0.3 \times 10}{10^5} \approx 0.026\ day^{-1} \quad (\text{Period} \approx 40\ days)$$

⇨ ***For the atmosphere***:

$$H \sim 10km, \qquad U \sim 10m.s^{-1}, \quad N \sim 10^{-2}s^{-1}$$

We then obtain:

$$\text{Rossby radius } L_R = \frac{NH}{f} \approx \frac{10^{-2} \times 10^4}{10^{-4}} \approx 1000km$$

$$\text{Instability scales } \sim 3.9 \times L_R \approx 4000km$$

$$\text{Growth rate } \sim 0.3\frac{U}{L_R} \approx \frac{0.3 \times 10}{10^6} \approx 0.26\ day^{-1} \quad (\text{Period} \approx 4\ days)$$

↳ The time scale is a few days for a ***weather system***

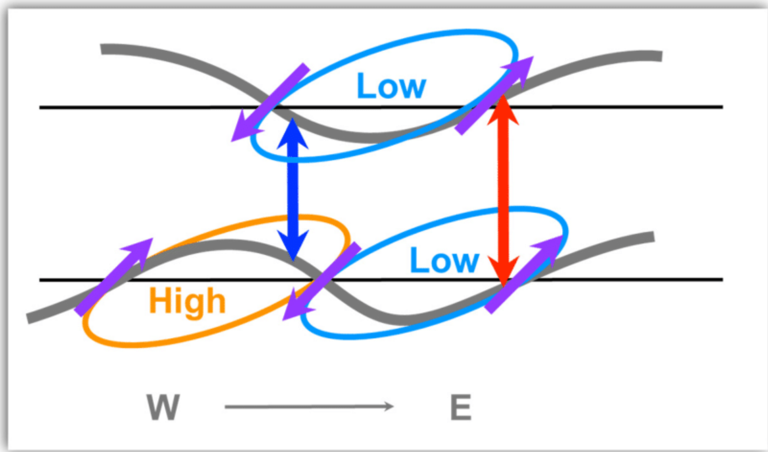
3.4.i) Heat transport in a baroclinic system

⇨ Baroclinic instabilities are important for the **climatic system**.

1) Consider how **radiative forcing** heats the equatorial region and cools at the poles. There is a **zonal jet** flowing horizontally between the two regions, consistent with **thermal wind balance**.

2) If this radiative forcing persists, the **jet will get stronger** as the equator gets warmer and the pole will get cooler.

3) At some point, the jet will **break out into eddies**. The origin of the growth is the **unstable profile** either in the horizontal or in the vertical direction (or both) - baroclinically unstable. The perturbations that grow will have **this shape**:



▪ A **low-pressure anomaly in the upper layer** with the associated pressure surface dipping downwards. In the lower layer, there is **a dipole of high and low-pressure**, slightly shifted.

▪ The distance/**thickness** between surface pressure level is indicative of the **temperature** ($\frac{\partial \psi}{\partial z}$):

- East of the upper level Low, they are far apart: the air is warm.
- West of the upper level Low, they are close together: the air is cold.

▪ The perturbation flow advects the **warm air towards the north**, while cold air is advected to the south. Such a weather system transports warm air upward and poleward and cold air downward and equatorward.

⇨ On the one hand, this configuration is the configuration that **perturbations need to exist** and grow. A configuration that has this **westward slope with height** leads to the **extraction of energy from the background state**.

⇨ On the other hand, this configuration is the configuration required to **transfer heat to the north** and thus **reduce the temperature gradient** between the equator and the pole, releasing the instability, flattening the isentropic slopes that are continually built up by the radiative forcing, and **dissipating the background jet**. This is an example of scale interaction (see **#GFD5**).

3.4.j) Baroclinic instability: summary

1) There is clear evidence of a preferred scale for turbulent motion in the ocean and the atmosphere.
2) Simple scaling arguments and more sophisticated stability analyses show that there is a preferred scale for growth to occur.
3) If this growth depends on extracting energy from sloping density surfaces (or equivalently vertical wind shear or horizontal temperature gradients) then there must be an interplay between vortex stretching and relative vorticity terms in the conservation of potential vorticity.
4) This naturally select structures around the Rossby radius scale.
5) These structures can grow exponentially provided certain criteria are met, notably if there are extrema (maxima and minima) in the potential vorticity of the basic state.