CHAPTER 2
Quasi-Geostrophic Theory

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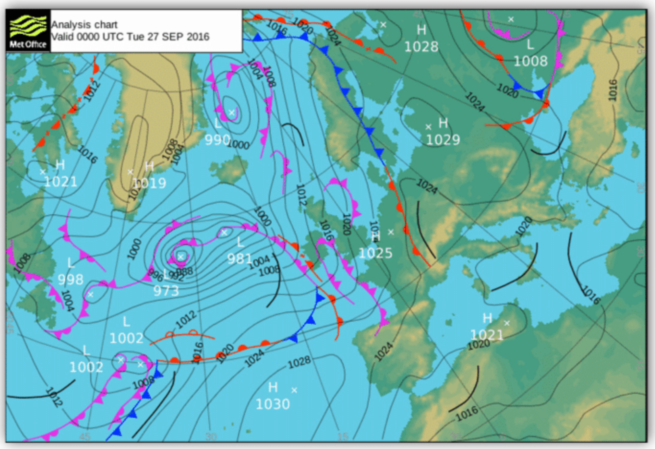
GFD2 Contents

In this chapter, we will tackle **quasi-geostrophic theory**. On the weather map on the right, surface pressure lines (isobars) are shown with black lines. If the wind follows the isobars exactly, it is in geostrophic balance. The closer together the isobars, the stronger the wind. But in this equilibrium, this pattern is not going to be transported/displaced. The wind circulation will remain as it is and never change. On the one hand, geostrophic balance is a very good way to **describe the flow**, but on the other hand to make **weather forecasts** (predict changes in the flow) we need to include more terms in our equation system than only geostrophic balance. In this chapter, we will consider the closest thing we can get to geostrophic balance, i.e **small departures from geostrophic balance**. It is called **quasi-geostrophic** because it is almost geostrophic but not quite.

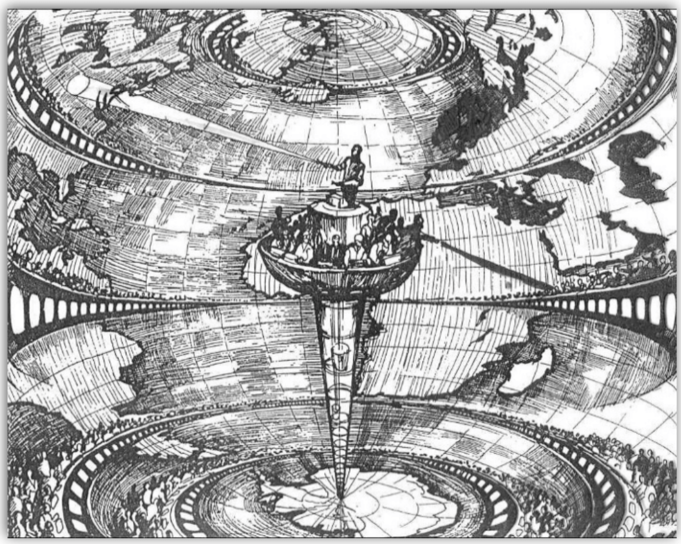


1) We start with an example of **steady departures from geostrophy**, with a flow that does not develop in time but in which we can describe the effects of non-linearities and drag (see **#GFD2.1**).

2) Then, we will discuss **ageostrophic flow** and the **importance of the divergent part of the flow**. We will introduce a new formulation of the conservation of the potential vorticity. We will start with the assumption that we are on an f**-plane** (see **#GFD2.2**) and then we will generalize to the situation where the planet has some **curvature** (see **#GFD2.3**).

3) We will finish by studying various **applications** of quasi-geostrophic theory (see **#GFD2.4**).

Quasi-geostrophic theory is very important because it was the basis of the first weather forecasts. It was the equation set used to predict the weather. The picture below refers to something called Richardson's dream. Lewis Fry Richardson (1881-1953) was one of the founders of the science of meteorology. Before the computer era, he had the idea that we could analyze the equations of motion to predict the weather. But they are so difficult to solve that you need lots of calculations. He dreamed about an amphitheater full of people making calculations with their pencil, paper, and their log tables, passing information to one another. He was ahead of his time, effectively imagining a massively parallel multi-core cluster. He anticipated the idea that we would solve the equations by some sort of multitude of calculations. And it is what we actually do nowadays, i.e. making weather predictions by discretizing (in space and time) and solving partial differential equations. And, of course, we do this on machines capable of performing very many calculations per second (super-calculators).



"Richardson's dream"

GFD2.1: Steady departures from Geostrophy

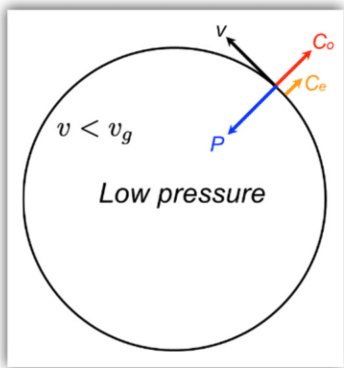
2.1.a) Gradient wind balance

⇨ We start by considering small ***steady departures from geostrophy***. Let's recall the zonal shallow water momentum equation, with the flow tendency, the advection terms, the Coriolis force, and the pressure gradient force expressed through the gradient of the layer thickness (see **#GFD1.2ef**):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \boxed{- fv = -g\frac{\partial h}{\partial x}}$$

↳ The two terms on the right are geostrophic balance (outlined in green), in which pressure gradient force balances the Coriolis force.

⇨ Now, let's consider **time-independent** (steady) **flow around a circle**. In a simple way, the nonlinear terms represent the **<u>local</u> centrifugal force** associated with this circular motion. This is **gradient wind balance** (without the Coriolis force it is "cyclostrophic" balance).



• On the schematic on the right, the flow is going around a **<u>low-pressure system</u>**, a perfect cyclonic motion. There is centrifugal force associated with this circular motion ≡ something extra compared to geostrophy.

↳ The **pressure force** ($\vec{P} \sim g\frac{dh}{dr}$) pushes the flow towards the center of the low pressure. It is balanced partly by the **Coriolis force** ($\vec{Co}$), which yields an anti-clockwise flow. Since the flow is spinning around, there is also a **centrifugal force** ($\vec{Ce}$) associated with the curvature of the flow. Notably, **both Coriolis and centrifugal forces are fictitious**, associated with the choice of reference frame.

↳ In this example, we consider the balance between these two fictitious forces and the real pressure gradient force. It follows:

$$fv + \frac{v^2}{r} = g\frac{dh}{dr} = fv_g$$

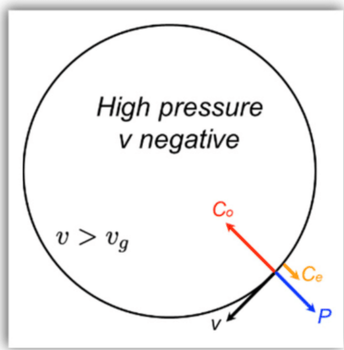
Coriolis term (fv)+centrifugal ($\frac{v^2}{r}$) is equal to the pressure gradient ($g\frac{dh}{dr}$). The latter is positive for a **cyclone** because the pressure is low in the center and increases outwards along the radius. If the flow were in geostrophic balance, the pressure gradient would be balanced by fv_g.

Since these two forces sum-up ($\vec{Co}$+$\vec{Ce}$) to compensate for the pressure gradient force ($\vec{P}$) and keep the flow parallel to the isobars, **the Coriolis force does not need to be as strong** as if it were in geostrophic balance. This means that **the flow does not have to be as fast** as the geostrophic velocity v_g. As a result, v is slightly weaker than v_g. We can just eliminate the pressure gradient in the equation above and rewrite the equation in terms of the difference between v and v_g. It follows:

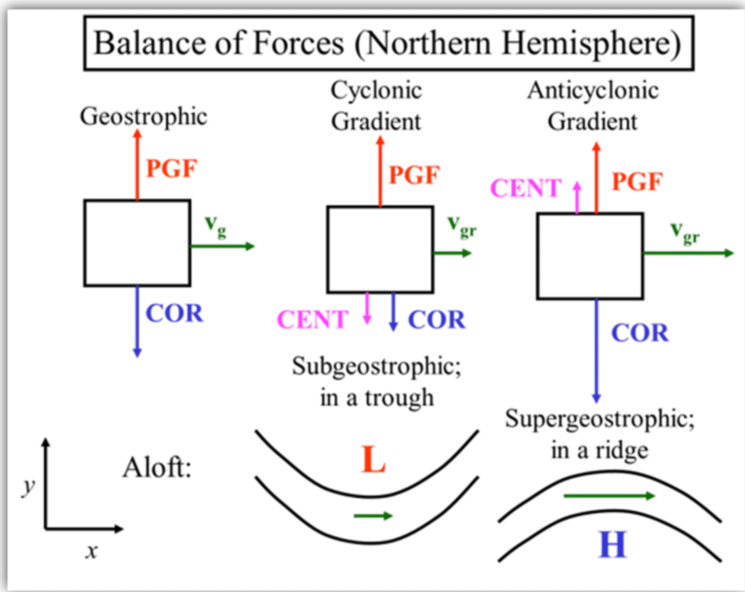
$$v\left(1 + \frac{v}{fr}\right) = v_g$$

$$R_0 = \frac{U}{fL} = \frac{v}{fr}$$

↳ With $\frac{v}{fr}$ positive, $1 + \frac{v}{fr} > 1$, confirms that v **is slightly weaker than its geostrophic counterpart** v_g.



• In the case of a **<u>high-pressure system</u>** (an anticyclone), the pressure is decreasing outwards and the flow is going the other way around – clockwise. There is now a balance between the sum of the centrifugal force and the pressure gradient force ($\vec{Ce}$+$\vec{P}$) and the Coriolis force ($\vec{Co}$). As a consequence, the Coriolis force needs to be stronger as if it were in geostrophic balance. So, v is slightly stronger than if the flow were geostrophic ($v > v_g$).



Credit: H.N. Shirer

• We can group these two cases: $|v| = \frac{|v_g|}{1 \pm R_o}$, with R_0 the Rossby number (see **#GFD1.2a**).

↳ If the flow is close to geostrophic, $v \sim v_g$ because R_0 is small.

⇨ The full solution to this quadratic equation is: $v = -\frac{fr}{2} \pm \sqrt{\frac{f^2r^2}{4} + rg\frac{dh}{dr}}$

Cyclone limit

▪ For a **cyclone**, as the pressure gradient is positive (pressure is increasing outwards), everything under the square root is positive and we can have real solutions. There is no limit to the strength of the pressure gradient.

▪ In the case of an **anticyclone**, the pressure gradient is negative. As a consequence, if the pressure gradient gets too strong, it leads to a square root of a negative quantity. In that case, the equation does not have real solutions. So, in the case of an anticyclone, there is a limit in the strength of the pressure gradient beyond which the equation does not have solutions: $\left|\frac{dh}{dr}\right| \leq \frac{f^2r}{4g}$

Asymmetry

↳ There is a limit on the strength of the pressure gradient associated with anticyclones compared to cyclones. This explains the **asymmetry between high- and low-pressure systems** we observed in **#GFD.intro**. On the previous weather map, there is a very intense cyclone south of Greenland, associated with very strong pressure gradients, while the anticyclone off the coast of Spain resembles a flat pattern and is associated with rather weak pressure gradients. This asymmetry is the result of a steady ageostrophic term, called **gradient wind balance**.

Wong way and Development

↳ In the full solution, there is a $\pm$, which means that in theory, the flow can go the wrong way around a cyclone. So, it is **mathematically** possible to find an equilibrium in which there is a **clockwise flow around a low-pressure system**, i.e. a solution for which pressure gradient and Coriolis forces are both directed towards the center of the pressure system, balanced by the centrifugal force.

In reality, it is not really possible. We never observe it, except maybe on very small scales maybe. Because you have to consider how the **equilibrium develops**. Starting from rest, a particle will accelerate towards low-pressure centers, while the Coriolis effect will deviate its trajectory to its right. The particle will end-up be turning around the low pressure anti-clockwise. So, the natural way in which these systems come into being favors the normal solutions rather than strange anomalous solutions.

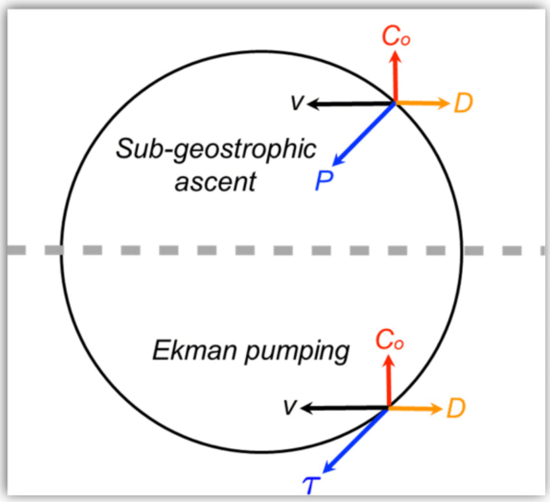
Sub-geostrophic flow

2.1.b) Boundary friction

⇨ Friction is another way to modify geostrophy in a steady flow, in which we add some drag to the system.

• Consider a **<u>cyclone</u>** in the **Atmosphere**. The **pressure gradient force** ($\vec{P}$) is directed toward the center (see illustration below). In addition to the **Coriolis force** ($\overrightarrow{Co}$), we add a **drag force** ($\vec{D}$) that points in the opposite direction to the flow, so this **drag and Coriolis forces balance the pressure gradient force**. For that to be possible, the flow has to **converge** into the cyclone. This is sub-geostrophic flow. If the wind converges towards the center of a cyclone, the mass will be gradually accumulated and the cyclone will decay as the low pressure fills at the surface.

But <u>before this happens</u>, the low-level flow (which experiences this drag and converges) will naturally give rise to upward motion by **conservation of mass**. We observe **ascent** in low-pressure/cyclone centers. This explains why a depression is always associated with cloudy weather (upward motion=condensation=clouds). Conversely, it is sunny in an anticyclone, associated with super-geostrophic (descending) flow. It is the opposite effect, the wind is blowing outwards and the flow experiences downward motion (=evaporation=clear skies).



Ekman convergence

• Let's focus on the **Ocean** and consider exactly the same diagram.

- Instead of being driven by a pressure gradient force, the flow is driven by surface wind-stress ($\vec{\tau}$). It is the stress which is acting on the upper surface layer of water.
- The same surface ocean current vector $\boldsymbol{v}$, directed at 45° from the wind forcing.
- It is being pulled back by friction with the water below, i.e. the drag ($\vec{D}$).
- The Coriolis force ($\overrightarrow{Co}$) is as usual perpendicular to the flow.

↳ We have the same balance but this is now **Ekman convergence**: wind-stress driving the flow, balanced by the Coriolis force and friction. It will be convergent and that will lead to downward motion in order to conserve mass. This is **Ekman pumping**. As each layer exerts a stress on the layer below, the movement of the upper layer will produce a stress on the next layer, creating the Ekman spiral.

In the case of the atmosphere, it is the slowing down of a flow that has already been established, while in the case of the ocean this is how the flow is forced.

⇨ But now we need to move away from these anecdotal cases and put together a system with advection and time dependence that is almost, but not quite geostrophic.

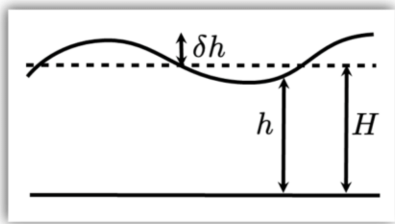
We do this essentially by separating the flow into a geostrophically balanced, nondivergent part, and the ageostrophic plus the divergent parts as a small perturbation. This small perturbation allows prognostic equations that lead to the evolution of the flow.

GFD2.2: Quasi-Geostrophic Theory I – the f-plane

2.2.a) Ageostrophic perturbations

⇨ The goal is to build a theory that is close to geostrophy but not quite geostrophic.

↳ We separate the flow in its geostrophic part and ageostrophic part, and we assume that the ageostrophic part is a small perturbation. We start with the shallow water momentum equations in a single layer (see **#GFD1.2f**):



$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0$$
$$\frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0$$

With $\frac{D}{Dt}$ the substantial derivative operator ($\frac{\partial}{\partial t}$+advection): $\left\{\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right\}$

⇨ What would happen if we just assume the **flow is geostrophic** ($\mathbf{v_g}$) and we substituted it into the momentum equations**?**

$$u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \quad v_g = \frac{g}{f}\frac{\partial h}{\partial x}$$

↳ We put it into the definition of the **substantial derivative**, using the geostrophic flow for the advection terms:

$$\frac{D}{Dt} \rightarrow \frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$$

✋ If we also put the geostrophic velocity ($\boldsymbol{v_g}$) into the Coriolis part instead of using the full flow ($\boldsymbol{v}$), this gives:

$$\frac{D_g}{Dt}u_g - fv_g + g\frac{\partial h}{\partial x} = 0$$

↳ The two terms on the right cancel each other and the geostrophic flow is non-divergent. This leads to a null tendency, which was expected as geostrophy is a balance. So, **we went one step too far**.

⇨ Instead, we make sure that the **equation remains linear in terms of the ageostrophic part** of the flow ($\mathbf{v}_{ag} = \mathbf{v} - \mathbf{v}_g$), i.e. $\mathbf{v}_{ag}$ contribution in the nonlinear terms (squared terms) will be small and we are going to neglected them, making the equations easier to solve. So, **the full flow is used in the linear Coriolis terms** ($\mathbf{v}$) and we advect with the geostrophic flow ($\mathbf{v}_g$). This is consistent with the idea that the ageostrophic part of the flow is small.

2.2.b) Quasi-geostrophic f-plane vorticity equation

⇨ Using geostrophic flow in the non-linear and tendency terms, and keeping the full flow in the Coriolis terms yields **quasi-geostrophic momentum equations**:

$$\frac{D_g}{Dt}u_g - fv + g\frac{\partial h}{\partial x} = 0 \quad (1)$$
$$\frac{D_g}{Dt}v_g + fu + g\frac{\partial h}{\partial y} = 0 \quad (2)$$

f-plane
$f = cst$

↳ As in **#GFD1.3b**, the **vorticity equation** is derived by cross differentiation: $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$

⇨ This process eliminates pressure gradient terms, and we get:

$$\frac{\partial}{\partial t}\xi_g + u_g\frac{\partial}{\partial x}\xi_g + v_g\frac{\partial}{\partial y}\xi_g + \xi_g\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = 0$$

f-plane ξ equation

⇨ We obtain an equation for the development of the **geostrophic vorticity**. Because **we have assumed that we are on an f-plane** (f=*cst*, i.e. $\frac{df}{dy}=0$), h is a stream function for u_g and v_g and **the divergence of the geostrophic flow** ($\nabla.\boldsymbol{v}_g=0$ on an f-plane) **remains zero**. The equation simplifies:

$$\frac{D_g}{Dt}(f+\xi_g)=-f\nabla.\mathbf{v}$$

↳ The geostrophic tendency of the **absolute geostrophic vorticity** is given by the **divergence of the ageostrophic flow**.

↳ ☞ If f **varies with latitude**, there is a divergent part to the geostrophic flow (see **#GFD2.3**).

2.2.c) Continuity equation

Here is the continuity equation in the form of the flux of layer thickness h (see **#GFD1.2d**):

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(uh)+\frac{\partial}{\partial y}(vh)=0$$

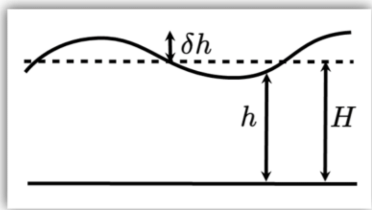
It can also be written as the substantial derivative of the layer thickness (see **#GFD1.3c**):

$$\frac{Dh}{Dt}+h\nabla.\mathbf{v}=0$$

Linearization of h

↳ As for the momentum equations (see **#GFD2.2b**), we replace the **substantial derivative** (D/Dt) by the **geostrophic operator** (D_g/Dt), and expand it. The continuity equation is then written:

$$\frac{\partial h}{\partial t}+\mathbf{v}_g.\nabla h+h\nabla.\mathbf{v}=0$$



☞ $h\nabla.\mathbf{v}$ is not a linear term, because h depends on $\mathbf{v}$. So, for consistency **we must make an approximation on h**:

$$h=H+\delta h,\quad \delta h\ll H$$

↳ We can now linearize the ageostrophic flow $h\nabla.\mathbf{v}$ by assuming (for this term only) it is approximated by H (a constant) times the divergence of the ageostrophic flow ($H\nabla.\mathbf{v}$). This is the equivalent to the approximation: $\mathbf{v}_g h\approx\mathbf{v}_g\delta h+\mathbf{v}H$.

f-plane continuity equation

↳ We obtain the following continuity equation: $\frac{\partial h}{\partial t}+\mathbf{v}_g.\nabla h+H\nabla.\mathbf{v}=0$

⇨ It can be written as the geostrophic substantial derivative of δh:

$$\frac{\partial}{\partial t}\delta h+\mathbf{v}_g.\nabla\delta h+H\nabla.\mathbf{v}=0 \quad \text{or} \quad \frac{D_g}{Dt}\delta h+H\nabla.\mathbf{v}=0$$

☞ To impose some linearity in the ageostrophic contributions, we had to make a strong approximation to the mean stratification: **the mean stratification, represented by the layer thickness (H), cannot vary in the horizontal**.

⇨ Finally, we can rewrite the continuity equation as: $\frac{f}{H}\frac{D_g}{Dt}\delta h=-f\nabla.\mathbf{v}$

2.2.d) Quasi-geostrophic potential vorticity

⇨ As in **#GFD1.3c**, we combine the vorticity (see **#GFD2.2b**) and the continuity (see **#GFD2.2d**) equations and eliminate the divergence, as follows:

$$\frac{D_g}{Dt}\left(f\frac{\delta h}{H}\right) = \frac{D_g}{Dt}(f+\xi_g)$$

↳ We obtain a <u>conservation principle</u>: $\frac{D_g}{Dt}\left\{f+\xi_g - f\frac{\delta h}{H}\right\} = 0$

⇨ This is the conservation law for ***f*-plane quasi-geostrophic potential vorticity**:

$$\frac{D_g}{Dt}q = 0, \quad q = f + \xi_g - f\frac{\delta h}{H}$$

q is conserved following the motion

In the absence of forcing or dissipation

⇨ It is not quite the same as the **Ertel potential vorticity** (see **#GFD1.3c**) <u>because there is a linearization of the stratification</u>. It does not even have the same units as the potential vorticity, but the **two quantities can be related**:

$$\frac{f+\xi}{h} = (f+\xi)\left(\frac{1}{H+\delta h}\right) = \left(\frac{f+\xi}{H}\right)\left(\frac{H}{H+\delta h}\right)$$

$$= \left(\frac{f+\xi}{H}\right)\left(\frac{H+\delta h-\delta h}{H+\delta h}\right) = \left(\frac{f+\xi}{H}\right)\left(1-\frac{\delta h}{H+\delta h}\right)$$

$$\frac{f+\xi}{h} \approx \left(\frac{f+\xi}{H}\right)\left(1-\frac{\delta h}{H}\right)$$

↳ As H is constant, with the **linearization of the stratification**, we obtain the conservation of the following quantity:

$$q = f + \xi - f\frac{\delta h}{H} - \xi\frac{\delta h}{H}$$

⇨ Then using scaling arguments, i.e. the **Rossby number is small**, the term involving relative vorticity is small compared to f.

$$Ro \ll 1, \quad \frac{U}{fL} \ll 1 \Rightarrow f \gg \xi, \quad \xi = \xi_g$$

↳ We can thus neglect the term on the right and recover the quasi-geostrophic potential vorticity formulation.

⇨ So, this **linearization of the layer thickness** is a surprising consequence of our <u>**insistence**</u> that the flow remains **close to geostrophic**. In a vertically continuous framework it means that the stratification is uniform in the horizontal (see **#GFD2.2b**).

GFD2.3: Quasi-Geostrophic Theory II – Expansion in small Rossby number

2.3.a) Adding curvature to the Earth

#GFD2.2 was pretty straightforward because we assumed that f was constant (f-plane). But many important dynamical phenomena depend on the **variation of f with latitude** (Rossby waves, for example, see **#GFD3**).

↳ On an f-plane, the geostrophic flow is strictly non-divergent, while on a planet with some curvature, the geostrophic stream function that contains f is not a proper stream function. It has departures associated with the **divergent part of the geostrophic flow**. So, allowing f to vary will complicate the theory as we have to deal with the divergent part of the geostrophic flow as well as the ageostrophic flow.

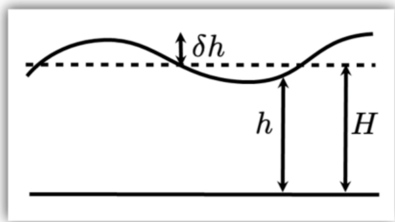
⇨ To proceed, we will derive the **quasi-geostrophic equation set** more formally than in **#GFD2.2**. We will do a **formal expansion** of these perturbations about a small parameter. We will naturally choose the **Rossby number** (see **#GFD1.2a**) for this small parameter.

2.3.b) Derivation of the quasi-geostrophic shallow-water momentum equations

• We recall the full 1-layer shallow water (see **#GFD1.2f**) momentum and continuity (in its divergence form, see **#GFD1.3c**, **#GFD2.2c**) equations using a vector notation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} + f\hat{\mathbf{k}}_\wedge\mathbf{v} + g\nabla h = 0$$

$$\frac{\partial h}{\partial t} + \mathbf{v}.\nabla h + h\nabla.\mathbf{v} = 0$$



Adimensionalization

• We now **non-dimensionalize** these equations.

- We use typical scaling values of length (L), speed (U), and time (T), to obtain non-dimensional variables noted with primes:

$$x' = x/L, \quad u' = u/U, \quad t' = t/T$$

- The layer thickness h can be written as $h = H + \delta h$. We non-dimensionalize the variations of the layer thickness (δh) by Δh a quantity typical of variations in the layer thickness (Δh, ✋ not H), as follows: $\eta' = \delta h/\Delta h$

↳ We substitute these non-dimensional variables into the shallow water equations, leading to:

$$\frac{U}{T}\frac{\partial \mathbf{v}'}{\partial t'} + \frac{U^2}{L}\mathbf{v}'.\nabla\mathbf{v}' + Uf\hat{\mathbf{k}}_\wedge\mathbf{v}' + g\frac{\Delta h}{L}\nabla\eta' = 0 \quad (1)$$

$$\frac{\Delta h}{T}\frac{\partial \eta'}{\partial t'} + \frac{U}{L}\Delta h\mathbf{v}'.\nabla\eta' + \frac{U}{L}(H + \Delta h\eta')\nabla.\mathbf{v}' = 0 \quad (2)$$

⇨ We obtain (messy) equations with scaling values in front of each term, in which the non-dimensional terms with prime ($\mathbf{v}'$ and η') are of **order 1**.

Relation between scalings

• So far, we have not made any assumptions or approximations. We now introduce the **quasi-geostrophic assumption** by **requiring that the relationship between the basic scalings** (L, U, T, and Δh) **conforms to geostrophic balance**, i.e. $f\mathbf{v} \sim g\nabla h$. In terms of typical scalings (with f_0 the value of f at a reference latitude), it follows that:

$$Uf_0 \sim g\frac{\Delta h}{L}$$

↳ We obtain an expression relating the value Δh to the other scaling parameters:

$$\Delta h = \frac{Uf_0L}{g} \quad (3)$$

• If we now define the **Rossby number** (see **#GFD1.2a**) and the **temporal Rossby number** and acknowledge that they will be small in the quasi-geostrophic approximation (see **#GFD2.3d** and **#GFD2.3f**). Note that if $U = L/T$, these two parameters are the same.

$$\epsilon = \frac{U}{f_0 L} \quad (4) \qquad \epsilon_T = \frac{1}{f_0 T} \quad (5)$$

⇨ For simplicity, <u>we now remove the primes</u> in the equations and rewrite them using our new scaling parameters (ε and ε_T).

↳ For the momentum equation, we divide (1) by $f_0 U$, replace Δh with (3), and then use (4) and (5). This leads to:

$$\epsilon_T \frac{\partial \mathbf{v}}{\partial t} + \epsilon \mathbf{v}.\nabla \mathbf{v} + \frac{f}{f_0} \hat{\mathbf{k}}_\wedge \mathbf{v} + \nabla \eta = 0$$

↳ **The last two terms constitute a non-dimensional form of geostrophic balance. The advection and development terms have epsilon in front (v, u, and η are of order 1).**

↳ It is worth mentioning that we **only made the hypothesis that the scales of the motion conform to geostrophic balance**. We just rewrote the equations using the ε and ε_T scaling parameters, we but have not yet assumed that these parameters are small. This will be done in **#GFD2.3f**.

2.3.c) Quasi-geostrophic continuity equation

⇨ For the continuity equation, we multiply (2) by L/UH, replace Δh by (3), and then use (4) and (5). It follows:

$$\epsilon_T \left(\frac{L^2 f_0^2}{gH}\right) \frac{\partial \eta}{\partial t} + \epsilon \left(\frac{L^2 f_0^2}{gH}\right) (\mathbf{v}.\nabla \eta + \eta \nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

Burger number

↳ The non-dimensional constant $\frac{L^2 f_0^2}{gH}$ that appears in brackets is the inverse of the Burger number (Bu^{-1}, see **#GFD1.2a**).

- *Bu* of order 1 means that Coriolis term and gravity/buoyancy effects are comparable or that vorticity advection and vortex stretching are equally important (see **#GFD5.5a**).
- *Bu* of order 1 means that we are dealing with typical synoptic systems, which can be amenable to quasi-geostrophic analysis.
- It is also associated with the length scale (L), such that $Bu^{-1} = L^2/L_R^2$. (L_R is the Rossby Radius, see **#GFD1.2a**).

↳ For simplicity, we call it $\boldsymbol{F}$ in the following.

⇨ We use F in the continuity equation and it writes:

$$F = \frac{L^2 f_0^2}{gH}$$

$$\epsilon_T F \frac{\partial \eta}{\partial t} + \epsilon F (\mathbf{v}.\nabla \eta + \eta \nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

Again, we **have not made any further approximations than the scales of movement conform to geostrophic balance** (✋).

↳ But we can already see from these two equations that to <u>**zero-order**</u> in our Rossby number parameters, the flow is geostrophic and non-divergent and that <u>first-order terms</u> concern advection divergence and time development.

2.3.d) The assumptions of quasi-geostrophic theory

⇨ Before doing a formal expansion in the Rossby number (see **#GFD2.3f**), we will set out our assumptions one by one:

• ***Assumption 1***: the Rossby number is small, i.e. close to geostrophy: $\varepsilon \ll 1$ with $\epsilon = \dfrac{U}{f_0 L}$

• ***Assumption 2***: the temporal Rossby number is also small. We consider that $\epsilon_T = \epsilon$

↳ This means that scaling for velocity (U) is consistent with our scaling for length (L) and time (T), i.e. the velocity is just the flow velocity (as opposed to wave velocity which could go much faster). This results in ***filtering the very fast surface gravity waves*** (ex: tides).

• ***Assumption 3***: Buoyancy/gravity-stratification effects are as important as the Coriolis effect, i.e. *Bu* is of order 1, and length scale ($\boldsymbol{L}$) is close to the Rossby radius. This means that the coefficient $\boldsymbol{F}$ in the momentum equation (see **#GFD2.3c**) is of order 1.

• A consequence of assumption 3 and assumption 1 is that departures (δh) from standard layer thickness (*H*) are small.

↳ This is the linearization of the continuity equation and of the quasi-geostrophic potential vorticity. *NB: In a continuously stratified case, this is equivalent to saying that Brunt Vaisala frequency squared (N^2) varies in the vertical but not in the horizontal.*

• ***Assumption 4***: Scales of motion are small compared to the radius of the Earth: $\dfrac{L}{r_e} \ll 1$

↳ In fact, we assume that $\dfrac{L}{r_e} = \epsilon$ is the same ε as the Rossby number. We keep only one small parameter ε.

✋ NB: Assumptions 3 (about the stratification) and 4 (about the scale compared to the size of the planet) have nothing to do with geostrophy. They are not the result of our intent to derive a system almost but not quite geostrophic. But they are necessary for our expansion to be self-consistent.

2.3.e) The beta effect

⇨ <u>**Assumption 4**</u> (length scales of the flow are small compared to r_e) indicates that the variation in f is non-zero but small.

⇨ A Taylor expansion of $f = 2\Omega \sin\phi$ about a reference latitude (ϕ_0) gives:

$$f = f_0 + \left.\frac{df}{dy}\right|_0 y + \left.\frac{d^2 f}{dy^2}\right|_0 \frac{y^2}{2} + \ldots = 2\Omega \sin\phi_0 + \frac{y' L}{r_e} 2\Omega\cos\phi_0 + \ldots = f_0 + \beta_0 y + \ldots$$

↳ It can be written <u>**non-dimensionally**</u> ($y = y'L$, see **#GFD2.3a**): $\dfrac{f}{f_0} = 1 + \dfrac{2\Omega cos\phi_0}{2\Omega sin\phi_0}\dfrac{L}{r_e} y' + \cdots$
$= 1 + cot\phi_0 \dfrac{L}{r_e} y' + \cdots$

⇨ At first order, with $\beta' = cot\phi_0$ and $\dfrac{L}{r_e} = \dfrac{U}{f_0 L} = \varepsilon$, it follows that: $\dfrac{f}{f_0} = 1 + \varepsilon\beta' y'$

↳ It represents the variation with latitude of the Coriolis parameter f, with a small parameter ε, in front of the β term. We introduced the β-plane, i.e. the function for $\boldsymbol{f}$ in $x - y$ space is linear and describes a plane.

- We cannot get too close to the equator where $cot\phi_0 \longrightarrow \infty$.
  It is thus an **extra-tropical beta approximation**.
- If $f = f_0$, it is an ***f-plane*** (as in **#GFD2.2**).
- β-plane is only in functional space, not in physical space.

β-plane

⇨ In the following, we are going to eliminate the prime in the notation (as in **#GFD2.3c**).

2.3.f) The expansion

⇨ Let's do the expansion. Using assumption in **#GFD2.3d**, we rewrite the momentum (**#GFD2.3b**) and continuity (**#GFD2.3c**) equations as **β-plane** (**#GFD2.3e**) non-dimensional equations, in which some terms are multiplied by ε and some terms are not:

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + (1+\epsilon\beta y)\hat{\mathbf{k}}_\wedge\mathbf{v} + \nabla\eta = 0$$

$$\epsilon F\frac{\partial \eta}{\partial t} + \epsilon F(\mathbf{v}.\nabla\eta + \eta\nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

↳ We expand the 3 variables (u, v and η, the departure of the layer thickness from the standard value) in increasing powers of ε (a zero-order part (ε^0)+ ε^1 × a first-order part + ε^2 × a second-order part, etc…):

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + ...$$
$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + ...$$

↳ We then substitute them into the equations. We will ***sort the terms in increasing order of epsilon***. We will focus here on zero-order terms and then on first-order terms (see **#GFD2.3g**).

For example, v_0 is a zero-order term in the last term of the continuity equation, while it is of first-order in the second term of the momentum equation. Likewise, εv_1 is a first-order term in the last term of the continuity equation, while it is of second-order in the second term of the momentum equation.

Zero-order

Zero-order: All the terms without any ε in front. At zero-order, the momentum equation yields geostrophic balance, while the continuity equation informs us that the flow is non-divergent:

$$\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_0 = 0 \quad (1) \qquad \nabla.\mathbf{v}_0 = 0 \quad (2)$$

↳ At zero-order, there is no development. The geostrophic non-divergent flow can only change with time if we include some first order (divergent) terms. The continuity equation is the equivalent of the momentum equations as $curl(1)$ gives (2).

η_0 acts as a stream function for the zero order (non-divergent) flow (u_0, v_0):

$$v_0 = \frac{\partial\eta_0}{\partial x}, \quad u_0 = -\frac{\partial\eta_0}{\partial y}$$

✋ It does not represent the geostrophic flow, it represents the **part of the geostrophic flow** you would have if *f* were constant.

2.3.g) First order in ε

⇨ First-order is what is left over when you select terms that have just ε in front of them (no second-order or higher-order terms). It follows that:

$$\frac{\partial\mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla\mathbf{v}_0 + \hat{\mathbf{k}}_\wedge\mathbf{v}_1 + \beta y\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_1 = 0 \quad (1)$$

$$F\frac{\partial\eta_0}{\partial t} + F(\mathbf{v}_0.\nabla\eta_0 + \eta_0\nabla.\mathbf{v}_0) + \nabla.\mathbf{v}_1 = 0 \quad (2)$$

Continuity

⇨ In the continuity equation (2), the ***second term is zero*** because the zero-order flow ($\mathbf{v_0}$) is perpendicular to the gradient of the stream function ($\nabla\eta_0$). And $\mathbf{v_0}$ is non-divergent, so:

$$F\frac{\partial\eta_0}{\partial t} = -\nabla.\mathbf{v}_1$$

↳ The rate of change of the zero-order layer thickness comes from the divergence of the first order flow.

Vorticity

⇨ We form the first-order vorticity equation by taking the curl of the momentum equations (1) $\frac{\partial}{\partial x}(y - equation) - \frac{\partial}{\partial y}(x - equation)$, as in **#GFD1.3b** and **#GFD2.2b**. It provides an equation for the **vorticity** of the flow (ξ, curl of the velocity):

$$\frac{\partial \xi_0}{\partial t} + \xi_0 \nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta y \nabla.\mathbf{v}_0 + \beta v_0 = 0$$

↳ The ***second and fifth terms are zero*** because the zero-order flow ($\mathbf{v_0}$) is non-divergent.

β-plane QG PV conservation

⇨ We then combine the vorticity equation with the continuity equation to get rid of the first-order divergence ($\nabla.\mathbf{v}_1$). This provides a conservation principle:

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \beta v_0 = -\nabla.\mathbf{v}_1 = F\frac{\partial \eta_0}{\partial t}$$

↳ Then taking into consideration that $\frac{\partial}{\partial t}(\beta y) = 0, \quad \mathbf{v}_0.\nabla(\beta y) = \beta v_0, \quad \mathbf{v}_0.\nabla\eta_0 = 0$ and using $\frac{\partial}{\partial t} + \mathbf{v}_0\nabla$ as the substantial derivative, we can factorize the equation, so that:

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = F\left[\frac{\partial \eta_0}{\partial t} + \mathbf{v}_0.\nabla\eta_0\right]$$

In the absence of forcing or dissipation

↳ It yields the conservation (following the flow) of the (non-dimentionalized) **quasi-geostrophic potential vorticity**:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_0.\nabla\right][\beta y + \xi_0 - F\eta_0] = 0$$

$F = \frac{L^2 f_0^2}{gH}$

2.3.h) Quasi-geostrophic potential vorticity on a β-plane

$$\frac{D}{Dt}[\beta y + \xi_0 - F\eta_0] = 0$$

⇨ If we now express the **zero-order (non-divergent) flow** ($\mathbf{v}_0$) in the advection terms **in terms of the stream function** η_0, so that $v_0 = \frac{\partial \eta_0}{\partial x}$ and $u_0 = -\frac{\partial \eta_0}{\partial y}$ (see **#GFD2.3f**), the substantial derivative of the potential quasi-geostrophic vorticity (q) is written:

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \boldsymbol{v}_0.\nabla q = \frac{\partial q}{\partial t} + \frac{\partial \eta_0}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial q}{\partial x}$$

⇨ Recalling that the vorticity is the *Laplacian* of the stream function (see **#GFD1.3a**), we can write the prognostic equation in terms of *one variable only*, the stream function (η_0):

$$\left[\frac{\partial}{\partial t} + \frac{\partial \eta_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial}{\partial x}\right][\beta y + \nabla^2\eta_0 - F\eta_0] = 0$$

Jacobian notation

⇨ This leads to an expression of conservation of (non-dimentionalized) **quasi-geostrophic potential vorticity q**, which can be written like this:

$$\frac{\partial q}{\partial t} + J(\eta_0, q) = 0 \quad \text{with} \quad q = \beta y + \nabla^2\eta_0 - F\eta_0$$

$F = \frac{L^2 f_0^2}{gH}$

↳ J is the Jacobian, i.e. a compact way of expressing advection, when you have a non-divergent flow, in terms of the stream function and the quantity being advected.

⇨ What we learned from this is that we have just *one variable in this system*. For the complete shallow water equations, we had three variables (u, v, and h, see **#GFD1.3d**). For the quasi-geostrophic theory, we can express everything in terms of η_0: **one equation - one variable**. This is rather useful to perform weather prediction.

⇨ This is all non-dimensional, so we now put the physical values back in (i.e. the opposite of non-dimensionalizing the equations, see **#GFD2.3b**) and dimensionalize the equation. This leads to the dimensional quasi-geostrophic potential vorticity:

$$q = \beta y + \nabla^2\psi - \frac{f_0}{H}\delta h$$

↳ i.e. β-term + relative vorticity + vortex stretching term.

⇨ If we define the quasi-geostrophic stream function as: $\psi = \frac{g}{f_0}\delta h$, it follows that:

The conservation of **quasi-geostrophic potential vorticity q on a β-plane**

$$q = \beta y + \nabla^2\psi - \left(\frac{f_0^2}{gH}\right)\psi \quad \text{or} \quad q = \beta y + \nabla^2\psi - \frac{1}{L_R^2}\psi$$

with the **conservation with the flow** as before: $$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

In the absence of forcing or dissipation

☝ As f_0 remains constant, it ***can be included in the definition of q***. It will not change the conservation principle.

☝ The **Rossby radius** (see **#GFD1.2a**) is the length scale on which ***relative vorticity*** and ***vortex stretching*** make equal contributions to ***potential vorticity*** (see **#GFD3.4c** and **#GFD5.5b**)

2.3.i) Continuously stratified fluid

⇨ Here, we provide the **quasi-geostrophic potential vorticity conservation principle** for more **realistic fluids**, with continuous (horizontal and vertical) variations of density.

↳ Up until now (**#GFD2.3a-h**), we have worked with discrete shallow water layers, each of which being homogeneous (constant density). The extension to **continuous stratification** requires that we abandon this formulation and reintroduce a vertical coordinate (see **details** on the next pages).

⇨ The **equation of conservation of quasi-geostrophic potential** vorticity remains the same:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

↳ It is the **definition of *q* that changes**:

⇨ In a flow where the stratification varies with the vertical and in which also the Coriolis parameter varies with latitude (β-plane), the stream function is defined in terms of pressure and f_0, such that:

$$\psi = \frac{p_0}{\rho_s f_0}$$

☝ Similarly, $\boldsymbol{\psi}$ is the stream function for the ***non-divergent part of the geostrophic flow***.

↳ The density varies in the vertical and horizontal, such that there is a reference value of density and a perturbation, function of (x, y, z, t). The expansion around a small Rossby number and the derivation of the **full quasi-geostrophic equation set** are very similar (detailed in the **following pages**).

• For the quite realistic **anelastic case** (see **#GFD1.1c**) which allows large variations of density with height, accounting for the static compressibility of the atmosphere, the quasi-geostrophic potential vorticity is:

$$q = \beta y + \nabla^2\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial\psi}{\partial z}\right)$$

↳ It is the same as before: the βy term, the relative vorticity, and the vortex stretching term. The latter is more complicated and depends on vertical gradients of the stream function.

• This definition can be simplified in the case of the **Boussinesq approximation** in which the reference density (ρ_0) is constant (independent of z, see **#GFD1.1c**). In this context, the density variable between the vertical derivatives cancels. It follows that:

$$q = \beta y + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)$$

⇨ The result is once again a **conservation law** for **quasi-geostrophic potential vorticity**, which is defined entirely in terms of a stream function, **so one equation**, **one variable**.

Derivation of the PV equation in a continuously stratified fluid

IV) EXTENSION TO A CONTINUOUSLY STRATIFIED FLUID
(with non-Boussinesq, static compressibility effects)

Three dimensional scalings for a compressible, baroclinic stratified fluid:

$$x, y \to L, \quad u, v \to U, \quad z \to H, \quad w \to \frac{UH}{L}, \quad t \to \frac{L}{U}$$

$$p = p_s(z) + \tilde{p}(x, y, z, t)$$
$$\rho = \rho_s(z) + \tilde{\rho}(x, y, z, t)$$

Geostrophic scaling for pressure

$$fv \sim \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x}$$

so

$$\tilde{p} \to f_0 U L \rho_s$$

Hydrostatic scaling for density

$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g$$

so

$$\tilde{\rho} \to \frac{f_0 U \rho_s L}{Hg} = \rho_s \epsilon F$$

so

$$\boxed{\rho = \rho_s(1 + \epsilon F \rho')}$$

recall

$$F = \frac{f_0^2 L^2}{gH}$$
$$\epsilon = \frac{U}{f_0 L}$$

also

$$\frac{f}{f_0} = 1 + \epsilon \beta' y'$$

where

$$\beta' = \frac{\beta_0 L^2}{U} = \cot \phi_0$$

as before.

Non-dimensional momentum equation:

$$\frac{\partial \mathbf{v}'}{\partial t}\frac{U^2}{L} + \mathbf{v}'.\nabla \mathbf{v}'\frac{U^2}{L} + w'\frac{\partial \mathbf{v}'}{\partial z}\frac{HU^2}{LH} + fU\hat{\mathbf{k}}_\wedge \mathbf{v}' = -\frac{1}{\rho_s(1+\epsilon F\rho')}\frac{\nabla p'}{L}ULf_0\rho_s$$

$$= -Uf_0\nabla p'(1 - \epsilon F\rho')$$

(to first order)

Divide by Uf_0, drop primes

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + \epsilon w\frac{\partial \mathbf{v}}{\partial z} + (1 + \epsilon\beta y)\hat{\mathbf{k}}_\wedge\mathbf{v} = -(1 - \epsilon F\rho)\nabla p$$

Non-dimensional continuity equation (non-Boussinesq)

$$\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla\rho + w\frac{\partial \rho}{\partial z} + \rho\nabla.\mathbf{v} + \rho\frac{\partial w}{\partial z} = 0$$

$$\rho_s\epsilon F\frac{U}{L}\frac{\partial \rho'}{\partial t'} + \rho_s\epsilon F\frac{U}{L}\mathbf{v}'.\nabla\rho' + \rho_s\epsilon F\frac{UH}{LH}w'\frac{\partial \rho'}{\partial z'}$$

$$+\frac{UH}{L}w'\left[\frac{\partial \rho_s}{\partial z}\right] + \rho_s(1+\epsilon F\rho')\left[\frac{U}{L}\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right)\right] = 0$$

$$\times\frac{L}{\rho_s U} \to$$

$$\epsilon F\frac{\partial \rho'}{\partial t'} + \epsilon F\mathbf{v}'.\nabla\rho' + \epsilon F w'\frac{\partial \rho'}{\partial z} + Hw'\left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z}\right] + (1+\epsilon F\rho')\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right) = 0$$

Note that the expression in square brackets resembles N^2, and note that z is dimensionless.

$$N^2 = \frac{g}{\theta_s}\frac{\partial \theta_s}{\partial z}$$

Define

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

then the fourth term above becomes

$$\left(\frac{HS^2}{g}\right)w'$$

This is the non-Boussinesq term.

So dropping primes

$$\epsilon F\left(\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla\rho + w\frac{\partial \rho}{\partial z}\right) + \frac{HS^2}{g}w + (1+\epsilon F\rho)\left(\nabla.\mathbf{v} + \frac{\partial w}{\partial z}\right) = 0$$

Expansion of non-dimensional variables

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + ...$$
$$w = w_0 + \epsilon w_1 + ...$$
$$\tilde{p} = p_0 + \epsilon p_1 + ...$$
$$\tilde{\rho} = \rho_0 + \epsilon\rho_1 + ...$$

Momentum equation to zero order

Geostrophic balance

$$\hat{\mathbf{k}}_\wedge\mathbf{v}_0 = -\nabla p_0$$

and

$$\nabla.\mathbf{v}_0 = 0$$

Continuity equation to zero order

$$\frac{HS^2}{g}w_0 + \nabla.\mathbf{v}_0 + \frac{\partial w_0}{\partial z} = 0$$

Therefore we can't generate w_0 in the body of the fluid by horizontal motion. At zero order, vertical motion can only be generated at the boundary.

Assume that the bottom vertical velocity

$$w_b = 0 + \epsilon w_{1b} + ...$$

(this is assumption (3): weak orography)
Integrate upwards, this implies

$$w_0 = 0$$

everywhere, so

$$w = \epsilon w_1 + ...$$

Momentum equation to first order

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla\mathbf{v}_0 + \hat{\mathbf{k}}_\wedge\mathbf{v}_1 + \beta y\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla p_1 - F\rho_0\nabla p_0 = 0$$

$$\hat{\mathbf{k}}.\nabla_\wedge(\text{this}) \rightarrow$$

vorticity equation:

$$\frac{\partial \xi_0}{\partial t} + \xi_0\nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta y\nabla.\mathbf{v}_0 + \beta\mathbf{v}_0 - F\left[\frac{\partial \rho_0}{\partial x}\frac{\partial p_0}{\partial y} - \frac{\partial \rho_0}{\partial y}\frac{\partial p_0}{\partial x}\right] = 0$$

Second and fifth terms disappear by nondivergence of the zero order flow, and the last term can be rewritten using geostrophy of the zero order flow to give

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta\mathbf{v}_0 + F\mathbf{v}_0.\nabla\rho_0 = 0$$

Continuity equation to first order

$$F\frac{\partial \rho_0}{\partial t} + F\mathbf{v}_0.\nabla\rho_0 + \left(\frac{HS^2}{g}\right)w_1 + \nabla.\mathbf{v}_1 + \frac{\partial w_1}{\partial z} = 0$$

Note that the second and fourth terms have just appeard in the vorticity equation. So we can eliminate them by combining the continuity and vorticity equations:

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \beta\mathbf{v}_0 = -F\mathbf{v}_0.\nabla\rho_0 - \nabla.\mathbf{v}_1$$

$$= F\frac{\partial \rho_0}{\partial t} + \left(\frac{HS^2}{g}\right)w_1 + \frac{\partial w_1}{\partial z}$$

At this stage we note that for synoptic scales $F \sim 0.1$ so we neglect the first term on the right hand side. This is because we have set

$$F = \frac{f_0^2L^2}{gH} = \frac{L^2}{R^2}$$

(remember, for the atmosphere:

$$R_{ext} \sim \frac{\sqrt{gH}}{f_0} = \frac{\sqrt{10 \times 10^4}}{10^{-4}} \sim 3 \times 10^6\,\text{m} = 3000\,\text{km}$$

$$F = \frac{L^2}{R^2} \sim \frac{1000^2}{3000^2} \sim 10^{-1}$$

)

So the vorticity equation is now

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = \frac{HS^2}{g}w_1 + \frac{\partial w_1}{\partial z}$$

Using

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z_*}$$

the right hand side can be written

$$= \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1)$$

We can evaluate the right hand side using the ...

Thermodynamic equation

$$\frac{D\theta}{Dt} = 0$$

scale

$$\theta = \theta_s(1 + \epsilon F(\theta_0 + ...))$$

as we did for density, so

$$\frac{U}{L}\frac{\partial \theta'}{\partial t'}\epsilon F\theta_s + \frac{U}{L}\mathbf{v}'.\nabla\theta'\epsilon F\theta_s + w'\frac{\partial \theta'}{\partial z'}\epsilon F\theta_s\frac{HU}{LH} + w\frac{\partial \theta_s}{\partial z_*}\frac{UH}{L} = 0$$

drop primes, get

$$\epsilon F\left(\frac{\partial \theta}{\partial t} + \mathbf{v}.\nabla\theta + w\frac{\partial \theta}{\partial z}\right) + w\frac{N^2H}{g} = 0$$

At zero order we recover

$$w_0 = 0$$

At first order:

$$F\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0.\nabla\theta_0\right) + w_1\frac{N^2H}{g} = 0$$

$$\rightarrow w_1 = -\frac{f_0^2L^2}{N^2H^2}\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0.\nabla\theta_0\right)$$

introduce

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_0.\nabla$$

so

$$w_1 = -\frac{f_0^2L^2}{N^2H^2}\left[\frac{D_0\theta_0}{Dt}\right]$$

Multiply by ρ_s, take vertical derivative and then divide by ρ_s, and exchange derivatives when possible. This gives

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1) = -\frac{D_0}{Dt}\left[\frac{f_0^2L^2}{H^2\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\theta_0}{N^2}\right)\right]$$

and we can use this to rewrite the vorticity equation as

$$\frac{D_0}{Dt}\left[\beta y + \xi_0 + \frac{f_0^2L^2}{H^2\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\theta_0}{N^2}\right)\right] = 0$$

Now we have one last thing to do...

Hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g$$

$$p = p_s + p_0\rho_s f_0UL$$

$$\rho = \rho_s + \rho_0\rho_s\epsilon F$$

$$\rightarrow \frac{\partial}{\partial z}(p_o\rho_s) = -\rho_0\rho_s$$

or

$$\rho_0 = -\frac{1}{\rho_s}\frac{\partial}{\partial z}(p_0\rho_s)$$

Now, define

$$\theta_* = \theta_s(z)(1 + \epsilon F\theta)$$

and define

$$\theta_0 = -\rho_0 + \frac{1}{\gamma}\left(\frac{\rho_s gH}{p_s}\right)p_0$$

ASIDE: So where does this come from ?
Its needed to ensure

$$\frac{d\theta}{\theta} = \frac{1}{\gamma}\frac{dp}{p} - \frac{d\rho}{\rho}$$

(where

$$\gamma = \frac{c_p}{c_v}$$

)

PROOF:
integrate this, gives

$$\log\theta_* = \frac{1}{\gamma}\log p_* - \log\rho_* + \text{const}$$

but

$$\theta_* = \theta_s(1 + \epsilon F\theta_0)$$

$$\rho_* = \rho_s(1 + \epsilon F\rho_0)$$

$$p_* = p_s + \rho_s f_0ULp_0 = p_s\left(1 + f_0UL\frac{\rho_s}{p_s}p_0\right)$$

$$= p_s\left(1 + \epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0\right)$$

The inner term in brackets is the reference hydrostatic scaling, ~ 1.
Substitute these expressions for ϑ, ρ* and p* into the log expression using the fact that to first order*

$$\log(1 + \epsilon x) = \epsilon x$$

$$\rightarrow \epsilon F\theta_0 = \frac{1}{\gamma}\epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0 - \epsilon F\rho_0$$

$$\rightarrow \theta_0 = \frac{1}{\gamma}\left(\frac{gH\rho_s}{p_s}\right)p_0 - \rho_0$$

END OF ASIDE

Substitute this into the hydrostatic relation to elimintate density

$$\rho_0 = -\theta_0 + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right) p_0 = -\frac{\partial p_0}{\partial z} - \frac{p_0}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

$$\theta_0 = \frac{\partial p_0}{\partial z} + p_0\left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right)\right]$$

From the reference hydrostatic relation, the second term in square brackets can be written

$$= -\frac{1}{\gamma}\frac{\partial p_s}{\partial z}$$

but this is just

$$\theta_0 = \frac{\partial p_0}{\partial z} - p_0\left(\frac{1}{\theta_s}\frac{\partial \theta_s}{\partial z}\right)$$

and the term in brackets

$$= \frac{N^2 H}{g} \sim \frac{g'}{g} \sim \epsilon$$

so we can write the perturbation hydrostatic relation in terms of perturbation potential temperature:

$$\boxed{\theta_0 = \frac{\partial p_0}{\partial z}}$$

... put this back into the vorticity equation:

$$\frac{D_0}{Dt}\left[\beta y + \xi_o + \frac{f_0^2 L^2}{H^2 \rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial p_0}{\partial z}\right)\right] = 0$$

This is the non-d quasi-geostrophic potential vorticity.

Redimensionalise:

$$q = \beta y + \xi_0 + \frac{f_0}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial}{\partial z}\left(\frac{p_0}{\rho_s}\right)\right)$$

introduce a dimensional geostrophic streamfunction

$$\psi = \frac{p_0}{\rho_s f_0}$$

get

$$\boxed{q = \beta y + \nabla^2\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial \psi}{\partial z}\right)}$$

This is the full quasi-geostrophic potential vorticity for a compressible stratified fluid.

Note: for stratified Boussinesq fluids this form reduces to

$$\boxed{q = \beta y + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)}$$

(this is OK for the ocean).

q is conserved following the flow:

$$\boxed{\frac{\partial q}{\partial t} + J(\psi, q) = 0}$$

Everything is represented in terms of one prognostic equation in one variable (the streamfunction).

2.3.j) One variable to rule them all

⇨ In **quasi-geostrophic theory**, we obtain only ***one variable*** in the system, the quasi-geostrophic stream function $\boldsymbol{\psi}$, that rules them all.

↳ Everything can be expressed in terms of ψ in the quasi-geostrophic set.

• The **horizontal velocity** can be expressed in terms of the stream function (it is the definition of the stream function), so:

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

• The **pressure** is a stream function for the non-divergent geostrophic flow (see **#GFD2.3i**):

$$p' = \rho_0 f_0 \psi$$

• The **density** is the vertical gradient of the stream function (hydrostatic balance):

$$\rho' = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}$$

• The **vertical velocity** is material tendency operator applied to the density, so it can also be expressed in terms of ψ:

$$w = \frac{f}{N^2}\left[\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right]\frac{\partial \psi}{\partial z}$$

⇨ With the quasi-geostrophic set of equations, it is easier to make **predictions** following this procedure:

The flow field ψ^{t_0} (ex: weather today) constitutes the **initial conditions**.

1) Compute the three-dimensional field of **potential vorticity**

$$q = \beta y + \nabla^2 \psi + \text{fn}(\psi_z)$$

2) q is conserved with the flow. But at one location, q changes as it is blown around by the wind. Thus, the next step consists of computing the advection terms and integrating the prognostic equation forward in time to **find the next state for $\boldsymbol{q}$** ($q^{t_{0+1}}$):

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = 0$$

In the quasi-geostrophic set, time steps can be quite long (half an hour or so) because gravity waves are filtered and nothing really fast is going on.

3) Invert the elliptic operator $q = \beta y + \nabla^2 \psi + \text{fn}(\psi_z)$ to estimate the stream function. This provides a new flow field $\psi^{t_{0+1}}$ that will constitute the initial conditions for the next time step.

↳ Using this **prediction system**, you can do it 48 times in a row. This will provide weather forecasts for tomorrow. The first weather predictions were done with the quasi-geostrophic set.

GFD2.4: Quasi-Geostrophic Theory III – Applications and Diagnostics

2.4.a) Development

⇨ In order to predict the weather without taking into account the potential vorticity, one can still consider directly the **time development** of ψ, i.e. **pressure** (focusing on pressure centers for instance). This means that we can remain in the quasi-geostrophic framework without going through this inversion process for the potential vorticity.

Consider the development equation for the potential vorticity, in which the formulation of the potential vorticity is developed in term of ψ (slightly simplified below):

$$\frac{\partial}{\partial t}\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) + \mathbf{v}.\nabla\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) = 0$$

We can **rearrange** it if we assume all the functions are well behaved (differentiable, etc) and that we can swap over the order of the derivatives. Instead of having $\frac{\partial}{\partial t}$ of a big elliptic function of ψ, we have a function of $\frac{\partial\psi}{\partial t}$, and we put the second term on the RHS and develop it, so that:

$$\left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial\psi}{\partial t} = -\mathbf{v}.\nabla\left(\nabla^2\psi + f\right) - \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

The two RHS terms control the tendency of the stream function, and by extension, they control the pressure development. **Pressure development** will thus be **determined** by 1) the absolute **vorticity advection** and 2) the **vertical gradient of the horizontal density advection**.

So, if the sum of their contributions is negative there will be low-pressure development. Except that there is an elliptic operator in front of the tendency term.

Let's assume that the functional form of ψ is wave-like in (x, y) and it changed sign once in the vertical (first-baroclinic mode). The effect of this operator (on this simple wave-like structure) is a multiplication by a constant (> 0, involving the wavenumbers) and more importantly a change of sign.

$$\psi \propto \sin lx \, \sin my \, \cos \pi z/H$$

In this context, the local rate of change of ψ, or change of pressure is proportional to the **absolute vorticity advection** and the **vertical gradient of the density/temperature advection**.

$$\frac{\partial\psi}{\partial t} \propto +\mathbf{v}.\nabla(\nabla^2\psi + f) + \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

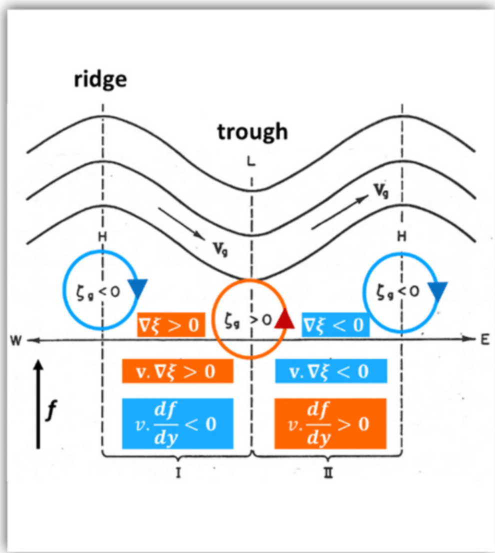
2.4.b) Advection of absolute vorticity

Advection of absolute vorticity is proportional to: $\frac{\partial\psi}{\partial t} \propto \mathbf{v}.\nabla(\nabla^2\psi + f)$

We study here an eastward flow with wave-type structure (see below), such that:

$$\nabla^2\psi = -(l^2 + m^2)\psi$$

In the advection of absolute vorticity, there are two terms: one associated with the relative vorticity and the other associated with the planetary vorticity.



• ***Advection of relative vorticity – short waves:***

In the ridge, the flow is clockwise and the relative vorticity is negative, while in the trough, the relative vorticity is positive. In region I: $v.\nabla\xi > 0 \Rightarrow \frac{\partial\psi}{\partial t} > 0$.

With the flow going eastwards, the zonal advection of relative vorticity will **send troughs and ridges eastwards**. This is the case for **short waves** for which ξ dominates.

• ***Advection of planetary vorticity*–long waves:**

As $\frac{df}{dy} > 0$, the meridional advection of planetary vorticity is controlled by the northward southward oscillation of the flow. It results in the opposite effect to the relative vorticity advection and will **send troughs and ridges west**. These are long Rossby waves (see **#GFD3**).

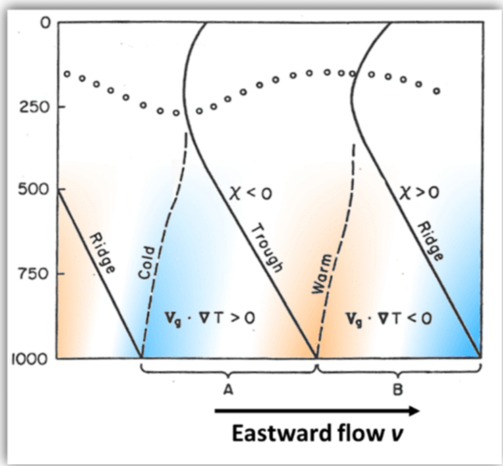
⇨ In conclusion, **there is a competition between the advection of planetary vorticity** (the Rossby wave term) **and the advection of relative vorticity** (the synoptic-scale term). Both will influence the way in which the pressure (weather) will develop.

2.4.c) Vertical gradient of temperature advection

⇨ The rate of change of the stream function, and by implication the pressure development, is proportional to the vertical gradient of the temperature advection:

$$\frac{\partial \psi}{\partial t} \propto \frac{\partial}{\partial z}\left(\mathbf{v}.\nabla \frac{\partial \psi}{\partial z}\right) \propto \frac{\partial}{\partial z}(\mathbf{v}.\nabla \theta) \qquad \left[\theta_0 = \frac{\partial p_0}{\partial z}\right]$$

The question is: "What is the vertical variation of the temperature advection?"



In the example on the side, we study an eastward flow going through the juxtaposition of cold and warm air masses in the x-direction. These air masses are associated with positive and negative zonal gradients of temperature which decrease with height. For instance:

In region A, there is cold advection at low-level ($v.\nabla\theta > 0$), then $\frac{\partial(v.\nabla\theta)}{\partial z} < 0$ and a trough develops.

In region B, there is warm advection at low-level ($v.\nabla\theta < 0$), then $\frac{\partial(v.\nabla\theta)}{\partial z} > 0$ and a ridge develops.

2.4.d) Vertical velocity: quasi-geostrophic omega equation

⇨ With the quasi-geostrophic set, it is also possible to make diagnostics for weather analysis, in particular, to diagnose the vertical velocity.

• We could deduce the vertical velocity by **integrating the continuity equation**:

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

✋ This is mathematically sound but it is ill-conditioned. $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are large terms which have cancellation between them (small differences between large terms). Such calculation for vertical velocity is not numerically accurate for real data sets.

• The quasi-geostrophic system to the rescue 😉. From the continually stratified version of the quasi-geostrophic theory (detailed in **#GFD2.3i**), the vertical velocity can be written:

$$w = -B_u\left(\frac{\partial}{\partial t}\frac{\partial p_o}{\partial z} + \mathbf{v}_0.\nabla\frac{\partial p_0}{\partial z}\right)$$

↳ We eliminate the tendency term by estimating the Laplacian of this formula and using the vertical gradient of the vorticity equation:

$$\nabla^2\left(\frac{\partial}{\partial z}\frac{\partial p_o}{\partial t}\right) = -\nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_0}{\partial z}\right) - B_u^{-1}\nabla^2 w_1$$

$$\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial t}(f+\xi_0) + \mathbf{v}_0.\nabla(f+\xi_0) = \frac{\partial w_1}{\partial z}\right\} \to \nabla^2\left(\frac{\partial}{\partial z}\frac{\partial p_0}{\partial t}\right) = \frac{\partial^2 w_1}{\partial z^2} - \frac{\partial}{\partial z}\left(\mathbf{v}_0.\nabla(f+\xi_0)\right)$$

⇨ Equating the two RHS yields a diagnostic equation for the vertical velocity in terms of the geostrophic stream function ($\xi_0 = \nabla^2 p_0$). It is called the **quasi-geostrophic omega equation**:

$$\left(B_u^{-1}\nabla^2 + \frac{\partial^2}{\partial z^2}\right) w_1 = \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f+\xi_0)) - \nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_o}{\partial z}\right)$$

↳ It is written as an elliptic operator on the vertical velocity, equal to the summed contribution of two terms.

- The first RHS term is related to the **vertical gradient of the absolute vorticity advection**.
- The second RHS term is the **Laplacian of the temperature advection**.

↳ This is the other way around to **#GFD2.4a**, in which we had vorticity advection and the vertical gradient of the temperature advection.

⇨ If we study a wave-type pattern, both elliptic operators can be represented by a simple change of sign. It follows that:

$$w \propto -\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla(f+\xi)\right) - \mathbf{v}.\nabla\theta$$

⇨ Note that this time we have eliminated the tendency term (rather than the vertical velocity term) between the vorticity and thermodynamic equations and obtained a diagnostic equation for w (rather than a prognostic equation for ψ). This equation is **usually derived in pressure coordinates**.

2.4.e) Application of the omega equation

⇨ Go to: https://www.meted.ucar.edu/labs/synoptic/qgoe_sample/qgoe_widget.htm

↳ It all gets very complicated and you have to sit and scratch your head a long time looking at these equations, making sure you have got the sign right… because if you get the sign wrong you get it all completely wrong.

2.4.f) Recap

⇨ Here is a summary for all these simplified quasi-geostrophic illustrations:

- The fall or rise of geopotential is proportional to:
  - positive or negative vorticity advection
  - the rate of decrease with height of the cold or warm advection
- For diagnosing the vertical velocity, rising or sinking motion is proportional to:
  - the rate of increase with height of the positive or negative vorticity advection
  - warm or cold advection