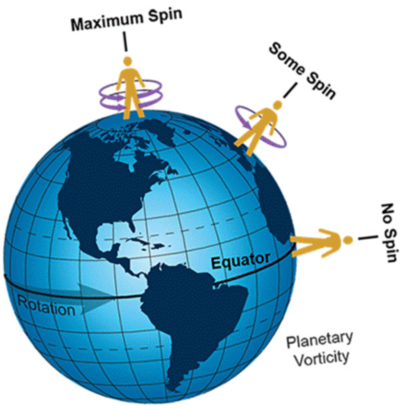
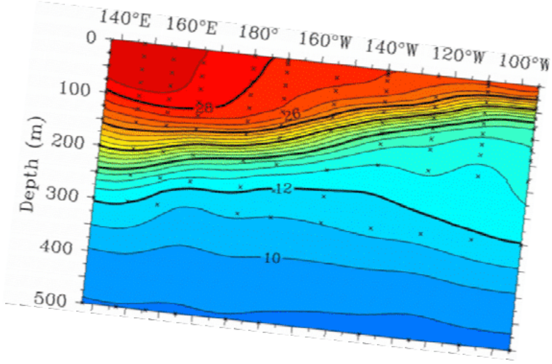
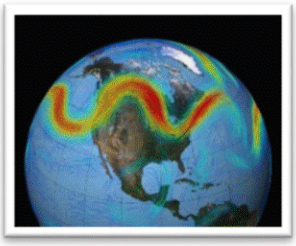
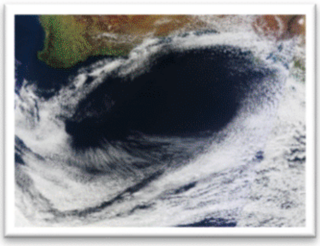
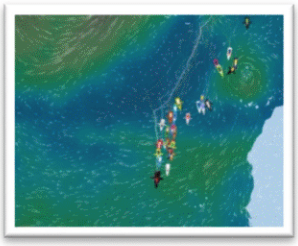
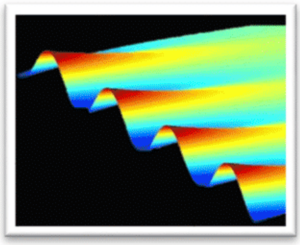
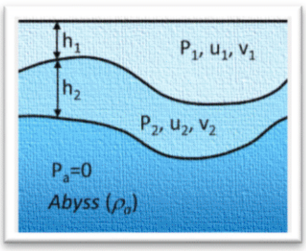



# Geophysical Fluid Dynamics



**Nick Hall – Serena Illig (September 2023)**

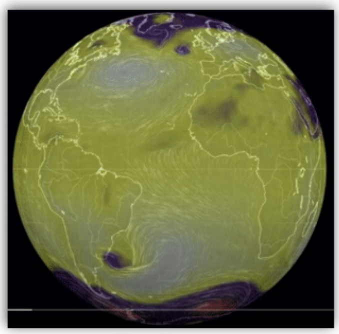
**M2 SOAC**

# Table of Contents

# INTRODUCTION
# Overview of Atmosphere and Ocean Dynamics

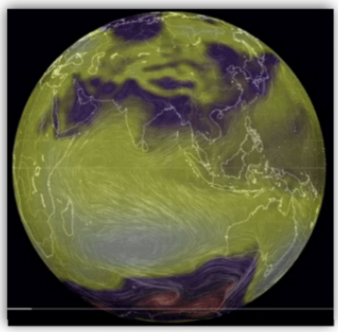
This is a course on **Geophysical Fluid Dynamics**. Before we start the course, let's go on a little tour in the Atmosphere and then the Ocean. This will give us a quick overview of the general circulation patterns we can observe in each system. We will highlight some similarities and differences between the **Atmosphere** and **Ocean circulation** that will provide insight on the associated dynamics. Below, snapshots were extracted from the *https://earth.nullschool.net/fr* web interface in Sept. 2016.

***Atmosphere dynamics***

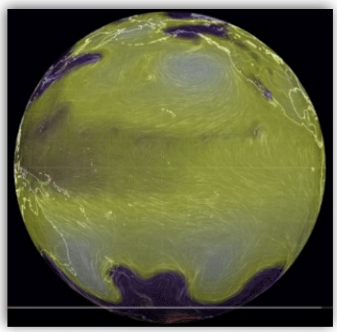


First, on the right are the low-level winds and surface pressure over the **Atlantic Ocean**. The arrows are the winds and the colors are the surface pressure. The first thing we notice is that there are winds swirling around maxima and minima of pressure. In the northern hemisphere, there is a high-pressure system – an anticyclone. The wind is going around it clockwise. In the southern hemisphere, there is another big anticyclone, around which the wind flows anti-clockwise. Around these high-pressure areas, there are a few little low-pressure areas. The wind is going around them in the other direction.

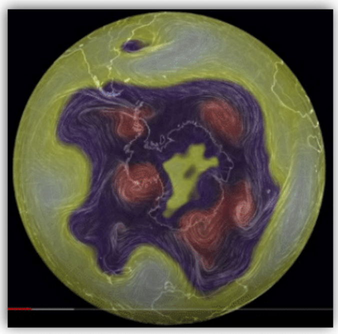
↳ We easily observe that **high-pressure** areas are much **bigger** than **low-pressure** areas, which is fairly normal.



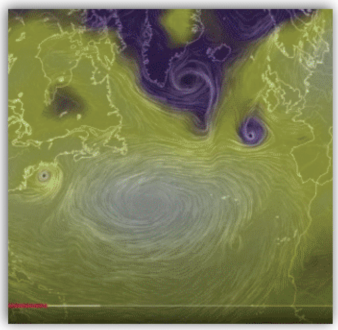
In the **Indian Ocean** (see figure on the left), there is another high-pressure area. The winds are flowing northwards along the coast of Africa. This is the northern Hemisphere summer.



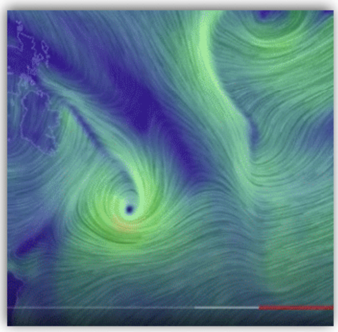
The snapshot on the right shows the **Pacific region**. There is another couple of anticyclones, along with some cyclonic features. We observe a convergence of the winds in the equatorial region - the Easterlies - a little bit north of the equator. This is the Inter-Tropical Convergence Zone (ITCZ).



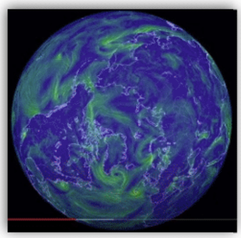
In the **South Pole region** (figure on the left), it is southern hemisphere winter. There is a large area of low-pressure, around which a jet is circling clockwise. The low-pressure system is surrounded by 5 intense cyclones, swirling around Antarctica. This is a fairly common configuration for winter: an intense winter jet, and 4-5 cyclones.



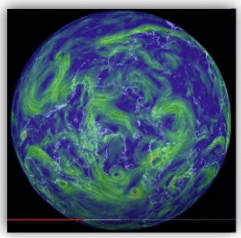
On the right is a zoom on the **North Atlantic**. We have a closer look at the big Azores High – the north Atlantic Subtropical Anticyclone – and a smaller more-intense cyclonic feature. The anticyclones are bigger than the cyclones and this asymmetry is fairly common. We will see that this is due to the nonlinear nature of atmospheric dynamics (see **#GFD2.1a**).



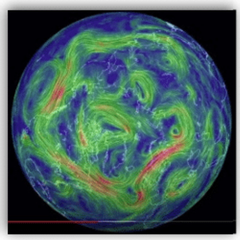
The Figure on the left shows the actual **wind strength** in color. It is easy to see that the strongest winds are around the cyclones. If we zoom a little bit more in the **North-East Atlantic sector**, we observe a very strong cyclone, strong enough to have a name – Petra. The winds are not only very strong, circling around the low-pressure system, but they are also changing direction very abruptly across a line. This line is a front, a temperature front. Surface winds change direction very suddenly where the temperature is changing very suddenly as well.



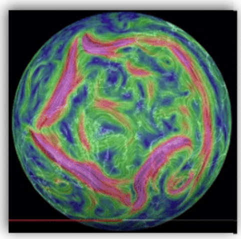
***Surface***



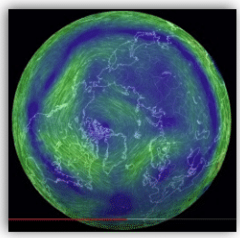
***850 hPa***



***500 hPa***

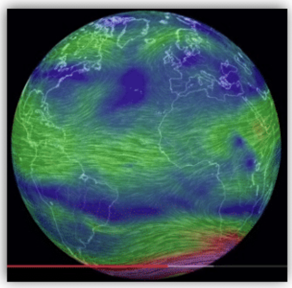


***250 hPa***

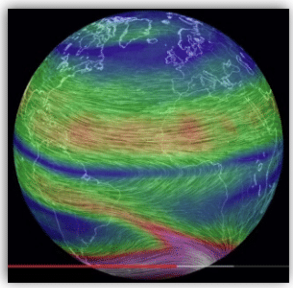


***70 hPa***

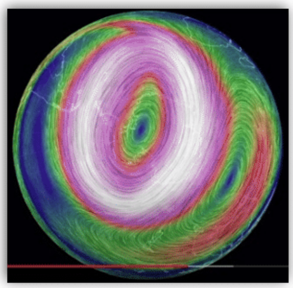
On the figures above, we inspect the summer winds at various height in the **northern hemisphere**. At the surface the winds are stronger over the Ocean than over land, due to the stronger drag over land. Going up to 850 millibars, we see the same structures, i.e. cyclonic and anticyclonic flows, but the winds are getting a bit stronger. At 500 millibars, winds are substantially stronger and they are flowing around the Earth, with the main Jets in the Pacific and Atlantic basins. On that particular day, there was also a jet over the Scandinavian region and a sort of vortex over the north pole. We still observe some small-scale features. At the top of the troposphere (at 250 millibars), where the winds are the strongest, we see the Pacific jet, the Atlantic jet, and strong winds all the way around the northern Hemisphere. In the stratosphere (at 70 millibars), the amplitude of the winds has dropped a little bit. We only observe very large-scale patterns. There are not so many smaller-scale details remaining.



***Tropics (70 hPA)***



***Tropics (10 hPA)***

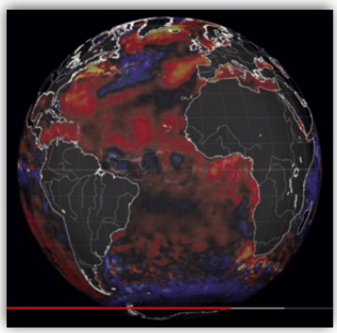


***South Pole (10 hPA)***

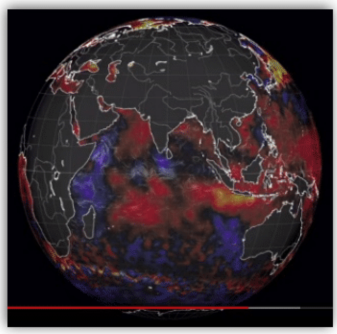
In the **tropical stratosphere**, we observe the tropical winds: the broad easterlies. Going higher, at 10 millibars, they get stronger. Stratospheric tropical winds tend to change sign with altitude and over long timescales as well.

In the **southern hemisphere** – winter stratosphere, we observe the stratospheric vortex which is much stronger than all the features we observed previously.

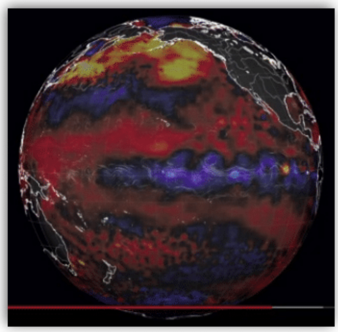
***Ocean circulation***



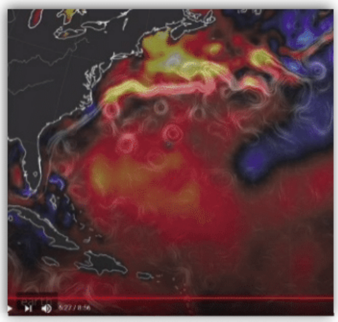
In these maps, surface currents are represented by these swirling features and the colors show the temperature anomalies (relative to the climatology). The Atlantic Ocean surface circulation is shown on the left figure. Most of the strong currents are either in the tropical region or in the western boundaries.



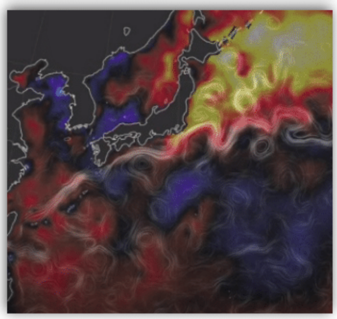
In the **Indian ocean** (figure on the right), we observe some tropical currents associated with a lot of perturbations. In the south, the strong and unstable Antarctic circumpolar current goes all the way around Antarctica.



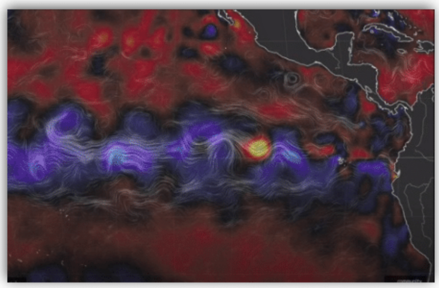
The figure on the left shows the circulation in the **Pacific Ocean**. Similarly, we observe these tropical systems, confined to the equatorial region, that we are going to explore in **#GFD4**. Along the Japanese coast, there is a strong western boundary current with lots of eddies - the Kuroshio.



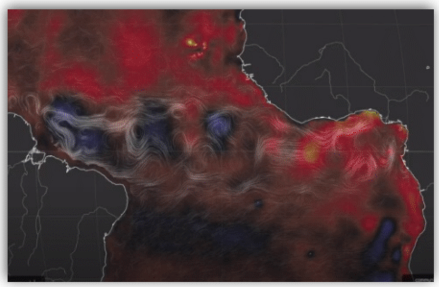
The **Gulf Stream** is a very strong current that flows along the Florida coast up to Cape Hatteras where it flows eastwards into the Atlantic Ocean. The Gulf Stream is meandering and shedding cyclonic and anticyclonic eddies as it goes. Gulf Stream is not just a steady river, it is a very unstable active meandering system. The equivalent western boundary current in the Pacific is called the **Kuroshio**. Likewise, it flows between warm water to the south and cold water to the north. It is a baroclinically unstable region, with lots of mesoscale features (see **#GFD3.4**). These eddies are equivalent to the cyclones and anticyclones we observed in the Atmosphere.



Below, a zoom on the **eastern Pacific equatorial region** provides a better view of the equatorially confined wave-like patterns. In **#GFD4.4**, we are going to describe a whole family of different equatorial waves. The equatorial waves also imprint the surface currents and the temperature anomalies in the equatorial Atlantic.

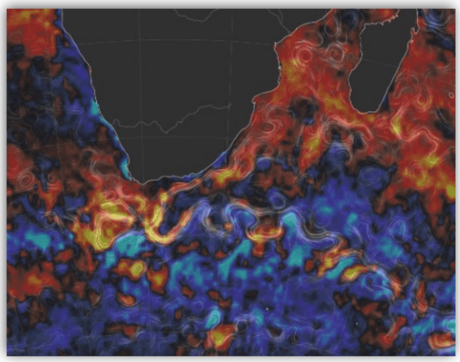


***Eastern Equatorial Pacific***



***Equatorial Atlantic***

In the **western tropical Atlantic**, the Brazil Current is a counter current. It is a very complicated current system in the mean currents associated with a lot of variability.



Finally, we focus on South Africa and the Antarctic circumpolar current. The ***Agulhas Current*** flows westwards along the eastern coast of South Africa. Then it retroflects, meanders, and sheds eddies. Periodically, it also emits large eddies (Agulhas rings) into the Atlantic Ocean.

***Objectives of this course***

These descriptive tours of the Atmosphere and Ocean circulation illustrates the ***large diversity of phenomena*** that we must get to grips with. In this course, the goal is not to analyze each one of them individually. On contrary, we will try to ***integrate*** all of them into a ***framework*** for understanding the ***dynamics of geophysical fluids*** in a ***rotating stratified*** environment. Welcome to GFD class😉

***Videos of the lectures***

Videos of the lectures in **French** and **English** are available on **Nick Hall's YouTube channel** at *https://www.youtube.com/channel/UCqjV8aiVVEvRdYf4DG6Br-w/videos*.

# COURSE OUTLINE

This class is composed of **five chapters**.

**1)** The first course will be devoted to a reminder of the governing equation systems (see **#GFD1.1**) that we will use throughout this class. We will formulate the **shallow water equations** (see **#GFD1.2**) and introduce some useful variable/coordinate transformations, along with some refresher on **vorticity** (see **#GFD1.3**).

**2)** The second course will provide a fair grounding in **quasi-geostrophic theory** (see **#GFD2**). You may have heard of it or even used it, and possibly have a limited understanding of it. After this course, I hope you will understand it perfectly.

**3)** The third GFD session focuses on the **Rossby waves** (see **#GFD3**). You have seen Rossby waves before. In this course, we are going to go into much more detail about Rossby waves and instabilities.

**4)** We will then go to the tropics and discuss gravity waves, along with many other types of tropical waves (see **#GFD4**).

**5)** The last lecture is more descriptive. We will discuss various nonlinear phenomena (in the Atmosphere and the Ocean) which are associated with scale interactions (see **#GFD5**).

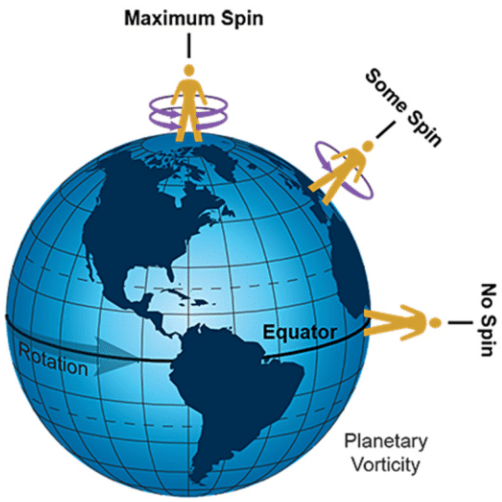
The **prerequisites** for this course are partial differential equations and vector calculus. You need to make do with derivatives, partial derivatives, divergence, and rotational. You have to remember what the Coriolis force is, along with geostrophic balance. **#GFD1** provides a little reminder of the basic equation of motion, flow vorticity, and divergence.

Here is a list of **books**. The course is taken out of these books.

Books:
*Introduction to GFD - Cushman-Roisin*
*Introduction to Dynamical Meteorology - Holton*
*Geophysical Fluid Dynamics - Pedlosky*
*Atmospheric and Oceanic Fluid Dynamics - Vallis*
*El Niño - Philander*

***GFD Keywords***

Geophysical fluid dynamics is the study of ***fluid dynamics*** on ***rotating planets***, often when there is some ***stratification*** in the fluid. So, the planet is rotating and there is a variation in density in the fluid. This is the essence of **GFD**.




The important **keywords** are ***rotation*** and ***stratification***. In this course we will also talk about ***balance*** and ***development***.

GFD pillars

⇨ Rotation. Everything changes when you put the fluid in rotation – on Earth.

⇨ Stratification. This means density varies in horizontal and vertical directions. It is the second pillar of the GFD after rotation.

⇨ Development is when things change with time, when there is a time derivative in the equations, often called tendency. The development is born from an imbalance.

⇨ Balance: It is basically any system of equations where there is no time derivative. Thus, it is a system that is not developing in time. Therefore, there is some sort of balance between the terms in the equation. There are numerous types of balance. The simplest type of balance that you know is geostrophic balance, i.e. the balance between pressure gradient force and Coriolis force. We can go far by supposing that the system is close to a certain equilibrium. But if we are in a state of equilibrium, we do not have any development. We cannot do forecasts.

⇨ Non-linearities are very important. There are nonlinear terms in the equations (advection terms) that give rise to interesting nonlinear dynamical systems, especially for scale interactions and asymmetry. For instance, we will see that non-linearities explain the difference in the size between cyclones and anticyclones.

⇨ We define barotropic and baroclinic and the consequences for the fluid dynamics.

⇨ The variables: In the primitive equations, we classically use **flow velocity** (3D wind/current, $u$,$v$,$w$), **pressure** ($p$), and **density** ($\rho$).

↳ We will try to simplify the systems of equations. We will try to reduce the number of variables or use alternative variables. With some approximations, we can solve less complicated sets of equations than the primitive equations. For instance, we will use the **layer thickness** ($h_n$). If we use a layer model, then the thickness of a layer will inform you about the stratification of the fluid. Vorticity ($\xi$) and divergence can also be used as state variables, instead of the flow velocity. We can use stream function ($\psi$) and potential vorticity ($\xi$).

These variables are all useful depending on which equation set you are solving. We will start by looking at the shallow water equations (see **#GFD1.2**). You may have seen the shallow water equations before or used them without much explanation. In **#GFD1**, we are going to derive them. We will perform a coordinate transformation and introduce the reduced gravity. We will discuss external and internal modes. We will finish a summary of circulation and vorticity.

# CHAPTER 1
# Shallow water and vorticity

# CHAPTER 1
# Shallow water and vorticity

**GFD1 Contents**

# GFD1.1: Basic Equations, Balance, and Flow Partition

### *1.1.a) The primitive equations*

⇨ The primitive equations are composed of a set of 5 equations.

• First, the $x$ **and** $y$ **momentum equations**:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

development nonlinear rotation balance

⇘ They are Newton's law: force equals mass times acceleration or force per unit mass equals acceleration. Conventionally, on the left-hand side, we have acceleration and on the right-hand side we have the force(s). But here, the term associated with the planet rotation - the ***Coriolis force*** (outlined in blue) - is traditionally put on the left-hand side of the equation. It is not a real force, it is just an artifact (fictitious force) of having changed our coordinate system.

▪ It is convenient to keep the ***Coriolis force*** on the LHS because it can be thought of as balancing the (real) pressure gradient force in the ***geostrophic balance*** (blue shading).

▪ The ***nonlinear terms*** (orange shading) – the advection terms - are included in a higher-order balance. We will see that this balance accounts for the difference between small intense cyclones and big flat anticyclones that cannot be explained by geostrophic balance (see **#GFD2.1a**). Higher-order terms (nonlinearities) need to be included to allow asymmetry in size and intensity between cyclones and anticyclones.

▪ The ***development*** is outlined in red. As mentioned before, without this term we cannot perform any forecasting, unless your forecast is persistence.

⇘ During this course, we will choose to keep or not the non-linear terms depending on their importance in the process we analyze. We will see that it is quite useful and interesting to just study a developing linear system. It will give rise to wave solutions that can still transport perturbation properties with the flow (see **#GFD4**).

• For the rest of the primitive equations, we start with the vertical momentum equation. It is the leading order balance, a balance between gravity and the vertical gradient of pressure, known as ***hydrostatic balance***:

$$\frac{\partial p}{\partial z} = -\rho g$$

• Then, we have the ***continuity equation*** or the ***mass conservation equation*** for an incompressible fluid, i.e. the three-dimensional non-divergence:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

• The last equation describes how the density of a fluid parcel will change following the flow. It is the ***density equation***:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = k\frac{\partial^2 \rho}{\partial z^2}$$

⇘ Basically, the only thing that is likely to change the density of the fluid is diffusion, unless it is in contact with the surface. If you chose to think in terms of temperature, you would have to add a state equation that combines pressure and temperature variations in the Atmosphere, or associates density with temperature, salinity, and pressure variations in the Ocean.

⇨ We have 5 state variables, the three wind/current ($u$, $v$, $w$) components, the pressure ($p$), and the density ($\rho$). And we have 5 equations: two prognostic momentum equations, a diagnostic hydrostatic equation (no development), a diagnostic continuity equation, and a prognostic equation for density (which is just the conservation of density but it remains a prognostic equation).

### *1.1.b) Geostrophic/hydrostatic flow - thermal wind balance - vertical coordinate transformation*



⇨ We start by considering a balanced situation in a ***homogeneous fluid*** (with constant density $\rho_0$) in all directions (no density variations) or a fluid that satisfies the Boussinesq approximation (see **#GFD1.1c**), i.e. the density only varies a little bit ($\rho(x,y,z,t) = \rho_0 + \rho'(x,y,z,t)$ with $\rho' \ll \rho_0$).

↳ ***Geostrophic flow*** can be possible because we can change the pressure in the fluid through variations in the height of the surface. If there is a tilted surface to the fluid, then we will have horizontal gradients of pressure throughout the fluid. With a horizontal pressure gradient, the flow can accelerate and establish geostrophic balance.

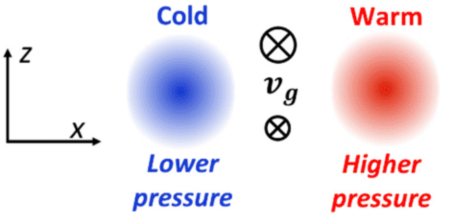
⇨ So, let's consider a Boussinesq fluid that satisfies geostrophic balance in the horizontal ($v_g$ is the geostrophic meridional flow) and hydrostatic balance in the vertical:

$$\frac{\partial p}{\partial x} = \rho_0 f v_g, \quad \frac{\partial p}{\partial z} = -\rho g$$

⇨ This leads to ***thermal wind*** balance: $\frac{\partial^2 p}{\partial x \partial z} = \boxed{\rho_0 f \frac{\partial v_g}{\partial z} = -g \frac{\partial \rho}{\partial x}}$

↳ The vertical gradient of the flow (shear) depends on the horizontal gradient of the density. This is **thermal wind balance**.

↳ This means that if there is no horizontal gradient in the density field, the flow will not change in the vertical direction ($z$), it will be **barotropic**.




⇨ To derive this result, we used $z$ as the vertical coordinate and **made a strong approximation** about the density field. We had to consider a reference density ($\rho_0$) that remained constant when differentiating the hydrostatic equation in $x$: we did not consider the horizontal gradient of density in this step of the calculus.



↳ Let's now consider a method where we can come to the same conclusion but for which we do not need to make any approximations in the density field. For this, we need to **change the vertical coordinate** of our set of equations. Previously, we used $z$, now we will use pressure as our vertical coordinate.

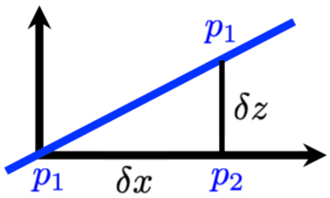
⇨ Let's consider a pressure surface in $(z, x)$ space with constant value $p_1$. We have ***geostrophic balance***:

$$\left.\frac{\partial p}{\partial x}\right|_z = \rho f v_g$$

↳ With a little bit of geometry, we can rewrite the horizontal pressure gradient as:

$$\left.\frac{\partial p}{\partial x}\right|_z = \frac{p_2 - p_1}{\delta x} = \frac{p_2 - p_1}{\delta z}\frac{\delta z}{\delta x} = -\frac{p_1 - p_2}{\delta z}\frac{\delta z}{\delta x}$$

$$\left.\frac{\partial p}{\partial x}\right|_z = -\frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_p = \rho f v_g$$




⇨ Using hydrostatic balance $\frac{\partial p}{\partial z} = -\rho g$, we get two formulae:

$$\left.\frac{\partial z}{\partial x}\right|_p = \frac{f v_g}{g} \qquad \frac{\partial z}{\partial p} = -\frac{1}{\rho g}$$

↳ Differentiating the first by $p$ and the second by $x$ leads to:

$$\frac{\partial^2 z}{\partial x \partial p} = \boxed{\frac{f}{g}\frac{\partial v_g}{\partial p} = -\frac{1}{g}\left.\frac{\partial}{\partial x}\right|_p \left(\frac{1}{\rho}\right)}$$

Barotropic flow definition

↳ The vertical gradient of the geostrophic flow in pressure coordinates is related to the gradient of $1/\rho$ along a pressure surface (horizontal gradient measured in pressure coordinates), i.e. a quantity related to density. This is ***thermal wind balance*** expressed in **pressure coordinates**. The big difference here is that **we have not made any approximations** for the density field and we end up with the same conclusion.

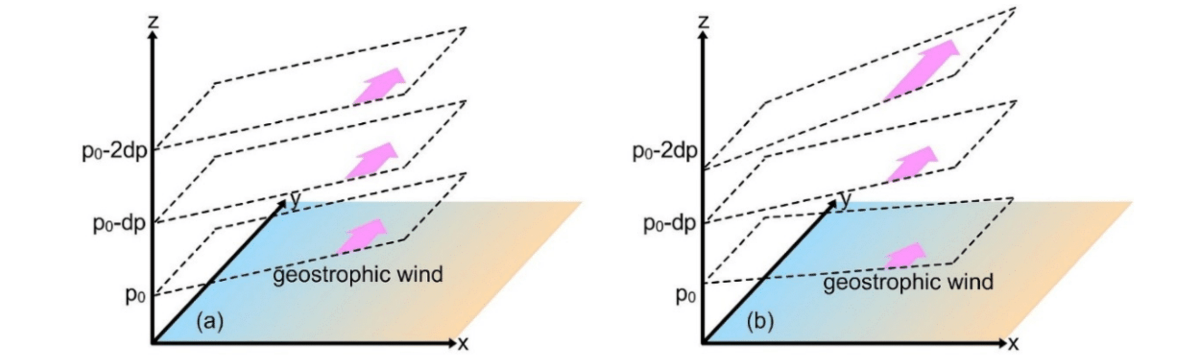
↳ The useful interpretation we can draw from this analysis is that the horizontal density variation along a pressure surface leads to a vertical gradient (shear) in the wind/current.

If { pressure and the density are functionally related; pressure and density surfaces are parallel; density does not change along a pressure surface }, the right-hand term would be zero,

↪ because there is no gradient of density along the pressure surface.

⇨ This kind of flow is called **barotropic flow**. This is the actual definition of barotropic flow where there is no change of density along a pressure surface. And in this case, there is no vertical gradient of the current/wind, i.e. barotropic flow (no vertical variation in the flow).

↳ This is illustrated in the figure below.




*The blue portion of the surface denotes a cold region while the orange portion denotes a warm region. This temperature structure is restricted to the surface in (a) but extends through the depth of the fluid in (b). The dotted lines enclose isobaric surfaces which remain at constant slope with increasing height in (a) and increase in slope with height in (b). Pink arrows illustrate the direction and amplitude of the horizontal wind. Only in the baroclinic atmosphere (b) do these vary with height.*

### *1.1.c) Density and its variations*

⇨ We mentioned variations of density. Let's now introduce some commonly-used levels of approximation - a useful glossary. Various approximations ordered by increasing levels of accuracy:

*Homogeneous:* $\rho = \rho_0 \quad (\rho' = 0), \quad p = p_0(z) + p'(x, y, t) \quad \Rightarrow \frac{\partial \mathbf{v}}{\partial z} = 0$

*Boussinesq:* $\rho = \rho_0 + \rho'(x, y, z, t), \quad p = p_0(z) + p'(x, y, z, t)$

*Anelastic:* $\rho = \rho_0(z) + \rho'(x, y, z, t)$

➢ First, we have **homogeneous flow**. It happens when there is no variation in density at all in the fluid. The density is the same everywhere. Similar to a bucket of water that is spinning around. Density is constant, but the pressure can vary. The pressure varies with $z$, as you go down the pressure increases (hydrostatic balance). Pressure can also vary in the horizontal $(x, y)$ and time, because of the variation of the free surface. But there are no horizontal gradients of density, which means that there will not be any variation of the flow with depth.

➢ **Boussinesq flow** is the next level up. There is a constant reference density ($\rho_0$) and a perturbation ($\rho'$) which is small compared to the reference density ($\rho' \ll \rho_0$). In this flow, the pressure is composed of a reference pressure ($p_0$) which is only a function of $z$ and some pressure perturbations associated with the dynamics.

- The Boussinesq approximation is a good approximation for the **Ocean**. The density of seawater near the surface is ~1025 kg/m$^3$. Sea water is not totally incompressible. At the bottom of the ocean, the seawater is denser (~1040 kg/m$^3$), but fractionally not very different. The variation in density associated with compression is small compared to the actual density (~1.5%). So, Boussinesq remains an excellent approximation in the Ocean
- In the **Atmosphere**, the density at the surface is ~1Kg/m$^3$. As you go up through the Atmosphere, the air parcel gets less and less dense and there are large variations of density. Yet the Boussinesq approximation is still useful in the atmosphere because these variations in density are mainly in the vertical, so they do not affect the horizontal gradients of density and they do not play into the dynamics of the atmosphere. We can study atmospheric dynamics using the Boussinesq approximation at large scales, and it works.

➢ If we want to analyze the variation of the density with height, we can use the **anelastic approximation** where the reference density is a function of $z$ ($\rho_0(z)$). This approximation is used for smaller-scale (mesoscale) meteorology.

### *1.1.d) Barotropic and baroclinic flow*

⇨ Then we recall the definition of **barotropic**, where the density is a function of pressure implying that there is no vertical variation of the geostrophic flow.

$$\rho = \rho(p) \quad \Rightarrow \quad \frac{\partial \mathbf{v}_g}{\partial z} = 0$$

↳ Barotropic: In some circumstances, the flow is ***vertically coherent***. Depth independent flow is associated with the barotropic component also referred to as the ***external mode***.

This terminology comes from changes in the level of the free surface and which are transmitted throughout the depth of the fluid, i.e. an external influence on the flow (see **#GFD1.2e**).

Barotropic flow can exhibit many phenomena: vortices, Rossby waves, jets, and instability. It is a good starting point for a lot of understanding of the dynamics of Atmospheric and Ocean. It is the foundation of large-scale ocean circulation theory (see **#GFD5.3c**). Many theories related to the large-scale Ocean circulation are barotropic (Stommel, Munk, …).

⇨ **Baroclinic** just means that ***it is not barotropic***. There are variations of density on a pressure surface with all sorts of interesting consequences.

$$\rho \neq \rho(p)$$

↳ Baroclinic: When density surfaces cross pressure surfaces the flow is baroclinic. The baroclinic component is associated with horizontal temperature gradients, fronts, developing cyclones, ocean eddies on the thermocline. Baroclinic processes are necessary to liberate potential energy and generate circulation (see **#GFD3.4**). The growth of geostrophic systems depends on baroclinic conversions of energy. Baroclinic instability occurs on a preferred scale (see **#GFD3.4j**) - the Rossby radius (see **#GFD1.2a**) - and this is important for generating geostrophic turbulence.

### *1.1.e) Stationary and transient flow*

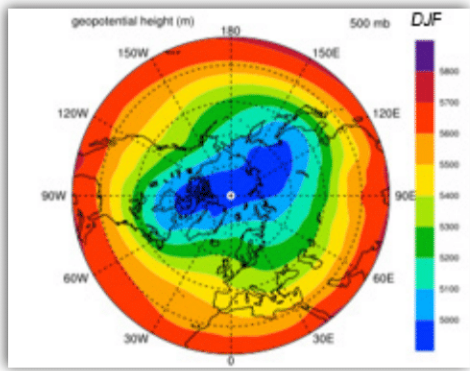
⇨ A quick word on different ways of dividing the flow, mainly in the Atmosphere.

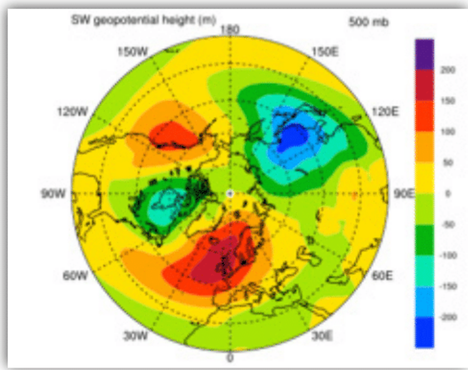
*What does an atmospheric scientist mean when he talks about **stationary waves**? He might be using the wrong **terminology**.*

• **<u>Stationary waves</u>**: A stationary wave is like a wave which has fixed nodes and it oscillates *in situ* without propagating (like when you move in the bath). But, when a meteorologist refers to stationary waves, it means something much simpler. It is just the departure from the **zonal average**:

$$\phi = [\phi] + \phi^*$$

↳ For instance, consider a variable (winter 500 millibar height) and estimate the zonal average (noted with squared brackets) by computing the mean value around latitude circles. Subtracting this zonal mean component from the original field yields the **stationary wave pattern**.




↳ If you are interested in the flux of temperature (or heat) which is effected by the circulation, then the zonal average of this flux will depend on two components:

$$[vT] = [v][T] + [v^*T^*]$$

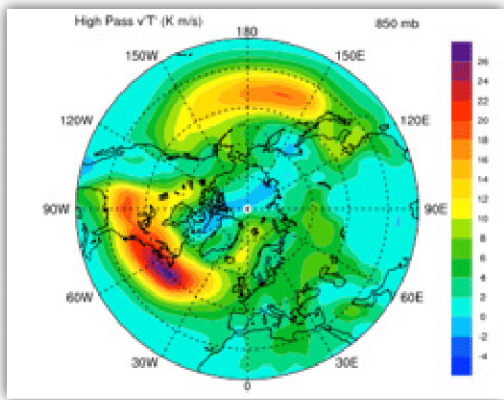
1) The flux of the zonal average temperature by the zonal average wind. This is the overturning circulation cells (Hadley/Ferrel cell).
2) The zonal average of the product of the stationary wave wind with the stationary wave temperature.

• The **<u>transient part of the flow</u>**. The time average of the flow is denoted by a bar. The instantaneous flow is equal to the average plus a perturbation. This is the ***Reynolds decomposition***.

$$\phi = \overline{\phi} + \phi', \quad \overline{vT} = \overline{v}\overline{T} + \overline{v'T'}$$

↳ In this case, the average flux is the flux by the average flow plus the transient eddy flux. The latter is the (*very important*) transfer of temperature and momentum which is carried out by these transient systems in the Atmosphere and the Ocean (see **#GFD5**).

On the right, the transient eddy flux of temperature 850 millibars reveals the "storm track" regions, in the western Atlantic and western Pacific, where we encounter the most intense transfer of heat associated with these synoptic transient systems.



⇨ Writing down the equation for the development of the temperature, i.e. development of temperature, plus advection terms equals forcing minus dissipation:

$$\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T = \mathcal{F} - \mathcal{D}$$

↳ Averaging the equation gives: $\overline{\mathbf{v}}.\nabla\overline{T} = -\overline{\mathbf{v}'.\nabla T'} + \overline{\mathcal{F}} - \overline{\mathcal{D}}$

↳ The flux by the average flow is equal to the average forcing and dissipation plus the transient flux. The transient flux term is put it on the right-hand side, but it is just a part of the flow (see **#GFD5.1c**). It is however often regarded as another type of forcing: the transient eddy forcing. It is the main term that balances the mean advection.

## GFD1.2: Density Coordinates and Shallow Water Equations

### *1.2.a) Some scaling parameters*

Let's look at some typical non-dimensional numbers which are important in geophysical fluid dynamics (rotation and stratification).

Rossby number

1) **The importance of rotation**. We compare the **advection term** with the **Coriolis force**.

$$u\frac{\partial u}{\partial x}\Big/ fu$$

↳ We compare this steady acceleration term with the Coriolis force, by computing the following ratio:

$$R_o = \frac{U}{fL}$$

typical wind $U$ divided by $f$ times a typical length scale $L$

↳ If the rotation/Coriolis force is important then this number, the **Rossby number**, will be small and the flow will be close to geostrophic.

2) **The importance of stratification**: **Froude number**. For steady non-rotating flow, the steady acceleration term (**advection term**) is balanced by the **pressure gradient force**:

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

↳ In this framework, we compare vertical divergence ($\frac{\partial w}{\partial z}$) with horizontal divergence ($\frac{\partial u}{\partial x}$). A scale analysis of this ratio (see Cushman-Roisin) gives the Froude number:

$$F_r = \frac{U}{NH} = \frac{U}{\sqrt{g'H}}$$

Recall

$$N^2 = \frac{g}{\rho}\frac{\partial \rho}{\partial z} = \frac{g'}{H}$$

$$(g' = g\Delta\rho/\rho)$$

Typical flow speed $U$ divided by $N$
times the typical thickness of your fluid $H$
$N$ is the Brunt-Vaisala frequency, related to the vertical gradient of density

↳ The Froude number can also be expressed as $U$ divided by the root of $g'H$ which is the gravity wave speed.

- If the Froude number is small then the stratification is strong and the gravity waves are very fast, i.e. a kind of rigid system where there are no big vertical excursions, fast gravity waves, and not much communication between the layers.
- If the Froude number is large then bigger vertical excursions of the flow can occur.

3) Which is the more important **rotation or stratification**? We can divide the Rossby number by the Froude number. This is the square of the **Burger number**:

$$(B_u)^{\frac{1}{2}} = \frac{R_o}{F_r} = \frac{NH}{fL} = \frac{\sqrt{g'H}}{fL}$$

↳ When the Burger number is close to 1 that is when both the rotation and stratification are important. There are many different types of scale analysis that lead to this situation. For example, you can say that **vorticity advection balances vortex stretching** (see **#GFD3.4c** and **#GFD5.5b**).

You can back out the typical length scale $L_R$ at which these two processes are both important.

$$L_R = \frac{NH}{f} = \frac{\sqrt{g'H}}{f}$$

↳ This scale is called the **Rossby radius**. It is a fundamental important quantity in geophysical fluid dynamics. It is the typical scale of the systems we described in **#GFDintro** (cyclones and anticyclones) in the Atmosphere and in **#GFDintro** in the Ocean. We observed that they are much smaller in the Ocean. This is due to the stratification.

The Rossby radius is also the scale of the coastal Kelvin waves (see **#GFD4.2b**). ✋ Close to the equator, the Rossby radius does not work as a useful scale because $f = 0$ (see **#GFD4.3a**)

### *1.2.b) Equation sets and variables*

⇨ Let's discuss how we can simplify the primitive equations (see **#GFD1.1a**). We have a set of 5 equations with 5 variables (*u*, *v*, *w*, *p* and ρ).

• The next step of simplification (see **#GFD1.2ef**) is to represent the stratification as finite layers of homogeneous density and derive the ***shallow water equations*** (see **#GFD1.2f**).

↳ You can stack these layers on top of one another to get a fairly intricate description of the flow but with only three variables: each layer has just *u*, *v* and *h*, the thickness of the layer.

↳ **Shallow water equations**: three variables + three prognostic equations

• Then we can go one step further (see **#GFD2**), with the ***quasi-geostrophic system***, in which there is only one variable, the stream function ($\psi$).

↳ **Quasi-geostrophic system**: one variable + one prognostic equation

⇨ Instead of considering the classical horizontal components of the wind/current $(u, v)$, we can express the horizontal flow in terms of its divergent (irrotational) and non-divergent part:

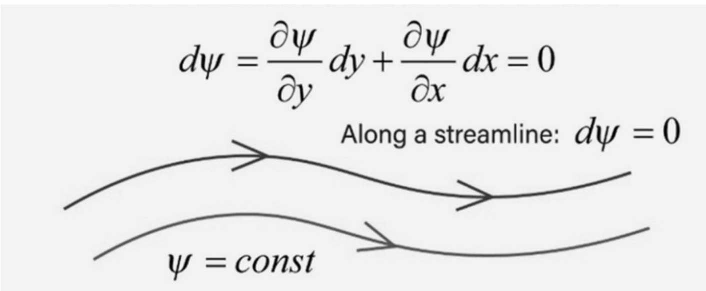
$$\mathbf{v} = (u, v) = -\nabla\phi + \hat{\mathbf{k}}_\wedge \nabla\psi$$

with $\phi$ the velocity potential and $\psi$ the stream function, such that:

$$u = -\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}$$

↳ Non-divergent flow can be defined entirely in term of the stream function, while the divergent part (irrotational flow) can be described entirely in terms of velocity potential.

The divergence of the flow is: $D = \nabla.\mathbf{v} = -\nabla^2\phi$
The relative vorticity is $\xi = \hat{\mathbf{k}}.\nabla_\wedge \mathbf{v} = \nabla^2\psi$



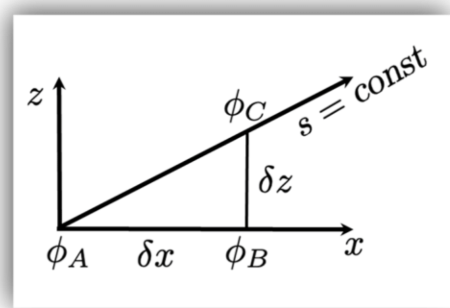
### *1.2.c) Alternative vertical coordinates*

⇨ We can ***simplify the equations*** if we use a conserved quantity as the vertical coordinate. In this frame of reference there is no "vertical velocity", rendering the system two-dimensional. So, we can reduce our equation set by using coordinate systems based on density in the ocean or potential temperature in the atmosphere. But the price we pay for this simplification is to complicate the boundary conditions: coordinate surfaces outcrop, they move in time, and our coordinates are no longer orthogonal.

↳ We will transform our primitive equations, using a vertical coordinate transformation. To do so, we first derive a couple of general rules that will be useful to transform the equation into density coordinates (see **#1.4.d**).

⇨ Let's consider an $(x,z)$ coordinate system in which a field $\phi$ varies. There is a line (or surface) on which a certain quantity $s$ (or density) remains constant.

• We want to transform the horizontal derivative of $\phi$ between $z$ and $s$ coordinates. With a little bit of geometry, we can show that:




$$\left.\frac{\partial \phi}{\partial x}\right|_s = \frac{\phi_C - \phi_A}{\delta x} \qquad \delta x,\ \delta z \to 0$$

$$= \frac{\phi_C - \phi_B}{\delta z}\left(\frac{\delta z}{\delta x}\right) + \frac{\phi_B - \phi_A}{\delta x}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_s = \frac{\partial \phi}{\partial z}\left(\left.\frac{\partial z}{\partial x}\right|_s\right) + \left.\frac{\partial \phi}{\partial x}\right|_z$$

$$\left.\frac{\partial \phi}{\partial x}\right|_z = \left.\frac{\partial \phi}{\partial x}\right|_s - \left.\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x}\right|_s \qquad [1]$$

⇨ Using the inclination of the surface iso-$s$ $\frac{\partial s}{\partial z}$, the vertical derivative becomes:

$$\frac{\partial \phi}{\partial z} = \frac{\partial s}{\partial z}\frac{\partial \phi}{\partial s} \qquad [2]$$

### *1.2.d) Density coordinates*

⇨ We transform the primitive equations (see **GFD1.3.a**) into a density coordinate system (see details of the calculous on the following page).

• The hydrostatic equation: $\frac{\partial p}{\partial z} = -\rho g$ , using rule [2] we get: $\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$

↳ At this point, we introduce a new variable $P$ (capital $P$), called the ***Montgomery potential***, defined as: $P = p + \rho g z$. The hydrostatic equation can be written neatly as:

$$\frac{\partial P}{\partial \rho} = gz$$

• The momentum equations: $\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\left.\frac{\partial p}{\partial x}\right|_z \overset{\text{Rule [1]}}{=} -\frac{1}{\rho_0}\left[\left.\frac{\partial p}{\partial x}\right|_\rho - \left.\frac{\partial p}{\partial z}\frac{\partial z}{\partial x}\right|_\rho\right]$

Hydrostatic balance

$$= -\frac{1}{\rho_0}\left[\left.\frac{\partial p}{\partial x}\right|_\rho + \rho g \left.\frac{\partial z}{\partial x}\right|_\rho\right] = -\frac{1}{\rho_0}\left.\frac{\partial P}{\partial x}\right|_\rho$$

↳ As density is conserved, there is no vertical velocity when developing $D/Dt$. The momentum equations in the density coordinates can be written (we lose $w$ and replace $p$ by $P$):

$$\frac{\partial u}{\partial t} + u\left.\frac{\partial u}{\partial x}\right|_\rho + v\left.\frac{\partial u}{\partial y}\right|_\rho - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\left.\frac{\partial v}{\partial x}\right|_\rho + v\left.\frac{\partial v}{\partial y}\right|_\rho + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

# Details for the transformation of the equations into a density coord syst

1) Hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g$$

$$[2] \Rightarrow \frac{\partial p}{\partial z} = \frac{\partial \rho}{\partial z}\frac{\partial p}{\partial \rho} = -\rho g$$

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$$

Define

$$P = p + \rho g z$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + gz + \rho g \frac{\partial z}{\partial \rho}$$

$$\frac{\partial P}{\partial \rho} = gz$$

Hydrostatic equation in terms of "Montgomery potential" $P$.

2) Thermodynamic equation (density equation)

in *any* coordinate system (Boussinesq fluid - incompressible)

$$\frac{D\rho}{Dt} = 0$$

in $z$ coordinates

$$\frac{\partial \rho}{\partial t} + u \left.\frac{\partial \rho}{\partial x}\right|_z + v \left.\frac{\partial \rho}{\partial y}\right|_z + w \frac{\partial \rho}{\partial z} = 0$$

i.e.

$$\frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} + \frac{dy}{dt}\frac{\partial \rho}{\partial y} + \frac{dz}{dt}\frac{\partial \rho}{\partial z} = 0$$

On a surface of constant $\rho$, $z$ varies. To make $z$ the variable and $\rho$ the coordinate, we rewrite this equation swapping the variables:

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + \frac{d\rho}{dt}\frac{\partial z}{\partial \rho} = w$$

and the last term is zero because $\rho$ is conserved. This gives us an equation for $w$.

3) $x$ momentum equation

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\left.\frac{\partial p}{\partial x}\right|_z$$

$$[1] \Rightarrow = -\frac{1}{\rho_0}\left[\left.\frac{\partial p}{\partial x}\right|_\rho - \frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\rho\right]$$

$$= -\frac{1}{\rho_0}\left.\frac{\partial}{\partial x}\right|_\rho (p + \rho g z) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

since

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left.\frac{\partial}{\partial x}\right|_\rho + v\left.\frac{\partial}{\partial y}\right|_\rho$$

(no vertical term because $\rho$ is conserved)

$$\rightarrow \frac{\partial u}{\partial t} + u\left.\frac{\partial u}{\partial x}\right|_\rho + v\left.\frac{\partial u}{\partial y}\right|_\rho - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

likewise

$$\frac{\partial v}{\partial t} + u\left.\frac{\partial v}{\partial x}\right|_\rho + v\left.\frac{\partial v}{\partial y}\right|_\rho + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

4) continuity equation

z coordinates:

$$\left.\frac{\partial u}{\partial x}\right|_z + \left.\frac{\partial v}{\partial y}\right|_z + \frac{\partial w}{\partial z} = 0$$

(ignore $dv/dy$ term for the moment)

$$[1] \text{ and } [2] \Rightarrow \left.\frac{\partial u}{\partial x}\right|_\rho - \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\rho + \frac{\partial w}{\partial \rho}\frac{\partial \rho}{\partial z} = 0$$

$$\text{multiply by } \frac{\partial z}{\partial \rho}$$

$$\frac{\partial z}{\partial \rho}\left.\frac{\partial u}{\partial x}\right|_\rho - \frac{\partial u}{\partial \rho}\left.\frac{\partial z}{\partial x}\right|_\rho + \frac{\partial w}{\partial \rho} = 0$$

the last term can be expanded

$$\frac{\partial w}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{Dz}{Dt}\right) = \frac{\partial}{\partial \rho}\left[\frac{\partial z}{\partial t} + u\left.\frac{\partial z}{\partial x}\right|_\rho\right]$$

$$= \frac{\partial^2 z}{\partial \rho \partial t} + \frac{\partial u}{\partial \rho}\left.\frac{\partial z}{\partial x}\right|_\rho + u\frac{\partial^2 z}{\partial \rho \partial x}$$

which leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \frac{\partial z}{\partial \rho}\left.\frac{\partial u}{\partial x}\right|_\rho + u\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \rho}\right) = 0$$

Putting the $y$ term back in gives

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial x}\right|_\rho \left(u\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial y}\right|_\rho \left(v\frac{\partial z}{\partial \rho}\right) = 0$$

This is the flux form of the continuity equation. The tendency of $dz/d\rho$ is given in terms of its flux along density surfaces. $dz/d\rho$ is a continuous form but this can be identified with mass conservation in terms of a flux of layer thickness

$$\frac{\partial z}{\partial \rho} = \frac{h}{\Delta \rho}$$

• The continuity equation: $\left.\frac{\partial u}{\partial x}\right|_z + \left.\frac{\partial v}{\partial y}\right|_z + \frac{\partial w}{\partial z} = 0$

↳ The transformation is slightly more complicated, but all the steps of the derivation are written on the previous page. Basically, we use **rule [1]** for the horizontal derivative, **rule [2]** for the vertical derivative, and at one point we use the fact that $w = Dz/Dt$. We get:

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial x}\right|_\rho \left(u\frac{\partial z}{\partial \rho}\right) + \left.\frac{\partial}{\partial y}\right|_\rho \left(v\frac{\partial z}{\partial \rho}\right) = 0$$

↳ We obtain a tendency equation expressed for the quantity $\frac{\partial z}{\partial \rho}$ which informs us about the stratification. It is mass conservation expressed as a convergence of the flux of stratification that leads to changes in stratification.

⇨ We can simplify this concept by considering a discreet representation with layers of constant density $\rho_0$, $\frac{\partial z}{\partial \rho}$ is going to become just $\frac{h}{\Delta\rho}$, with $h$ the thickness of a layer, and $\Delta\rho$ a standard density difference between two adjacent layers. That how we are going to derive the **shallow water equations** (see **#GFD1.2e**).

### *1.2.e) Shallow water layers*

⇨ Let's put together a **system** in which:

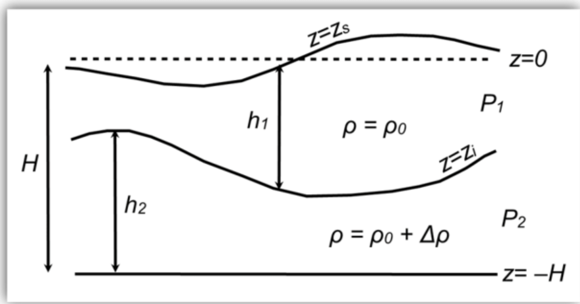
• There is a flat bottom, with fluid flowing over. There is a free-surface that can vary in time and position. The average depth of this fluid (or reference depth) is a constant $H$.

• First, consider a simple case with **two homogeneous layers of fluid**:

➲ The thickness of the upper layer is $h_1$ and lower layer $h_2$.

➲ The density of the upper layer is constant and equal to $\rho_0$. The lower layer is slightly denser ($\rho_0+\Delta\rho$).

➲ Since the density in each layer is constant, the horizontal gradient of the pressure in each layer is also constant, as are the flow speed $u$ and $v$.




↳ At any point in space, $\rho$, $u$, $v$, $P$ (and $\frac{\partial P}{\partial x}$) are discretized in the vertical and are vertically constant in each layer. On the contrary, $z$ and pressure $p$ remain continuous with $p$ increasing from the surface with $z$. ✋ As the sea level varies in space, $u$, $v$, $P$ also vary in space.

⇨ We estimate the **horizontal gradient of pressure** ($\frac{\partial P}{\partial x}$) in each layer.

• The ***Montgomery potential*** in the upper layer ($P_1$) is estimated from the hydrostatic equation (see **#GFD1.2d**) at the interface between the ocean and the atmosphere (at $z = z_S$):

$$\frac{\partial P}{\partial \rho} = \frac{\Delta P}{\Delta \rho} = g z_S \Leftrightarrow \frac{P_1 - P_a}{\rho_0 - \rho_a} = g z_S$$

$$P_1 = P_a + \rho_0 g z_S \Leftrightarrow P_1 = P_a + \rho_0 g(-H + h_2 + h_1)$$

The atmospheric pressure ($P_a$), assumed to be constant, will be neglected in the following because we are only interested in pressure gradients.

↳ The pressure gradient force in the upper layer takes the following form:

$$\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} = g\frac{\partial D}{\partial x} \quad \text{(where } \boldsymbol{D} \text{ is } h_1 + h_2\text{)}$$

↳ The horizontal gradient of $P_1$ depends on the horizontal gradient of the total depth of the fluid $\boldsymbol{D}$. *i.e.* the flow in the top layer depends only on the height of the free surface. This makes sense: pressure gradients in the top layer depend on how far from the surface the particle of fluid is.

• The Montgomery potential in the lower layer ($P_2$) is estimated from the hydrostatic equation at the interface between the top and the bottom layer:

$$\frac{\partial P}{\partial \rho} = \frac{\Delta P}{\Delta \rho} = \frac{P_2 - P_1}{\Delta \rho} = g z_i = g(-H + h_2)$$

↳ This yields: $P_2 - P_1 = \Delta\rho g(-H + h_2)$

↳ The pressure gradient force in the lower layer is given by: $\frac{1}{\rho_0}\frac{\partial P_2}{\partial x} = g\frac{\partial D}{\partial x} + g'\frac{\partial h_2}{\partial x}$

⇨ The flow in the lower layer can be decomposed in two contributions:

- i) a ***barotropic mode*** or external mode
- ii) a ***baroclinic contribution*** $g'\frac{\partial h_2}{\partial x}$.

$$g' = \frac{\Delta\rho}{\rho_0} g$$

↳ This can be generalized: throughout the fluid the flow undergoes the effect of the free surface variations, a barotropic external mode, and going down layer by layer different contributions from the stratification (the baroclinic part) add up.

⇨ The important difference between these two contributions is that the baroclinic term is multiplied by **the reduced gravity $g'$** $\left(= \frac{\Delta\rho}{\rho_0} g\right)$ which is **much smaller than $g$**. This means that:

- Tiny upper surface movements generate strong currents.
- To generate strong vertical variations in the currents (baroclinic part), the interface between the density layers has to move substantially more (significantly stronger horizontal gradients).

↳ This is realistic. The Ocean free-surface varies only by a few centimeters, while the thermocline displacements are of the order of tens of meters.

⇨ This system can be generalized to a multiple layer ($N$-layer) equation:

With $\boldsymbol{P}$ and $\boldsymbol{h}$ column vectors of Montgomery potential and layer thicknesses for all the layers: the system can be written:

$$\boldsymbol{P} = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} \qquad \boldsymbol{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix}$$

$$\frac{1}{\rho_0}\frac{\partial \mathbf{P}}{\partial x} = g\frac{\partial D}{\partial x} + g'\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}$$ with

*for 2 layers*

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \\ & & & N-1 \end{pmatrix}$$

Coupling between layers

The pressure gradient is the sum of the ***barotropic mode*** associated with the gradient of the free surface and the ***baroclinic contributions***. The latter is equal to $g'$ multiplied by the gradient of the layer thicknesses ($\boldsymbol{h}$) multiplied by a squared symmetric matrix **C**, which ***couples*** the layers together.

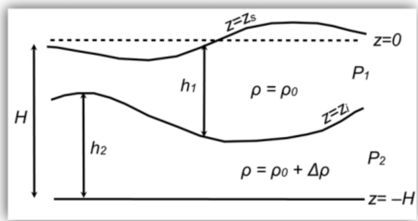
- The coupling term between layers for the two-layer model is very simple. Just $0\ 0\ 0\ 1$, giving the two-layer equation from before.
- For an N-layer model, **C** consists in zeros that correspond to the barotropic mode. Then there a are a bunch of ones, twos … etc, all the way to $N - 1$ in the lower right corner.

These $N$ equations are **strongly coupled**. One cannot just take one layer and solve for the flow in this particular layer. We need to know about the thicknesses of every other layer above before solving it. To solve the system, instead of solving the equations layer by layer, we solve them mode by mode. This involves finding the eigenvectors of the **C** matrix and transforming the variables to get a set of decoupled equations. **We will see that in an exercise at the end of the class**.

### *1.2.f) The shallow water equations*

⇨ Now that we have expressions for the horizontal gradients of the Montgomery potential, we can put them into the momentum equations. We discretize the stratification and write the equation set in terms of 3 variables *u*, *v* and *h*.

**For a two-layer system**:




$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} + v_i\frac{\partial u_i}{\partial y} - fv_i = -g\frac{\partial}{\partial x}(h_1+h_2) - g'\frac{\partial h_2}{\partial x}$$

$$\frac{\partial v_i}{\partial t} + u_i\frac{\partial v_i}{\partial x} + v_i\frac{\partial v_i}{\partial y} + fu_i = -g\frac{\partial}{\partial y}(h_1+h_2) - g'\frac{\partial h_2}{\partial y}$$

*(only for i=2)*

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x}(u_i h_i) + \frac{\partial}{\partial y}(v_i h_i) = 0$$

$i$ is the index that refers to the layer number.

- The momentum equations can be decomposed into acceleration, Coriolis force, and barotropic/baroclinic modes. For $i > 1$, there are the baroclinic internal mode contributions/terms.
- The continuity equation is just the same for every layer, as the mass is conserved in each layer. Convergence and divergence generate tendencies of layer thickness, separately in each layer.

⇨ For an N-layer model, the shallow water equations can be written:

$$\frac{Du_i}{Dt} - fv_i = -g\frac{\partial D}{\partial x} - g'\left[\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}\right]_{i,i>1}$$

$$\frac{Dv_i}{Dt} + fu_i = -g\frac{\partial D}{\partial y} - g'\left[\mathbf{C}\frac{\partial \mathbf{h}}{\partial y}\right]_{i,i>1}$$

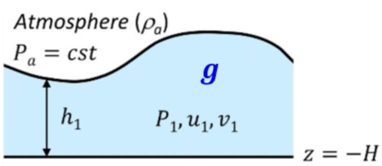
$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x}(u_i h_i) + \frac{\partial}{\partial y}(v_i h_i) = 0$$

### *1.2.g) The thermocline and the abyss*

Single layer

⇨ Let's consider a single layer model with a flat bottom and a free surface. It could be a barotropic ocean or a barotropic atmosphere. Just a layer of fluid which flows on a flat bottom.

⇘ Here are the 3 equations that describe the dynamics: two momentum equations, and the continuity equation.



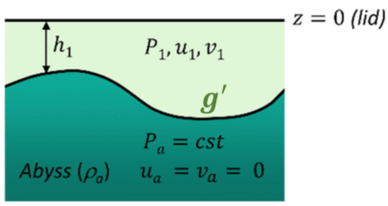

$$\frac{Du}{Dt} - fv = -g^{(\prime)}\frac{\partial h}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -g^{(\prime)}\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

Motionless abyss

⇨ Let's **flip it over**, to consider a rigid lid and a motionless abyss over which the thermocline can move. We have the same number of degrees of freedom. It is a one-layer system. The equations remain the same, except that instead of having the gravity $g$, we now use the reduced gravity $g'$.




⇘ Sometimes this is called a ***one-and-a-half-layer system*** because you have a fluid below the active layer which does not move (motionless abyss).

⇨ The ***N-layer*** version of this representation (rigid lid + many layers in the thermocline + the motionless abyss) has a different development equation. The barotropic part has disappeared, as we cancelled the effect of the free-surface. There is only the $g'\boldsymbol{C}$ matrix term and the **C** matrix has been **flipped**. It does not have zeros anymore and it has one extra baroclinic mode instead of the barotropic/external mode. All the gravity waves are slow.

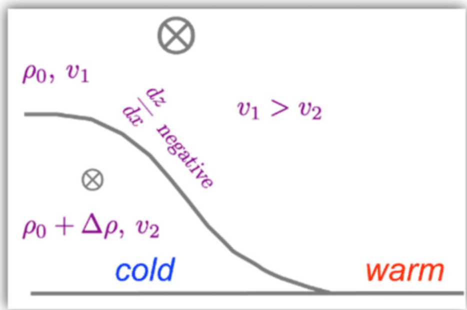
$$\frac{Du_i}{Dt} - fv_i = -g'\left[\mathbf{C}\frac{\partial \mathbf{h}}{\partial x}\right]_i \qquad \mathbf{C} = \begin{pmatrix} N & N-1 & N-2 & \dots & 1 \\ N-1 & N-1 & N-2 & \dots & 1 \\ N-2 & N-2 & N-2 & \dots & 1 \\ \vdots & \vdots & \vdots & & \\ 1 & 1 & 1 & & 1 \end{pmatrix}$$

### *1.2.h) Thermal wind revisited*

⇨ It is interesting to think of thermal wind balance (geostrophic hydrostatic balance) in terms of the change of density across the surface.

• ***Front in the atmosphere***:

Let's imagine a layer boundary, on the left of which there is cold air and, on the right, there is warm air. Density is thus slightly greater on the left.




The thermal wind equation in density coordinates (as the second derivative of Montgomery potential by $x$ and $z$) is written:

$$\frac{\partial^2 P}{\partial x \partial \rho} = \rho_0 f \frac{\partial v}{\partial \rho} = g\left.\frac{\partial z}{\partial x}\right|_\rho$$

⇨ This balance shows that difference in the wind between the warm and the cold layer ($\frac{\partial v}{\partial \rho}$) depends on the slope of the boundary ($\frac{\partial z}{\partial x}$):

$$\frac{\partial v}{\partial \rho} = \frac{g}{\rho_0 f}\left.\frac{\partial z}{\partial x}\right|_\rho, \qquad v_1 - v_2 = -\frac{g'}{f}\left.\frac{\partial z}{\partial x}\right|_\rho \qquad g' = \frac{\Delta\rho}{\rho_0}g$$

↳ The greater the frontal slope, the stronger the wind shear across it. This is called the **Margules relation**.

• ***Geostrophic currents***:

⇨ We can also think from an oceanographic point of view.

Imagine you are in a boat, on the surface of the ocean. You can measure the slope of the thermocline to deduce information about the currents.

↳ So, you navigate to one place and measure the depth of the thermocline there, then you go to another place and measure the depth of the thermocline there. You calculate the slope of the thermocline and then estimate the difference in currents across this density surface, using the thermal wind balance equation. But it will not give you the actual current amplitude, but only the difference across the density surface and that is the problem. Often, oceanographers call that the geostrophic current, but in reality, it is the **vertical gradient of the geostrophic current** and they do not know what the actual current is. They would need the slope of the surface, but from a boat you cannot get this information (you need an altimetric satellite to work that out). In order to overcome this limitation, oceanographers make the assumption that the abyssal flow is very weak, and thus deduce what the thermocline flow is.

• ***Vorticity***:

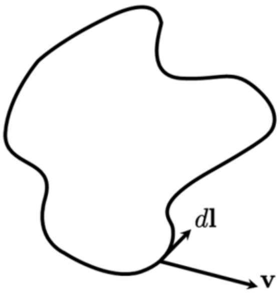
⇨ Usually, when we think of the vorticity of the fluid, we focus on the horizontal flow. The horizontal vorticity being a vector which points upwards. Vorticity can actually point in other directions. We can refer to horizontal component of the vorticity which describes the overturning flow.

The thermal wind balance expresses the horizontal component of the vorticity equation. It does not have anywhere near the same amount of liberty as the vertical component because it is locked into the strong hydrostatic balance so there is no direct development in this horizontal component of vorticity.

## GFD1.3: Circulation, Vorticity, and Potential Vorticity

### *1.3.a) Definitions of Circulation and Vorticity*

• ***Circulation***




⇨ The circulation over a closed contour is defined as the integral around the contour of the component of the flow parallel to the path:

$$C = \oint \mathbf{v}.dl$$

↳ We are interested in how to ***generate circulation***, i.e. to produce a tendency/time derivative of the circulation ($\frac{\partial C}{\partial t}$).

⇨ The time derivative of the circulation is: $$\frac{dC}{dt} = \oint \frac{d\mathbf{v}}{dt}.d\mathbf{l} + \frac{1}{2}\oint d(\mathbf{v}.\mathbf{v})$$

↳ The second term on the right-hand side is the integral of a perfect differential over a closed path and is therefore zero. So we get: $$\frac{dC}{dt} = \oint \frac{d\mathbf{v}}{dt}.d\mathbf{l}$$

⇨ In an inertial reference frame ($f = 0$, no rotation), if we can assume that the acceleration depends only on pressure gradients ($\frac{dv}{dt} = -\frac{1}{\rho}\nabla p$), then the rate of change of circulation is reduced to:

$$\frac{dC}{dt} = -\oint \frac{dp}{\rho}$$

↳ In order to generate circulation, we cannot have $\frac{dp}{\rho}$ equal to zero. So, if $\rho$ is a function of $p$, the time derivative of the circulation would be the integral of a constant along the contour and will cancel itself. We thus need $\rho$ to vary with respect to $p$ in order for the integral to be nonzero, i.e. $\rho$ not a function of $p$ – baroclinic flow.

↳ In a barotropic flow, there will not be any rate of change of circulation around a given boundary. Circulation can only be generated by baroclinic processes. This the ***Kelvin's theorem***.

• ***The vorticity***:

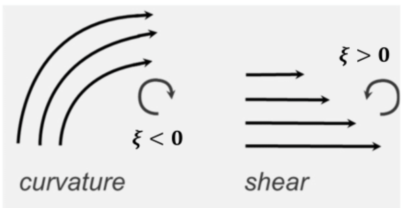
⇨ Using Stokes' theorem (or divergence theorem), the circulation of the flow can be converted through the area integral of the curl of the flow bounded by the circulation loop:

$$C = \oint \mathbf{v}.d\mathbf{l} = \iint_A (\nabla_\wedge \mathbf{v}) dA$$

with $\nabla_\wedge \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi = \nabla^2 \psi$

↳ The circulation is the area integral of the vorticity, which fits with the definition of vorticity being the curl of the flow. It is known as $\xi$, the ***relative vorticity*** and it is the Laplacian of the stream function ($\psi$).

*What does relative vorticity mean?*




▪ Imagine throwing a paddlewheel into the flow. Is it going to turn on its own axis or not? If it turns clockwise then the flow has negative vorticity.

▪ In a shear flow too. If you drop something in, it will spin on its own axis anti-clockwise in this example, i.e. with positive vorticity.

↳ Note that, in the rotating frame of the Earth, we get $C = \iint (\nabla \wedge \boldsymbol{v} + 2\Omega \sin\varphi)\, dA$, with $\xi_a = (\nabla \wedge \boldsymbol{v} + f)$ the ***absolute vorticity***. The circulation of a solid body in rotation is $2\pi\Omega r^2$, with $\Omega$ the rate of rotation of the Earth.

⇨ We showed that the circulation cannot change except through baroclinic processes, but the vorticity can. If there is convergence, the area will change size and shrink. A **barotropic flow** has to conserve its circulation, which means that the vorticity will be concentrated into a smaller area. So, the absolute value of the vorticity must locally increase. We will now see how vorticity can be generated through divergence (see **#GFD1.3b** and **#GFD1.3c**).

### *1.3.b) The vorticity equation*

We are going to derive the **vorticity equation**, analyse each term and see what it is useful for. We estimate the curl of the momentum equations, i.e. take the *x* derivative of the meridional momentum equation and subtract the *y* derivative of the zonal momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = \mathcal{F}\left[\frac{\partial \mathbf{h}}{\partial x}\right] + \tau_x - \mathcal{D}_x \quad (1)$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = \mathcal{F}\left[\frac{\partial \mathbf{h}}{\partial y}\right] + \tau_y - \mathcal{D}_y \quad (2)$$

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$$

↳ The horizontal pressure gradient terms are eliminated in the process (see details of the calculous on the following page) and this leads to the barotropic vorticity equation:

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left[f + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = \nabla_\wedge(\tau - \mathcal{D})$$

- The first term on the left-hand side it is the rate of change of ***relative vorticity***. It contains all the advection terms.
- The second term is the ***absolute vorticity*** (relative vorticity + planetary vorticity) multiplied by the ***divergence***. This is not a surprise as we mentioned that vorticity can be generated through divergence.
- There is a β ($\frac{\partial f}{\partial y}$) term, that can be brought in the first term to create the rate of change of the ***absolute vorticity***.
- On the right-hand side, we have the ***momentum forcing***: wind stress and frictional dissipation. Usually these terms only exist at the boundaries. This means that vorticity can only be generated by forcing only at the boundaries.

⇨ In vector form, the **barotropic vorticity equation** can be written:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right](f + \xi) = -(f + \xi)\nabla.\mathbf{v} + \nabla_\wedge(\tau - \mathcal{D})$$

↳ The substantial derivative of absolute vorticity ($\xi_a = f + \xi$) equals to **absolute vorticity** times the divergence plus the curl of the momentum forcing.

• For **non-divergent flow**, with no forcing, $\left[\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right](f + \xi) = 0$, the ***absolute vorticity*** is conserved following the motion.

• If we add some **momentum forcing**, $\nabla_\wedge(\tau - \mathcal{D})$, we add a **source of *absolute vorticity***.

↳ An example is the Sverdrup balance, in which advection of planetary vorticity is balanced by the wind stress forcing $\beta v = \nabla_\wedge \tau$ . This is the basis of large-scale ocean theory.

• Vorticity can be generated by divergence: $-(f + \xi)\nabla.\mathbf{v}$

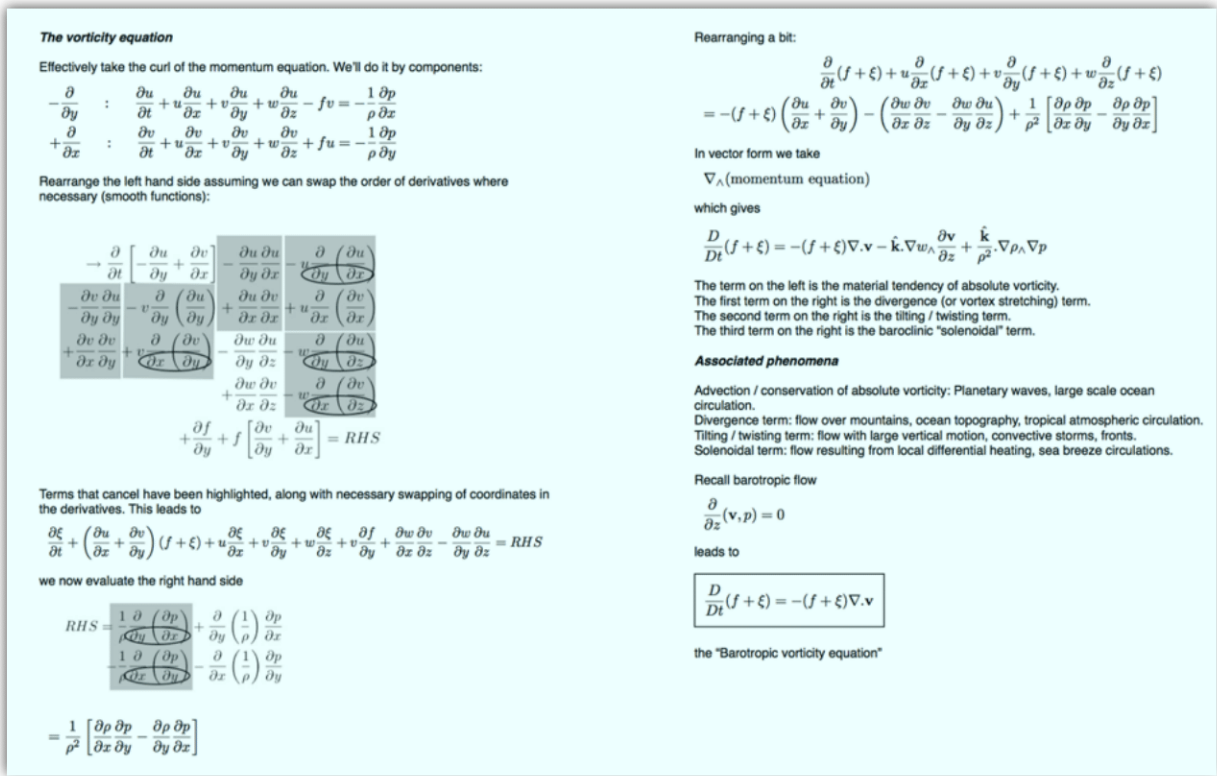
↳ Interestingly, divergence also changes the thickness of a layer, which gives us the link between changing layer thickness and changing vorticity. A convergent flow yields a thicker layer that will spin-up some more vorticity. This is called the vortex stretching effect. This term yields coupling between the layers.

• There is a term that has been cancelled because we used the ***Boussinesq approximation*** when we derived the shallow water equations. It is a ***baroclinic vorticity generation term***:

$$\left(\frac{1}{\rho_0}\frac{\partial p}{\partial x} \rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x}\right) \implies \frac{1}{\rho^2}J(\rho, p)$$

↳ It comes from the generation of circulation associated with the baroclinicity of the flow (see circulation theorem in **#GFD1.3a**). This term is important on smaller scales and not particularly important for very large-scale dynamics (see the following slide for a detailed derivation).

## *Details of the calculous for the vorticity equation*



***The vorticity equation***

Effectively take the curl of the momentum equation. We'll do it by components:

$$-\frac{\partial}{\partial y} \quad : \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$+\frac{\partial}{\partial x} \quad : \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Rearrange the left hand side assuming we can swap the order of derivatives where necessary (smooth functions):

$$\rightarrow \frac{\partial}{\partial t}\left[-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$
$$-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)$$
$$+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - w\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)$$
$$+\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - w\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)$$
$$+\frac{\partial f}{\partial y} + f\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] = RHS$$

Terms that cancel have been highlighted, along with necessary swapping of coordinates in the derivatives. This leads to

$$\frac{\partial \xi}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\xi) + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + w\frac{\partial \xi}{\partial z} + v\frac{\partial f}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} = RHS$$

we now evaluate the right hand side

$$RHS = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x}$$
$$-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial y}$$

$$= \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

Rearranging a bit:

$$\frac{\partial}{\partial t}(f+\xi) + u\frac{\partial}{\partial x}(f+\xi) + v\frac{\partial}{\partial y}(f+\xi) + w\frac{\partial}{\partial z}(f+\xi)$$
$$= -(f+\xi)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

In vector form we take

$$\nabla_\wedge(\text{momentum equation})$$

which gives

$$\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v} - \hat{\mathbf{k}}.\nabla w_\wedge\frac{\partial \mathbf{v}}{\partial z} + \frac{\hat{\mathbf{k}}}{\rho^2}.\nabla\rho_\wedge\nabla p$$

The term on the left is the material tendency of absolute vorticity.
The first term on the right is the divergence (or vortex stretching) term.
The second term on the right is the tilting / twisting term.
The third term on the right is the baroclinic "solenoidal" term.

***Associated phenomena***

Advection / conservation of absolute vorticity: Planetary waves, large scale ocean circulation.
Divergence term: flow over mountains, ocean topography, tropical atmospheric circulation.
Tilting / twisting term: flow with large vertical motion, convective storms, fronts.
Solenoidal term: flow resulting from local differential heating, sea breeze circulations.

Recall barotropic flow

$$\frac{\partial}{\partial z}(\mathbf{v}, p) = 0$$

leads to

$$\boxed{\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v}}$$

the "Barotropic vorticity equation"

Ertel potential vorticity conservation

### *1.3.c) Generation of vorticity by divergence*

⇨ The **continuity equation** can be written in the form of fluxes of mass or the flux of layer thickness determining the rate of change of layer thickness. It can be developed in terms of substantial rate of change of layer thickness ($D/Dt$):

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{Dh}{Dt} = -h\nabla.\mathbf{v}$$

↳ The substantial rate of change of layer thickness is given by $h$ times the divergence.

⇨ This is very similar to the vorticity equation (in the absence of momentum forcing, see **#GFD1.3b**).

$$\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\mathbf{v}$$

⇨ We can take advantage of this similarity. In the case of barotropic non-divergent flow (see **#GFD1.3b**), we do not have $-(f+\xi)\nabla.\mathbf{v}$ and there is a conservation of the ***absolute vorticity***.

↳ In order to include this vortex stretching term, we need to combine the vorticity and the continuity equations and eliminate the divergence. Isolated from any source or sink of potential vorticity forcing, we can write:

$$-\nabla.\mathbf{v} = \frac{1}{h}\frac{Dh}{Dt} = \frac{1}{(f+\xi)}\frac{D}{Dt}(f+\xi)$$

In the absence of forcing or dissipation

⇨ It leads to another conservation principle: $\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = 0$

- For <u>**non-divergent flow**</u>, the ***absolute vorticity*** was conserved following the motion.
- In a <u>divergent flow</u>, the absolute vorticity divided by the layer thickness is conserved on density layers. This is called the **potential vorticity** (**PV**) or the Ertel potential vorticity.

↳ It is another way of thinking about how divergence affects vorticity. It is a very compact way to express the dynamics.

- It can be generalized to more complicated flows. For example, PV is conserved in the Atmosphere on potential temperature surfaces.

*The steps of the calculous are*:

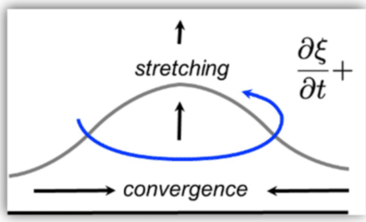
$$\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = (f+\xi)\frac{D}{Dt}\left(\frac{1}{h}\right) + \frac{1}{h}\frac{D}{Dt}(f+\xi) = -\frac{(f+\xi)}{h^2}\left(\frac{Dh}{Dt}\right) + \frac{1}{h}\frac{D}{Dt}(f+\xi) = 0$$

### *1.3.d) Potential vorticity conservation*

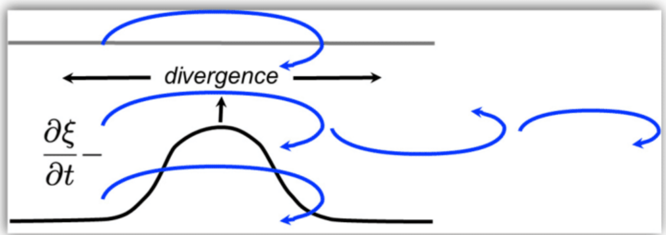
⇨ Let's think about a couple of cases in which changing layer thickness generates vorticity.

**1)** <u>**Low-level convergence**</u> expands the layer thickness ($h$), creating a cold lump between density surfaces. With $(f+\xi) > 0$, to conserve the potential vorticity, the relative vorticity increases ($f$ remains constant):

$$(f+\xi) > 0, \qquad \frac{f+\xi}{h} = cst, \qquad h \nearrow \implies \xi +$$




↳ If the flow was non-rotating before, in the northern hemisphere ($f > 0$), it will develop a positive relative vorticity and spin anti-clockwise around the cold dome. This is <u>***vortex stretching***</u>.




**2)** A flow encounters a **seamount bump** at the bottom of the ocean. The layer will get thinner and by the same argument, with $(f+\xi) > 0$ and a constant planetary vorticity ($f = cst$), there will be ***divergence*** and negative vorticity will be generated.

- If planetary vorticity is very important (it dominates the absolute vorticity) – say a rapidly rotating planet or very large-scales – then $f/h$ has to be conserved. One solution is simply to not change $h$ and the flow will follow the isobaths. This gives rise to rotating fluid systems, spinning around the bottom topography. These are called ***Taylor columns***.
- At large-scales, $f$ starts to vary in latitude with the flow. If $f$ varies, $h$ has to vary, or the relative vorticity will change and an oscillation in the flow can develop. This a ***Rossby wave*** (see **#GFD3**).

### *1.3.e) Conservation laws and potential quantities*

⇨ We overview conservation laws, namely mass conservation, conservation of absolute vorticity of a barotropic non-divergent flow, and **conservation of potential vorticity** for a barotropic divergent flow (see **#GFD1.3c**).

⇨ **Potential vorticity** is not vorticity. It does not even have the same units! It is called potential vorticity as if it had a label. You attach a **label** to a parcel of fluid and let it move away to different latitudes or different depths changing its relative vorticity as it goes. So, you could take it to the equator where there is no planetary vorticity. But as it has got this label, its **potential vorticity** **will not change**, it will be **conserved following the flow**. This means that if the parcel is brought back to its original location, its relative vorticity will recover its original value.

In the absence of forcing or dissipation


Consider a dynamics lecturer with an **English passport**. Basically, this means he drinks **tea**. But if he travels to France, he will drink red wine and coffee. If he goes to Mexico, he will drink Tequila and if he flies to Russia he will imbibe Vodka. But he will always carry his English passport, which means that back at his reference position in England, he will always enjoy drinking tea.

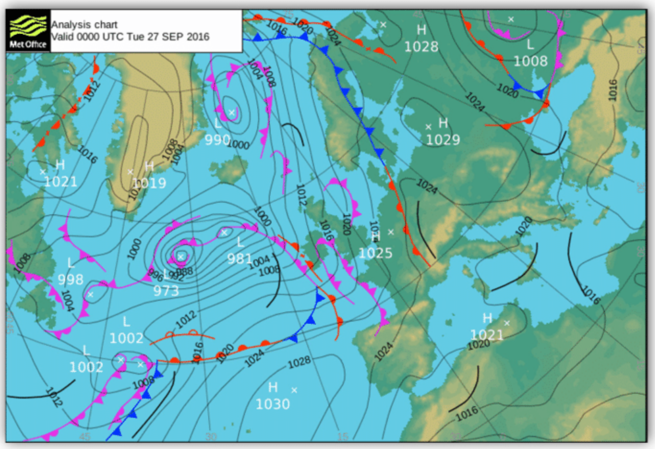
Similarly, in the atmosphere, **potential temperature** is the temperature at 1000 mb. If the parcel of air goes up (adiabatically) it will get colder, and when it comes down it will warm up again. It does not conserve its temperature, but it conserves its potential temperature. Brought back to 1000 mb, its temperature will be back to its original value.

# CHAPTER 2
# Quasi-Geostrophic Theory

# CHAPTER 2
# Quasi-Geostrophic Theory

**GFD2 Contents**

In this chapter, we will tackle **quasi-geostrophic theory**. On the weather map on the right, surface pressure lines (isobars) are shown with black lines. If the wind follows the isobars exactly, it is in geostrophic balance. The closer together the isobars, the stronger the wind. But in this equilibrium, this pattern is not going to be transported/displaced. The wind circulation will remain as it is and never change. On the one hand, geostrophic balance is a very good way to **describe the flow**, but on the other hand to make **weather forecasts** (predict changes in the flow) we need to include more terms in our equation system than only geostrophic balance. In this chapter, we will consider the closest thing we can get to geostrophic balance, i.e **small departures from geostrophic balance**. It is called **quasi-geostrophic** because it is almost geostrophic but not quite.

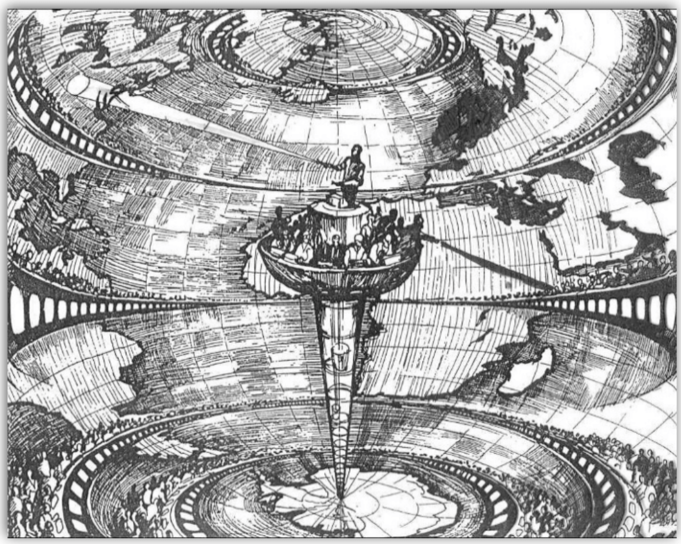



1) We start with an example of **steady departures from geostrophy**, with a flow that does not develop in time but in which we can describe the effects of non-linearities and drag (see **#GFD2.1**).

2) Then, we will discuss **ageostrophic flow** and the **importance of the divergent part of the flow**. We will introduce a new formulation of the conservation of the potential vorticity. We will start with the assumption that we are on an $f$**-plane** (see **#GFD2.2**) and then we will generalize to the situation where the planet has some **curvature** (see **#GFD2.3**).

3) We will finish by studying various **applications** of quasi-geostrophic theory (see **#GFD2.4**).

**Quasi-geostrophic theory** is very important because it was the basis of the first weather forecasts. It was the equation set used to predict the weather. The picture below refers to something called Richardson's dream. Lewis Fry Richardson (1881-1953) was one of the founders of the science of meteorology. Before the computer era, he had the idea that we could analyze the equations of motion to predict the weather. But they are so difficult to solve that you need lots of calculations. He dreamed about an amphitheater full of people making calculations with their pencil, paper, and their log tables, passing information to one another. He was ahead of his time, effectively imagining a massively parallel multi-core cluster. He anticipated the idea that we would solve the equations by some sort of multitude of calculations. And it is what we actually do nowadays, i.e. making weather predictions by discretizing (in space and time) and solving partial differential equations. And, of course, we do this on machines capable of performing very many calculations per second (super-calculators).



*"Richardson's dream"*

# GFD2.1: Steady departures from Geostrophy

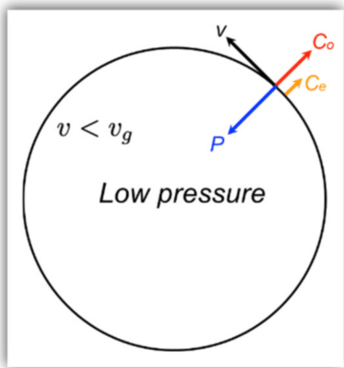
### *2.1.a) Gradient wind balance*

⇨ We start by considering small ***steady departures from geostrophy***. Let's recall the zonal shallow water momentum equation, with the flow tendency, the advection terms, the Coriolis force, and the pressure gradient force expressed through the gradient of the layer thickness (see **#GFD1.2ef**):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \boxed{- fv = -g\frac{\partial h}{\partial x}}$$

↳ The two terms on the right are geostrophic balance (outlined in green), in which pressure gradient force balances the Coriolis force.

⇨ Now, let's consider **time-independent** (steady) **flow around a circle**. In a simple way, the nonlinear terms represent the **local centrifugal force** associated with this circular motion. This is **gradient wind balance** (without the Coriolis force it is "cyclostrophic" balance).




• On the schematic on the right, the flow is going around a **low-pressure system**, a perfect cyclonic motion. There is centrifugal force associated with this circular motion ≡ something extra compared to geostrophy.

↳ The **pressure force** ($\vec{P} \sim g\,\frac{dh}{dr}$) pushes the flow towards the center of the low pressure. It is balanced partly by the **Coriolis force** ($\vec{Co}$), which yields an anti-clockwise flow. Since the flow is spinning around, there is also a **centrifugal force** ($\vec{Ce}$) associated with the curvature of the flow. Notably, **both Coriolis and centrifugal forces are fictitious**, associated with the choice of reference frame.

↳ In this example, we consider the balance between these two fictitious forces and the real pressure gradient force. It follows:

$$fv + \frac{v^2}{r} = g\frac{dh}{dr} = fv_g$$

Coriolis term ($fv$)+centrifugal ($\frac{v^2}{r}$) is equal to the pressure gradient ($g\,\frac{dh}{dr}$). The latter is positive for a **cyclone** because the pressure is low in the center and increases outwards along the radius. If the flow were in geostrophic balance, the pressure gradient would be balanced by $fv_g$.



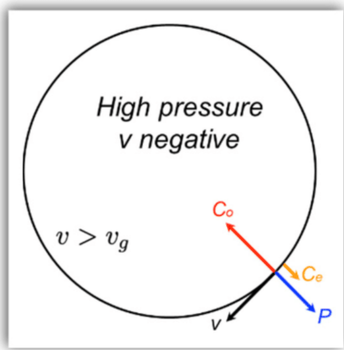
**Since these two forces sum-up** ($\vec{Co}$+$\vec{Ce}$) to compensate for the pressure gradient force ($\vec{P}$) and keep the flow parallel to the isobars, **the Coriolis force does not need to be as strong** as if it were in geostrophic balance. This means that **the flow does not have to be as fast** as the geostrophic velocity $v_g$. As a result, $v$ is slightly weaker than $v_g$. We can just eliminate the pressure gradient in the equation above and rewrite the equation in terms of the difference between $v$ and $v_g$. It follows:

$$v\left(1 + \frac{v}{fr}\right) = v_g$$

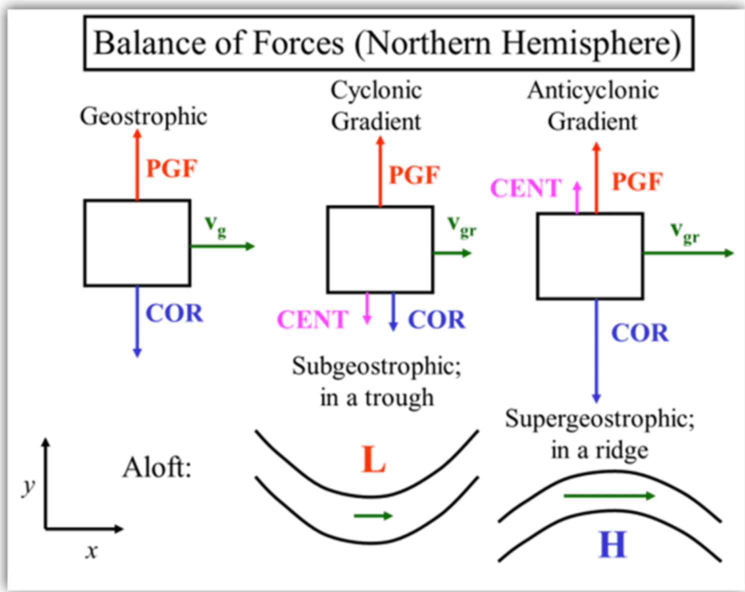
$R_0 = \frac{U}{fL} = \frac{v}{fr}$

↳ With $\frac{v}{fr}$ positive, $1 + \frac{v}{fr} > 1$, confirms that $v$ **is slightly weaker than its geostrophic counterpart** $v_g$.






• In the case of a **high-pressure system** (an anticyclone), the pressure is decreasing outwards and the flow is going the other way around – clockwise. There is now a balance between the sum of the centrifugal force and the pressure gradient force ($\vec{Ce}$+$\vec{P}$) and the Coriolis force ($\vec{Co}$). As a consequence, the Coriolis force needs to be stronger as if it were in geostrophic balance. So, $v$ is slightly stronger than if the flow were geostrophic ($v > v_g$).




*Credit: H.N. Shirer*

• We can group these two cases: $|v| = \dfrac{|v_g|}{1 \pm R_o}$, with $R_0$ the Rossby number (see **#GFD1.2a**).

↳ If the flow is close to geostrophic, $v \sim v_g$ because $R_0$ is small.

⇨ The full solution to this quadratic equation is: $v = -\dfrac{fr}{2} \pm \sqrt{\dfrac{f^2r^2}{4} + rg\dfrac{dh}{dr}}$

Cyclone limit

▪ For a **cyclone**, as the pressure gradient is positive (pressure is increasing outwards), everything under the square root is positive and we can have real solutions. There is no limit to the strength of the pressure gradient.

▪ In the case of an **anticyclone**, the pressure gradient is negative. As a consequence, if the pressure gradient gets too strong, it leads to a square root of a negative quantity. In that case, the equation does not have real solutions. So, in the case of an anticyclone, there is a limit in the strength of the pressure gradient beyond which the equation does not have solutions: $\left|\dfrac{dh}{dr}\right| \leq \dfrac{f^2 r}{4g}$

Asymmetry

↳ There is a limit on the strength of the pressure gradient associated with anticyclones compared to cyclones. This explains the **asymmetry between high- and low-pressure systems** we observed in **#GFD.intro**. On the previous weather map, there is a very intense cyclone south of Greenland, associated with very strong pressure gradients, while the anticyclone off the coast of Spain resembles a flat pattern and is associated with rather weak pressure gradients. This asymmetry is the result of a steady ageostrophic term, called **gradient wind balance**.

Wong way and Development

↳ In the full solution, there is a $\pm$, which means that in theory, the flow can go the wrong way around a cyclone. So, it is **mathematically** possible to find an equilibrium in which there is a **clockwise flow around a low-pressure system**, i.e. a solution for which pressure gradient and Coriolis forces are both directed towards the center of the pressure system, balanced by the centrifugal force.

**In reality**, it is not really possible. We never observe it, except maybe on very small scales maybe. Because you have to consider how the **equilibrium develops**. Starting from rest, a particle will accelerate towards low-pressure centers, while the Coriolis effect will deviate its trajectory to its right. The particle will end-up be turning around the low pressure anti-clockwise. So, the natural way in which these systems come into being favors the normal solutions rather than strange anomalous solutions.

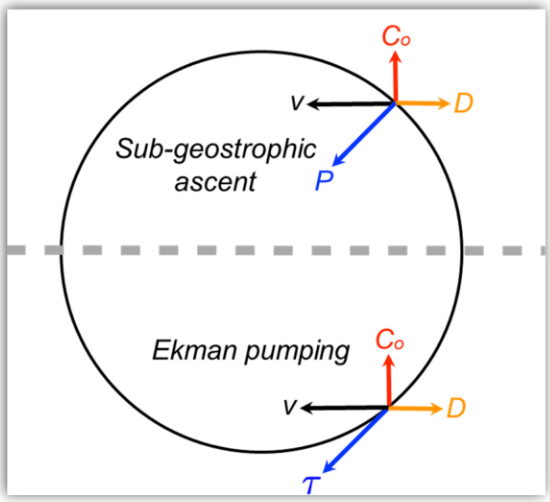
Sub-geostrophic flow

### *2.1.b) Boundary friction*

⇨ Friction is another way to modify geostrophy in a steady flow, in which we add some drag to the system.

• Consider a **cyclone** in the **Atmosphere**. The **pressure gradient force** ($\vec{P}$) is directed toward the center (see illustration below). In addition to the **Coriolis force** ($\overrightarrow{Co}$), we add a **drag force** ($\vec{D}$) that points in the opposite direction to the flow, so this **drag and Coriolis forces balance the pressure gradient force**. For that to be possible, the flow has to **converge** into the cyclone. This is sub-geostrophic flow. If the wind converges towards the center of a cyclone, the mass will be gradually accumulated and the cyclone will decay as the low pressure fills at the surface.

But before this happens, the low-level flow (which experiences this drag and converges) will naturally give rise to upward motion by **conservation of mass**. We observe **ascent** in low-pressure/cyclone centers. This explains why a depression is always associated with cloudy weather (upward motion=condensation=clouds). Conversely, it is sunny in an anticyclone, associated with super-geostrophic (descending) flow. It is the opposite effect, the wind is blowing outwards and the flow experiences downward motion (=evaporation=clear skies).




Ekman convergence

• Let's focus on the **Ocean** and consider exactly the same diagram.
- Instead of being driven by a pressure gradient force, the flow is driven by surface wind-stress ($\vec{\tau}$). It is the stress which is acting on the upper surface layer of water.
- The same surface ocean current vector $\boldsymbol{v}$, directed at 45° from the wind forcing.
- It is being pulled back by friction with the water below, i.e. the drag ($\vec{D}$).
- The Coriolis force ($\overrightarrow{Co}$) is as usual perpendicular to the flow.

↳ We have the same balance but this is now **Ekman convergence**: wind-stress driving the flow, balanced by the Coriolis force and friction. It will be convergent and that will lead to downward motion in order to conserve mass. This is **Ekman pumping**. As each layer exerts a stress on the layer below, the movement of the upper layer will produce a stress on the next layer, creating the Ekman spiral.

In the case of the atmosphere, it is the slowing down of a flow that has already been established, while in the case of the ocean this is how the flow is forced.

⇨ But now we need to move away from these anecdotal cases and put together a system with advection and time dependence that is almost, but not quite geostrophic.

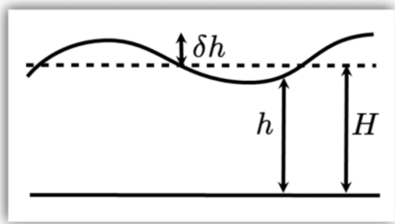
*We do this essentially by separating the flow into a geostrophically balanced, nondivergent part, and the ageostrophic plus the divergent parts as a small perturbation. This small perturbation allows prognostic equations that lead to the evolution of the flow.*

# GFD2.2: Quasi-Geostrophic Theory I – the $f$-plane

### *2.2.a) Ageostrophic perturbations*

⇨ The goal is to build a theory that is close to geostrophy but not quite geostrophic.

↳ We separate the flow in its geostrophic part and ageostrophic part, and we assume that the ageostrophic part is a small perturbation. We start with the shallow water momentum equations in a single layer (see **#GFD1.2f**):




$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0$$
$$\frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0$$

With $\frac{D}{Dt}$ the substantial derivative operator ($\frac{\partial}{\partial t}$+advection): $\left\{ \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right\}$

⇨ What would happen if we just assume the **flow is geostrophic** ($\mathbf{v_g}$) and we substituted it into the momentum equations**?**

$$u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \quad v_g = \frac{g}{f}\frac{\partial h}{\partial x}$$

↳ We put it into the definition of the **substantial derivative**, using the geostrophic flow for the advection terms:

$$\frac{D}{Dt} \rightarrow \frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$$

✋ If we also put the geostrophic velocity ($\boldsymbol{v_g}$) into the Coriolis part instead of using the full flow ($\boldsymbol{v}$), this gives:

$$\frac{D_g}{Dt}u_g - fv_g + g\frac{\partial h}{\partial x} = 0$$

↳ The two terms on the right cancel each other and the geostrophic flow is non-divergent. This leads to a null tendency, which was expected as geostrophy is a balance. So, **we went one step too far**.

⇨ Instead, we make sure that the **equation remains linear in terms of the ageostrophic part** of the flow ($\mathbf{v}_{ag} = \mathbf{v} - \mathbf{v}_g$), i.e. $\mathbf{v}_{ag}$ contribution in the nonlinear terms (squared terms) will be small and we are going to neglected them, making the equations easier to solve. So, **the full flow is used in the linear Coriolis terms** ($\mathbf{v}$) and we advect with the geostrophic flow ($\mathbf{v}_g$). This is consistent with the idea that the ageostrophic part of the flow is small.



### *2.2.b) Quasi-geostrophic $f$-plane vorticity equation*

⇨ Using geostrophic flow in the non-linear and tendency terms, and keeping the full flow in the Coriolis terms yields **quasi-geostrophic momentum equations**:

$$\frac{D_g}{Dt}u_g - fv + g\frac{\partial h}{\partial x} = 0 \quad (1)$$
$$\frac{D_g}{Dt}v_g + fu + g\frac{\partial h}{\partial y} = 0 \quad (2)$$

$f$-plane
$f = cst$

↳ As in **#GFD1.3b**, the **vorticity equation** is derived by cross differentiation: $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$

⇨ This process eliminates pressure gradient terms, and we get:

$$\frac{\partial}{\partial t}\xi_g + u_g\frac{\partial}{\partial x}\xi_g + v_g\frac{\partial}{\partial y}\xi_g + \xi_g\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = 0$$

*f*-plane ξ equation

⇨ We obtain an equation for the development of the **geostrophic vorticity**. Because **we have assumed that we are on an *f*-plane** ($f$=*cst*, i.e. $\frac{df}{dy}=0$), $h$ is a stream function for $u_g$ and $v_g$ and **the divergence of the geostrophic flow** ($\nabla.\boldsymbol{v}_g=0$ on an $f$-plane) **remains zero**. The equation simplifies:

$$\frac{D_g}{Dt}(f+\xi_g)=-f\nabla.\mathbf{v}$$

↳ The geostrophic tendency of the **absolute geostrophic vorticity** is given by the **divergence of the ageostrophic flow**.

↳ ☝ If ***f* varies with latitude**, there is a divergent part to the geostrophic flow (see **#GFD2.3**).

### *2.2.c) Continuity equation*

Here is the continuity equation in the form of the flux of layer thickness $h$ (see **#GFD1.2d**):

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(uh)+\frac{\partial}{\partial y}(vh)=0$$

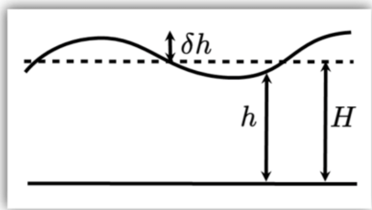
It can also be written as the substantial derivative of the layer thickness (see **#GFD1.3c**):

$$\frac{Dh}{Dt}+h\nabla.\mathbf{v}=0$$

Linearization of ***h***

↳ As for the momentum equations (see **#GFD2.2b**), we replace the **substantial derivative** ($D/Dt$) by the **geostrophic operator** ($D_g/Dt$), and expand it. The continuity equation is then written:

$$\frac{\partial h}{\partial t}+\mathbf{v}_g.\nabla h+h\nabla.\mathbf{v}=0$$




☝ $h\nabla.\mathbf{v}$ is not a linear term, because $h$ depends on $\mathbf{v}$. So, for consistency **we must make an approximation on *h***:

$$h=H+\delta h,\qquad \delta h\ll H$$

↳ We can now linearize the ageostrophic flow $h\nabla.\mathbf{v}$ by assuming (for this term only) it is approximated by $H$ (a constant) times the divergence of the ageostrophic flow ( $H\nabla.\mathbf{v}$ ). This is the equivalent to the approximation: $\mathbf{v}_g h\approx\mathbf{v}_g\delta h+\mathbf{v}H$.

*f*-plane continuity equation

↳ We obtain the following continuity equation: $\frac{\partial h}{\partial t}+\mathbf{v}_g.\nabla h+H\nabla.\mathbf{v}=0$

⇨ It can be written as the geostrophic substantial derivative of $\delta h$:

$$\frac{\partial}{\partial t}\delta h+\mathbf{v}_g.\nabla\delta h+H\nabla.\mathbf{v}=0\quad\text{or}\quad\frac{D_g}{Dt}\delta h+H\nabla.\mathbf{v}=0$$

☝ To impose some linearity in the ageostrophic contributions, we had to make a strong approximation to the mean stratification: **the mean stratification, represented by the layer thickness ($H$), cannot vary in the horizontal**.

⇨ Finally, we can rewrite the continuity equation as: $\frac{f}{H}\frac{D_g}{Dt}\delta h=-f\nabla.\mathbf{v}$

$f$-plane QG PV conservation

### *2.2.d) Quasi-geostrophic potential vorticity*

⇨ As in **#GFD1.3c**, we combine the vorticity (see **#GFD2.2b**) and the continuity (see **#GFD2.2d**) equations and eliminate the divergence, as follows:

$$\frac{D_g}{Dt}\left(f\frac{\delta h}{H}\right) = \frac{D_g}{Dt}(f+\xi_g)$$

↳ We obtain a conservation principle: $\frac{D_g}{Dt}\left\{f+\xi_g - f\frac{\delta h}{H}\right\} = 0$

⇨ This is the conservation law for **$f$-plane quasi-geostrophic potential vorticity**:

$$\frac{D_g}{Dt}q = 0, \quad q = f + \xi_g - f\frac{\delta h}{H}$$

**$q$ is conserved following the motion**

In the absence of forcing or dissipation

⇨ It is not quite the same as the **Ertel potential vorticity** (see **#GFD1.3c**) because there is a linearization of the stratification. It does not even have the same units as the potential vorticity, but the **two quantities can be related**:

$$\frac{f+\xi}{h} = (f+\xi)\left(\frac{1}{H+\delta h}\right) = \left(\frac{f+\xi}{H}\right)\left(\frac{H}{H+\delta h}\right)$$

$$= \left(\frac{f+\xi}{H}\right)\left(\frac{H+\delta h-\delta h}{H+\delta h}\right) = \left(\frac{f+\xi}{H}\right)\left(1-\frac{\delta h}{H+\delta h}\right)$$

$$\frac{f+\xi}{h} \approx \left(\frac{f+\xi}{H}\right)\left(1-\frac{\delta h}{H}\right)$$

↳ As $H$ is constant, with the **linearization of the stratification**, we obtain the conservation of the following quantity:

$$q = f + \xi - f\frac{\delta h}{H} - \xi\frac{\delta h}{H}$$

⇨ Then using scaling arguments, i.e. the **Rossby number is small**, the term involving relative vorticity is small compared to $f$.

$$Ro \ll 1, \quad \frac{U}{fL} \ll 1 \Rightarrow f \gg \xi, \quad \xi = \xi_g$$

↳ We can thus neglect the term on the right and recover the quasi-geostrophic potential vorticity formulation.

⇨ So, this **linearization of the layer thickness** is a surprising consequence of our **insistence** that the flow remains **close to geostrophic**. In a vertically continuous framework it means that the stratification is uniform in the horizontal (see **#GFD2.2b**).

# GFD2.3: Quasi-Geostrophic Theory II – Expansion in small Rossby number

### *2.3.a) Adding curvature to the Earth*

**#GFD2.2** was pretty straightforward because we assumed that $f$ was constant ($f$-plane). But many important dynamical phenomena depend on the **variation of $f$ with latitude** (Rossby waves, for example, see **#GFD3**).

↳ On an $f$-plane, the geostrophic flow is strictly non-divergent, while on a planet with some curvature, the geostrophic stream function that contains $f$ is not a proper stream function. It has departures associated with the **divergent part of the geostrophic flow**. So, allowing $f$ to vary will complicate the theory as we have to deal with the divergent part of the geostrophic flow as well as the ageostrophic flow.

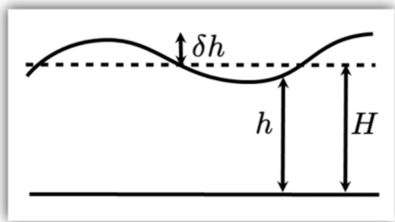
⇨ To proceed, we will derive the **quasi-geostrophic equation set** more formally than in **#GFD2.2**. We will do a **formal expansion** of these perturbations about a small parameter. We will naturally choose the **Rossby number** (see **#GFD1.2a**) for this small parameter.

### *2.3.b) Derivation of the quasi-geostrophic shallow-water momentum equations*

• We recall the full 1-layer shallow water (see **#GFD1.2f**) momentum and continuity (in its divergence form, see **#GFD1.3c**, **#GFD2.2c**) equations using a <u>vector notation</u>:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} + f\hat{\mathbf{k}}_\wedge\mathbf{v} + g\nabla h = 0$$

$$\frac{\partial h}{\partial t} + \mathbf{v}.\nabla h + h\nabla.\mathbf{v} = 0$$




Adimensionalization

• We now **<u>non-dimensionalize</u>** these equations.

- We use typical scaling values of length ($L$), speed ($U$), and time ($T$), to obtain non-dimensional variables noted with primes:

$$x' = x/L, \quad u' = u/U, \quad t' = t/T$$

- The layer thickness $h$ can be written as $h = H + \delta h$. We non-dimensionalize the variations of the layer thickness ($\delta h$) by $\Delta h$ a quantity typical of variations in the layer thickness ($\Delta h$, ✋ not $H$), as follows: $\eta' = \delta h/\Delta h$

↳ We substitute these non-dimensional variables into the shallow water equations, leading to:

$$\frac{U}{T}\frac{\partial \mathbf{v}'}{\partial t'} + \frac{U^2}{L}\mathbf{v}'.\nabla\mathbf{v}' + Uf\hat{\mathbf{k}}_\wedge\mathbf{v}' + g\frac{\Delta h}{L}\nabla\eta' = 0 \quad (1)$$

$$\frac{\Delta h}{T}\frac{\partial \eta'}{\partial t'} + \frac{U}{L}\Delta h\mathbf{v}'.\nabla\eta' + \frac{U}{L}(H + \Delta h\eta')\nabla.\mathbf{v}' = 0 \quad (2)$$

⇨ We obtain (messy) equations with scaling values in front of each term, in which the non-dimensional terms with prime ($\mathbf{v}'$ and $\eta'$) are of **<u>order 1</u>**.

Relation between scalings

• So far, we have not made any assumptions or approximations. We now introduce the **quasi-geostrophic assumption** by **requiring that the <u>relationship between the basic scalings</u>** ($L$, $U$, $T$, and $\Delta h$) **conforms to geostrophic balance**, i.e. $f\mathbf{v} \sim g\nabla h$. In terms of typical scalings (with $f_0$ the value of $f$ at a reference latitude), it follows that:

$$Uf_0 \sim g\frac{\Delta h}{L}$$

↳ We obtain an expression relating the value $\Delta h$ to the other scaling parameters:

$$\Delta h = \frac{Uf_0L}{g} \quad (3)$$

• If we now define the **Rossby number** (see **#GFD1.2a**) and the **temporal Rossby number** and acknowledge that they will be small in the quasi-geostrophic approximation (see **#GFD2.3d** and **#GFD2.3f**). Note that if $U = L/T$, these two parameters are the same.

$$\epsilon = \frac{U}{f_0 L} \quad (4) \qquad \epsilon_T = \frac{1}{f_0 T} \quad (5)$$

⇨ For simplicity, <u>we now remove the primes</u> in the equations and rewrite them using our new scaling parameters ($\varepsilon$ and $\varepsilon_T$).

↳ For the momentum equation, we divide $(1)$ by $f_0 U$, replace $\Delta h$ with $(3)$, and then use $(4)$ and $(5)$. This leads to:

$$\epsilon_T \frac{\partial \mathbf{v}}{\partial t} + \epsilon \mathbf{v}.\nabla \mathbf{v} + \frac{f}{f_0} \hat{\mathbf{k}}_\wedge \mathbf{v} + \nabla \eta = 0$$

↳ **The last two terms constitute a non-dimensional form of geostrophic balance. The advection and development terms have epsilon in front ($v$, $u$, and $\eta$ are of order 1).**

↳ It is worth mentioning that we **only made the hypothesis that the scales of the motion conform to geostrophic balance**. We just rewrote the equations using the $\varepsilon$ and $\varepsilon_T$ scaling parameters, we but have not yet assumed that these parameters are small. This will be done in **#GFD2.3f**.

### *2.3.c) Quasi-geostrophic continuity equation*

⇨ For the continuity equation, we multiply $(2)$ by $L/UH$, replace $\Delta h$ by $(3)$, and then use $(4)$ and $(5)$. It follows:

$$\epsilon_T \left(\frac{L^2 f_0^2}{gH}\right) \frac{\partial \eta}{\partial t} + \epsilon \left(\frac{L^2 f_0^2}{gH}\right) (\mathbf{v}.\nabla \eta + \eta \nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

Burger number

↳ The non-dimensional constant $\frac{L^2 f_0^2}{gH}$ that appears in brackets is the inverse of the Burger number ($Bu^{-1}$, see **#GFD1.2a**).

- *Bu* of order 1 means that Coriolis term and gravity/buoyancy effects are comparable or that vorticity advection and vortex stretching are equally important (see **#GFD5.5a**).
- *Bu* of order 1 means that we are dealing with typical synoptic systems, which can be amenable to quasi-geostrophic analysis.
- It is also associated with the length scale ($L$), such that $Bu^{-1} = L^2/L_R^2$. ($L_R$ is the Rossby Radius, see **#GFD1.2a**).

↳ For simplicity, we call it $\boldsymbol{F}$ in the following.

⇨ We use $F$ in the continuity equation and it writes:

$$F = \frac{L^2 f_0^2}{gH}$$

$$\epsilon_T F \frac{\partial \eta}{\partial t} + \epsilon F (\mathbf{v}.\nabla \eta + \eta \nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

Again, we **have not made any further approximations than the scales of movement conform to geostrophic balance** (✋).

↳ But we can already see from these two equations that to **<u>zero-order</u>** in our Rossby number parameters, the flow is geostrophic and non-divergent and that <u>first-order terms</u> concern advection divergence and time development.

#### 2.3.d) The assumptions of quasi-geostrophic theory

⇨ Before doing a formal expansion in the Rossby number (see **#GFD2.3f**), we will set out our assumptions one by one:

• ***Assumption 1***: the Rossby number is small, i.e. close to geostrophy: $\varepsilon \ll 1$ with $\epsilon = \frac{U}{f_0 L}$

• ***Assumption 2***: the temporal Rossby number is also small. We consider that $\epsilon_T = \epsilon$

↳ This means that scaling for velocity ($U$) is consistent with our scaling for length ($L$) and time ($T$), i.e. the velocity is just the flow velocity (as opposed to wave velocity which could go much faster). This results in ***filtering the very fast surface gravity waves*** (ex: tides).

• ***Assumption 3***: Buoyancy/gravity-stratification effects are as important as the Coriolis effect, i.e. *Bu* is of order 1, and length scale ($\boldsymbol{L}$) is close to the Rossby radius. This means that the coefficient $\boldsymbol{F}$ in the momentum equation (see **#GFD2.3c**) is of order 1.

• A consequence of assumption 3 and assumption 1 is that departures ($\delta h$) from standard layer thickness (*H*) are small.

↳ This is the linearization of the continuity equation and of the quasi-geostrophic potential vorticity. *NB: In a continuously stratified case, this is equivalent to saying that Brunt Vaisala frequency squared ($N^2$) varies in the vertical but not in the horizontal.*

• ***Assumption 4***: Scales of motion are small compared to the radius of the Earth: $\frac{L}{r_e} \ll 1$

↳ In fact, we assume that $\frac{L}{r_e} = \epsilon$ is the same ε as the Rossby number. We keep only one small parameter ε.

✋ NB: Assumptions 3 (about the stratification) and 4 (about the scale compared to the size of the planet) have nothing to do with geostrophy. They are not the result of our intent to derive a system almost but not quite geostrophic. But they are necessary for our expansion to be self-consistent.

#### 2.3.e) The beta effect

⇨ **Assumption 4** (length scales of the flow are small compared to $r_e$) indicates that the variation in $f$ is non-zero but small.

⇨ A Taylor expansion of $f = 2\Omega \sin\phi$ about a reference latitude ($\phi_0$) gives:

$$f = f_0 + \left.\frac{df}{dy}\right|_0 y + \left.\frac{d^2 f}{dy^2}\right|_0 \frac{y^2}{2} + \ldots = 2\Omega \sin\phi_0 + \frac{y' L}{r_e} 2\Omega \cos\phi_0 + \ldots = f_0 + \beta_0 y + \ldots$$

↳ It can be written **non-dimensionally** ($y = y'L$, see **#GFD2.3a**): $\frac{f}{f_0} = 1 + \frac{2\Omega cos\phi_0}{2\Omega sin\phi_0}\frac{L}{r_e} y' + \cdots$

$$= 1 + cot\phi_0 \frac{L}{r_e} y' + \cdots$$

⇨ At first order, with $\beta' = cot\phi_0$ and $\frac{L}{r_e} = \frac{U}{f_0 L} = \varepsilon$, it follows that: $\frac{f}{f_0} = 1 + \varepsilon\beta' y'$

↳ It represents the variation with latitude of the Coriolis parameter $f$, with a small parameter ε, in front of the $\beta$ term. We introduced the $\beta$-plane, i.e. the function for $\boldsymbol{f}$ in $x - y$ space is linear and describes a plane.

- We cannot get too close to the equator where $cot\phi_0 \longrightarrow \infty$ .
  It is thus an **extra-tropical beta approximation**.
- If $f = f_0$, it is an ***f-plane*** (as in **#GFD2.2**).
- $\beta$-plane is only in functional space, not in physical space.

$\beta$-plane

⇨ In the following, we are going to eliminate the prime in the notation (as in **#GFD2.3c**).

### *2.3.f) The expansion*

⇨ Let's do the expansion. Using assumption in **#GFD2.3d**, we rewrite the momentum (**#GFD2.3b**) and continuity (**#GFD2.3c**) equations as **$\beta$-plane** (**#GFD2.3e**) non-dimensional equations, in which some terms are multiplied by ε and some terms are not:

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon\mathbf{v}.\nabla\mathbf{v} + (1+\epsilon\beta y)\hat{\mathbf{k}}_\wedge\mathbf{v} + \nabla\eta = 0$$

$$\epsilon F\frac{\partial \eta}{\partial t} + \epsilon F(\mathbf{v}.\nabla\eta + \eta\nabla.\mathbf{v}) + \nabla.\mathbf{v} = 0$$

We expand the 3 variables ($u$, $v$ and $\eta$, the departure of the layer thickness from the standard value) in increasing powers of $\varepsilon$ (a zero-order part ($\varepsilon^0$)+ $\varepsilon^1$ × a first-order part + $\varepsilon^2$ × a second-order part, etc…):

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + ...$$

$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + ...$$

We then substitute them into the equations. We will ***sort the terms in increasing order of epsilon***. We will focus here on zero-order terms and then on first-order terms (see **#GFD2.3g**).

**For example, $v_0$ is a zero-order term in the last term of the continuity equation, while it is of first-order in the second term of the momentum equation. Likewise, $\varepsilon v_1$ is a first-order term in the last term of the continuity equation, while it is of second-order in the second term of the momentum equation.**

Zero-order

**Zero-order**: All the terms without any $\varepsilon$ in front. At zero-order, the momentum equation yields geostrophic balance, while the continuity equation informs us that the flow is non-divergent:

$$\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_0 = 0 \quad (1) \qquad \nabla.\mathbf{v}_0 = 0 \quad (2)$$

At zero-order, there is no development. The geostrophic non-divergent flow can only change with time if we include some first order (divergent) terms. The continuity equation is the equivalent of the momentum equations as $curl(1)$ gives $(2)$.

***$\eta_0$* acts as a stream function for the zero order** (non-divergent) flow $(u_0, v_0)$:

$$v_0 = \frac{\partial\eta_0}{\partial x}, \quad u_0 = -\frac{\partial\eta_0}{\partial y}$$

It does not represent the geostrophic flow, it represents the **part of the geostrophic flow** you would have if *f* were constant.

### *2.3.g) First order in $\varepsilon$*

⇨ First-order is what is left over when you select terms that have just $\varepsilon$ in front of them (no second-order or higher-order terms). It follows that:

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla\mathbf{v}_0 + \hat{\mathbf{k}}_\wedge\mathbf{v}_1 + \beta y\hat{\mathbf{k}}_\wedge\mathbf{v}_0 + \nabla\eta_1 = 0 \quad (1)$$

$$F\frac{\partial\eta_0}{\partial t} + F(\mathbf{v}_0.\nabla\eta_0 + \eta_0\nabla.\mathbf{v}_0) + \nabla.\mathbf{v}_1 = 0 \quad (2)$$

Continuity

⇨ In the continuity equation $(2)$, the ***second term is zero*** because the zero-order flow ($\mathbf{v_0}$) is perpendicular to the gradient of the stream function ($\nabla\eta_0$). And $\mathbf{v_0}$ is non-divergent, so:

$$F\frac{\partial\eta_0}{\partial t} = -\nabla.\mathbf{v}_1$$

The rate of change of the zero-order layer thickness comes from the divergence of the first order flow.

Vorticity

⇨ We form the first-order vorticity equation by taking the curl of the momentum equations (1) $\frac{\partial}{\partial x}(y - equation) - \frac{\partial}{\partial y}(x - equation)$, as in **#GFD1.3b** and **#GFD2.2b**. It provides an equation for the **vorticity** of the flow ($\xi$, curl of the velocity):

$$\frac{\partial \xi_0}{\partial t} + \xi_0 \nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla\xi_0 + \nabla.\mathbf{v}_1 + \beta y \nabla.\mathbf{v}_0 + \beta v_0 = 0$$

↳ The ***second and fifth terms are zero*** because the zero-order flow ($\mathbf{v_0}$) is non-divergent.

β-plane QG PV conservation

⇨ We then combine the vorticity equation with the continuity equation to get rid of the first-order divergence ($\nabla.\mathbf{v}_1$). This provides a conservation principle:

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla\xi_0 + \beta v_0 = -\nabla.\mathbf{v}_1 = F\frac{\partial \eta_0}{\partial t}$$

↳ Then taking into consideration that $\frac{\partial}{\partial t}(\beta y) = 0, \quad \mathbf{v}_0.\nabla(\beta y) = \beta v_0, \quad \mathbf{v}_0.\nabla\eta_0 = 0$ and using $\frac{\partial}{\partial t} + \mathbf{v}_0\nabla$ as the substantial derivative, we can factorize the equation, so that:

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = F\left[\frac{\partial \eta_0}{\partial t} + \mathbf{v}_0.\nabla\eta_0\right]$$

In the absence of forcing or dissipation

↳ It yields the conservation (following the flow) of the (non-dimentionalized) quasi-geostrophic potential vorticity:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_0.\nabla\right][\beta y + \xi_0 - F\eta_0] = 0$$

$F = \frac{L^2 f_0^2}{gH}$

#### *2.3.h) Quasi-geostrophic potential vorticity on a β-plane*

$$\frac{D}{Dt}[\beta y + \xi_0 - F\eta_0] = 0$$

⇨ If we now express the **zero-order (non-divergent) flow** ($\mathbf{v}_0$) in the advection terms **in terms of the stream function** $\eta_0$, so that $v_0 = \frac{\partial \eta_0}{\partial x}$ and $u_0 = -\frac{\partial \eta_0}{\partial y}$ (see **#GFD2.3f**), the substantial derivative of the potential quasi-geostrophic vorticity ($q$) is written:

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \boldsymbol{v}_0.\nabla q = \frac{\partial q}{\partial t} + \frac{\partial \eta_0}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial q}{\partial x}$$

⇨ Recalling that the vorticity is the *Laplacian* of the stream function (see **#GFD1.3a**), we can write the prognostic equation in terms of *one variable only*, the stream function ($\eta_0$):

$$\left[\frac{\partial}{\partial t} + \frac{\partial \eta_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial}{\partial x}\right][\beta y + \nabla^2\eta_0 - F\eta_0] = 0$$

Jacobian notation

⇨ This leads to an expression of conservation of (non-dimentionalized) quasi-geostrophic potential vorticity $q$, which can be written like this:

$$\frac{\partial q}{\partial t} + J(\eta_0, q) = 0 \quad \text{with} \quad q = \beta y + \nabla^2\eta_0 - F\eta_0$$

$F = \frac{L^2 f_0^2}{gH}$

↳ $J$ is the Jacobian, i.e. a compact way of expressing advection, when you have a non-divergent flow, in terms of the stream function and the quantity being advected.

⇨ What we learned from this is that we have just *one variable in this system*. For the complete shallow water equations, we had three variables ($u$, $v$, and $h$, see **#GFD1.3d**). For the quasi-geostrophic theory, we can express everything in terms of $\eta_0$: **one equation - one variable**. This is rather useful to perform weather prediction.

⇨ This is all non-dimensional, so we now put the physical values back in (i.e. the opposite of non-dimensionalizing the equations, see **#GFD2.3b**) and dimensionalize the equation. This leads to the dimensional quasi-geostrophic potential vorticity:

$$q = \beta y + \nabla^2 \psi - \frac{f_0}{H}\delta h$$

↳ i.e. $\beta$-term + relative vorticity + vortex stretching term.

⇨ If we define the quasi-geostrophic stream function as: $\psi = \dfrac{g}{f_0}\delta h$, it follows that:

The conservation of **quasi-geostrophic potential vorticity $q$ on a $\beta$-plane**

$$q = \beta y + \nabla^2 \psi - \left(\frac{f_0^2}{gH}\right)\psi \quad \text{or} \quad q = \beta y + \nabla^2 \psi - \frac{1}{L_R^2}\psi$$

with the **conservation with the flow** as before: $$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

In the absence of forcing or dissipation

✋ As $f_0$ remains constant, it ***can be included in the definition of q***. It will not change the conservation principle.

✋ The **Rossby radius** (see **#GFD1.2a**) is the length scale on which ***relative vorticity*** and ***vortex stretching*** make equal contributions to ***potential vorticity*** (see **#GFD3.4c** and **#GFD5.5b**)

### *2.3.i) Continuously stratified fluid*

⇨ Here, we provide the **quasi-geostrophic potential vorticity conservation principle** for more **realistic fluids**, with continuous (horizontal and vertical) variations of density.

↳ Up until now (**#GFD2.3a-h**), we have worked with discrete shallow water layers, each of which being homogeneous (constant density). The extension to **continuous stratification** requires that we abandon this formulation and reintroduce a vertical coordinate (see **details** on the next pages).

⇨ The **equation of conservation of quasi-geostrophic potential** vorticity remains the same:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

↳ It is the **definition of $q$ that changes**:

⇨ In a flow where the stratification varies with the vertical and in which also the Coriolis parameter varies with latitude ($\beta$-plane), the stream function is defined in terms of pressure and $f_0$, such that:

$$\psi = \frac{p_0}{\rho_s f_0}$$

✋ Similarly, $\boldsymbol{\psi}$ is the stream function for the ***non-divergent part of the geostrophic flow***.

↳ The density varies in the vertical and horizontal, such that there is a reference value of density and a perturbation, function of $(x, y, z, t)$. The expansion around a small Rossby number and the derivation of the **full quasi-geostrophic equation set** are very similar (detailed in the **following pages**).

• For the quite realistic **anelastic case** (see **#GFD1.1c**) which allows large variations of density with height, accounting for the static compressibility of the atmosphere, the quasi-geostrophic potential vorticity is:

$$q = \beta y + \nabla^2 \psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s \frac{\partial \psi}{\partial z}\right)$$

↳ It is the same as before: the $\beta y$ term, the relative vorticity, and the vortex stretching term. The latter is more complicated and depends on vertical gradients of the stream function.

• This definition can be simplified in the case of the **Boussinesq approximation** in which the reference density ($\rho_0$) is constant (independent of $z$, see **#GFD1.1c**). In this context, the density variable between the vertical derivatives cancels. It follows that:

$$q = \beta y + \nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)$$

⇨ The result is once again a **conservation law** for **quasi-geostrophic potential vorticity**, which is defined entirely in terms of a stream function, **so one equation**, **one variable**.

# Derivation of the PV equation in a continuously stratified fluid

***IV) EXTENSION TO A CONTINUOUSLY STRATIFIED FLUID***
***(with non-Boussinesq, static compressibility effects)***

Three dimensional scalings for a compressible, baroclinic stratified fluid:

$$x, y \to L, \quad u, v \to U, \quad z \to H, \quad w \to \frac{UH}{L}, \quad t \to \frac{L}{U}$$

$$p = p_s(z) + \tilde{p}(x, y, z, t)$$
$$\rho = \rho_s(z) + \tilde{\rho}(x, y, z, t)$$

Geostrophic scaling for pressure

$$fv \sim \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x}$$

so

$$\tilde{p} \to f_0 U L \rho_s$$

Hydrostatic scaling for density

$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g$$

so

$$\tilde{\rho} \to \frac{f_0 U \rho_s L}{Hg} = \rho_s \epsilon F$$

so

$$\boxed{\rho = \rho_s(1 + \epsilon F \rho')}$$

recall

$$F = \frac{f_0^2 L^2}{gH}$$
$$\epsilon = \frac{U}{f_0 L}$$

also

$$\frac{f}{f_0} = 1 + \epsilon \beta' y'$$

where

$$\beta' = \frac{\beta_0 L^2}{U} = \cot \phi_0$$

as before.

***Non-dimensional momentum equation:***

$$\frac{\partial \mathbf{v}'}{\partial t}\frac{U^2}{L} + \mathbf{v}'.\nabla \mathbf{v}'\frac{U^2}{L} + w'\frac{\partial \mathbf{v}'}{\partial z}\frac{HU^2}{LH} + fU\hat{\mathbf{k}}_\wedge \mathbf{v}' = -\frac{1}{\rho_s(1+\epsilon F\rho')}\frac{\nabla p'}{L} U L f_0 \rho_s$$

$$= -U f_0 \nabla p'(1 - \epsilon F \rho')$$

(to first order)

Divide by $Uf_0$, drop primes

$$\epsilon\frac{\partial \mathbf{v}}{\partial t} + \epsilon \mathbf{v}.\nabla \mathbf{v} + \epsilon w \frac{\partial \mathbf{v}}{\partial z} + (1 + \epsilon \beta y)\hat{\mathbf{k}}_\wedge \mathbf{v} = -(1 - \epsilon F \rho)\nabla p$$

***Non-dimensional continuity equation (non-Boussinesq)***

$$\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla \rho + w\frac{\partial \rho}{\partial z} + \rho \nabla.\mathbf{v} + \rho\frac{\partial w}{\partial z} = 0$$

$$\rho_s \epsilon F \frac{U}{L}\frac{\partial \rho'}{\partial t'} + \rho_s \epsilon F \frac{U}{L}\mathbf{v}'.\nabla \rho' + \rho_s \epsilon F \frac{UH}{LH} w' \frac{\partial \rho'}{\partial z'}$$

$$+ \frac{UH}{L} w' \left[\frac{\partial \rho_s}{\partial z}\right] + \rho_s(1 + \epsilon F \rho')\left[\frac{U}{L}\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right)\right] = 0$$

$$\times \frac{L}{\rho_s U} \to$$

$$\epsilon F \frac{\partial \rho'}{\partial t'} + \epsilon F \mathbf{v}'.\nabla \rho' + \epsilon F w' \frac{\partial \rho'}{\partial z} + H w' \left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z}\right] + (1 + \epsilon F \rho')\left(\nabla.\mathbf{v}' + \frac{\partial w'}{\partial z'}\right) = 0$$

Note that the expression in square brackets resembles $N^2$, and note that $z$ is dimensionless.

$$N^2 = \frac{g}{\theta_s}\frac{\partial \theta_s}{\partial z}$$

Define

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

then the fourth term above becomes

$$\left(\frac{HS^2}{g}\right) w'$$

This is the non-Boussinesq term.

So dropping primes

$$\epsilon F\left(\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla \rho + w\frac{\partial \rho}{\partial z}\right) + \frac{HS^2}{g} w + (1 + \epsilon F \rho)\left(\nabla.\mathbf{v} + \frac{\partial w}{\partial z}\right) = 0$$

***Expansion of non-dimensional variables***

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + ...$$
$$w = w_0 + \epsilon w_1 + ...$$
$$\tilde{p} = p_0 + \epsilon p_1 + ...$$
$$\tilde{\rho} = \rho_0 + \epsilon \rho_1 + ...$$

***Momentum equation to zero order***

Geostrophic balance

$$\hat{\mathbf{k}}_\wedge \mathbf{v}_0 = -\nabla p_0$$

and

$$\nabla.\mathbf{v}_0 = 0$$

***Continuity equation to zero order***

$$\frac{HS^2}{g} w_0 + \nabla.\mathbf{v}_0 + \frac{\partial w_0}{\partial z} = 0$$

Therefore we can't generate $w_0$ in the body of the fluid by horizontal motion. At zero order, vertical motion can only be generated at the boundary.

Assume that the bottom vertical velocity

$$w_b = 0 + \epsilon w_{1b} + ...$$

(this is assumption (3): weak orography)
Integrate upwards, this implies

$$w_0 = 0$$

everywhere, so

$$w = \epsilon w_1 + ...$$

***Momentum equation to first order***

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0.\nabla \mathbf{v}_0 + \hat{\mathbf{k}}_\wedge \mathbf{v}_1 + \beta y \hat{\mathbf{k}}_\wedge \mathbf{v}_0 + \nabla p_1 - F\rho_0 \nabla p_0 = 0$$

$$\hat{\mathbf{k}}.\nabla_\wedge(\text{this}) \rightarrow$$

vorticity equation:

$$\frac{\partial \xi_0}{\partial t} + \xi_0 \nabla.\mathbf{v}_0 + \mathbf{v}_0.\nabla \xi_0 + \nabla.\mathbf{v}_1 + \beta y \nabla.\mathbf{v}_0 + \beta \mathbf{v}_0 - F\left[\frac{\partial \rho_0}{\partial x}\frac{\partial p_0}{\partial y} - \frac{\partial \rho_0}{\partial y}\frac{\partial p_0}{\partial x}\right] = 0$$

Second and fifth terms disappear by nondivergence of the zero order flow, and the last term can be rewritten using geostrophy of the zero order flow to give

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla \xi_0 + \nabla.\mathbf{v}_1 + \beta \mathbf{v}_0 + F\mathbf{v}_0.\nabla \rho_0 = 0$$

***Continuity equation to first order***

$$F\frac{\partial \rho_0}{\partial t} + F\mathbf{v}_0.\nabla \rho_0 + \left(\frac{HS^2}{g}\right) w_1 + \nabla.\mathbf{v}_1 + \frac{\partial w_1}{\partial z} = 0$$

Note that the second and fourth terms have just appeard in the vorticity equation. So we can eliminate them by combining the continuity and vorticity equations:

$$\frac{\partial \xi_0}{\partial t} + \mathbf{v}_0.\nabla \xi_0 + \beta \mathbf{v}_0 = -F\mathbf{v}_0.\nabla \rho_0 - \nabla.\mathbf{v}_1$$

$$= F\frac{\partial \rho_0}{\partial t} + \left(\frac{HS^2}{g}\right) w_1 + \frac{\partial w_1}{\partial z}$$

At this stage we note that for synoptic scales $F \sim 0.1$ so we neglect the first term on the right hand side. This is because we have set

$$F = \frac{f_0^2 L^2}{gH} = \frac{L^2}{R^2}$$

*(remember, for the atmosphere:*

$$R_{ext} \sim \frac{\sqrt{gH}}{f_0} = \frac{\sqrt{10 \times 10^4}}{10^{-4}} \sim 3 \times 10^6\,\mathrm{m} = 3000\,\mathrm{km}$$

$$F = \frac{L^2}{R^2} \sim \frac{1000^2}{3000^2} \sim 10^{-1}$$

*)*

So the vorticity equation is now

$$\frac{\partial}{\partial t}(\beta y + \xi_0) + \mathbf{v}_0.\nabla(\beta y + \xi_0) = \frac{HS^2}{g} w_1 + \frac{\partial w_1}{\partial z}$$

Using

$$S^2 = \frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z_*}$$

the right hand side can be written

$$= \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1)$$

We can evaluate the right hand side using the ...

***Thermodynamic equation***

$$\frac{D\theta}{Dt} = 0$$

scale

$$\theta = \theta_s(1 + \epsilon F(\theta_0 + ...))$$

as we did for density, so

$$\frac{U}{L}\frac{\partial \theta'}{\partial t'}\epsilon F\theta_s + \frac{U}{L}\mathbf{v}'.\nabla\theta'\epsilon F\theta_s + w'\frac{\partial \theta'}{\partial z'}\epsilon F\theta_s\frac{HU}{LH} + w\frac{\partial \theta_s}{\partial z_*}\frac{UH}{L} = 0$$

drop primes, get

$$\epsilon F\left(\frac{\partial \theta}{\partial t} + \mathbf{v}.\nabla\theta + w\frac{\partial \theta}{\partial z}\right) + w\frac{N^2 H}{g} = 0$$

At zero order we recover

$$w_0 = 0$$

At first order:

$$F\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0.\nabla\theta_0\right) + w_1\frac{N^2 H}{g} = 0$$

$$\rightarrow w_1 = -\frac{f_0^2 L^2}{N^2 H^2}\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0.\nabla\theta_0\right)$$

introduce

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_0.\nabla$$

so

$$w_1 = -\frac{f_0^2 L^2}{N^2 H^2}\left[\frac{D_0 \theta_0}{Dt}\right]$$

Multiply by $\rho_s$, take vertical derivative and then divide by $\rho_s$, and exchange derivatives when possible. This gives

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1) = -\frac{D_0}{Dt}\left[\frac{f_0^2 L^2}{H^2 \rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s \theta_0}{N^2}\right)\right]$$

and we can use this to rewrite the vorticity equation as

$$\frac{D_0}{Dt}\left[\beta y + \xi_0 + \frac{f_0^2 L^2}{H^2 \rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s \theta_0}{N^2}\right)\right] = 0$$

Now we have one last thing to do...

***Hydrostatic equation***

$$\frac{\partial p}{\partial z} = -\rho g$$

$$p = p_s + p_0 \rho_s f_0 U L$$
$$\rho = \rho_s + \rho_0 \rho_s \epsilon F$$
$$\rightarrow \frac{\partial}{\partial z}(p_o \rho_s) = -\rho_0 \rho_s$$

or

$$\rho_0 = -\frac{1}{\rho_s}\frac{\partial}{\partial z}(p_0 \rho_s)$$

Now, define

$$\theta_* = \theta_s(z)(1 + \epsilon F\theta)$$

and define

$$\theta_0 = -\rho_0 + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right) p_0$$

*ASIDE: So where does this come from ?*
*Its needed to ensure*

$$\frac{d\theta}{\theta} = \frac{1}{\gamma}\frac{dp}{p} - \frac{d\rho}{\rho}$$

*(where*

$$\gamma = \frac{c_p}{c_v}$$

*)*

*PROOF:*
*integrate this, gives*

$$\log\theta_* = \frac{1}{\gamma}\log p_* - \log\rho_* + \text{const}$$

*but*

$$\theta_* = \theta_s(1 + \epsilon F\theta_0)$$
$$\rho_* = \rho_s(1 + \epsilon F\rho_0)$$
$$p_* = p_s + \rho_s f_0 U L p_0 = p_s\left(1 + f_0 U L\frac{\rho_s}{p_s}p_0\right)$$
$$= p_s\left(1 + \epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0\right)$$

*The inner term in brackets is the reference hydrostatic scaling, ~ 1.*
*Substitute these expressions for $\theta^*$, $\rho^*$ and $p^*$ into the log expression using the fact that to first order*

$$\log(1 + \epsilon x) = \epsilon x$$

$$\rightarrow \epsilon F\theta_0 = \frac{1}{\gamma}\epsilon F\left(\frac{gH\rho_s}{p_s}\right)p_0 - \epsilon F\rho_0$$

$$\rightarrow \theta_0 = \frac{1}{\gamma}\left(\frac{gH\rho_s}{p_s}\right)p_0 - \rho_0$$

*END OF ASIDE*

Substitute this into the hydrostatic relation to elimintate density

$$\rho_0 = -\theta_0 + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right) p_0 = -\frac{\partial p_0}{\partial z} - \frac{p_0}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

$$\theta_0 = \frac{\partial p_0}{\partial z} + p_0\left[\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} + \frac{1}{\gamma}\left(\frac{\rho_s g H}{p_s}\right)\right]$$

From the reference hydrostatic relation, the second term in square brackets can be written

$$= -\frac{1}{\gamma}\frac{\partial p_s}{\partial z}$$

but this is just

$$\theta_0 = \frac{\partial p_0}{\partial z} - p_0\left(\frac{1}{\theta_s}\frac{\partial \theta_s}{\partial z}\right)$$

and the term in brackets

$$= \frac{N^2 H}{g} \sim \frac{g'}{g} \sim \epsilon$$

so we can write the perturbation hydrostatic relation in terms of perturbation potential temperature:

$$\boxed{\theta_0 = \frac{\partial p_0}{\partial z}}$$

... put this back into the vorticity equation:

$$\frac{D_0}{Dt}\left[\beta y + \xi_o + \frac{f_0^2 L^2}{H^2 \rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial p_0}{\partial z}\right)\right] = 0$$

This is the non-d quasi-geostrophic potential vorticity.

Redimensionalise:

$$q = \beta y + \xi_0 + \frac{f_0}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial}{\partial z}\left(\frac{p_0}{\rho_s}\right)\right)$$

introduce a dimensional geostrophic streamfunction

$$\psi = \frac{p_0}{\rho_s f_0}$$

get

$$\boxed{q = \beta y + \nabla^2\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial \psi}{\partial z}\right)}$$

This is the full quasi-geostrophic potential vorticity for a compressible stratified fluid.

Note: for stratified Boussinesq fluids this form reduces to

$$\boxed{q = \beta y + \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)}$$

(this is OK for the ocean).

*q* is conserved following the flow:

$$\boxed{\frac{\partial q}{\partial t} + J(\psi, q) = 0}$$

Everything is represented in terms of one prognostic equation in one variable (the streamfunction).

*2.3.j) One variable to rule them all*

⇨ In **quasi-geostrophic theory**, we obtain only ***one variable*** in the system, the quasi-geostrophic stream function $\boldsymbol{\psi}$, that rules them all.

↳ Everything can be expressed in terms of $\psi$ in the quasi-geostrophic set.

• The **horizontal velocity** can be expressed in terms of the stream function (it is the definition of the stream function), so:

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

• The **pressure** is a stream function for the non-divergent geostrophic flow (see **#GFD2.3i**):

$$p' = \rho_0 f_0 \psi$$

• The **density** is the vertical gradient of the stream function (hydrostatic balance):

$$\rho' = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}$$

• The **vertical velocity** is material tendency operator applied to the density, so it can also be expressed in terms of $\psi$:

$$w = \frac{f}{N^2}\left[\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right]\frac{\partial \psi}{\partial z}$$

⇨ With the quasi-geostrophic set of equations, it is easier to make **predictions** following this procedure:

The flow field $\psi^{t_0}$ (ex: weather today) constitutes the **initial conditions**.

1) Compute the three-dimensional field of **potential vorticity**

$$q = \beta y + \nabla^2 \psi + \mathrm{fn}(\psi_z)$$

2) $q$ is conserved with the flow. But at one location, $q$ changes as it is blown around by the wind. Thus, the next step consists of computing the advection terms and integrating the prognostic equation forward in time to **find the next state for $q$** ($q^{t_{0+1}}$):

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = 0$$

*In the quasi-geostrophic set, time steps can be quite long (half an hour or so) because gravity waves are filtered and nothing really fast is going on.*

3) Invert the elliptic operator $q = \beta y + \nabla^2 \psi + \mathrm{fn}(\psi_z)$ to estimate the stream function. This provides a new flow field $\psi^{t_{0+1}}$ that will constitute the initial conditions for the next time step.

↳ Using this **prediction system**, you can do it 48 times in a row. This will provide weather forecasts for tomorrow. The first weather predictions were done with the quasi-geostrophic set.

## GFD2.4: Quasi-Geostrophic Theory III – Applications and Diagnostics

*2.4.a) Development*

⇨ In order to predict the weather without taking into account the potential vorticity, one can still consider directly the **time development** of $\psi$, i.e. **pressure** (focusing on pressure centers for instance). This means that we can remain in the quasi-geostrophic framework without going through this inversion process for the potential vorticity.

↳ Consider the development equation for the potential vorticity, in which the formulation of the potential vorticity is developed in term of $\psi$ (slightly simplified below):

$$\frac{\partial}{\partial t}\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) + \mathbf{v}.\nabla\left(\nabla^2\psi + f + \frac{f^2}{N^2}\psi_{zz}\right) = 0$$

⇨ We can **rearrange** it if we assume all the functions are well behaved (differentiable, etc) and that we can swap over the order of the derivatives. Instead of having $\frac{\partial}{\partial t}$ of a big elliptic function of $\psi$, we have a function of $\frac{\partial\psi}{\partial t}$, and we put the second term on the RHS and develop it, so that:

$$\left(\nabla^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial\psi}{\partial t} = -\mathbf{v}.\nabla\left(\nabla^2\psi + f\right) - \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

↳ The two RHS terms control the tendency of the stream function, and by extension, they control the pressure development. **Pressure development** will thus be **determined** by 1) the absolute **vorticity advection** and 2) the **vertical gradient of the horizontal density advection**.

So, if the sum of their contributions is negative there will be low-pressure development. ✋ Except that there is an elliptic operator in front of the tendency term.

↳ Let's assume that the functional form of $\psi$ is wave-like in $(x, y)$ and it changed sign once in the vertical (first-baroclinic mode). The effect of this operator (on this simple wave-like structure) is a multiplication by a constant ($> 0$, involving the wavenumbers) and more importantly a change of sign.

$$\psi \propto \sin lx \, \sin my \, \cos \pi z/H$$

⇨ In this context, the local rate of change of $\psi$, or change of pressure is proportional to the **absolute vorticity advection** and the **vertical gradient of the density/temperature advection**.

$$\frac{\partial\psi}{\partial t} \propto +\mathbf{v}.\nabla(\nabla^2\psi + f) + \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla\frac{\partial\psi}{\partial z}\right)$$

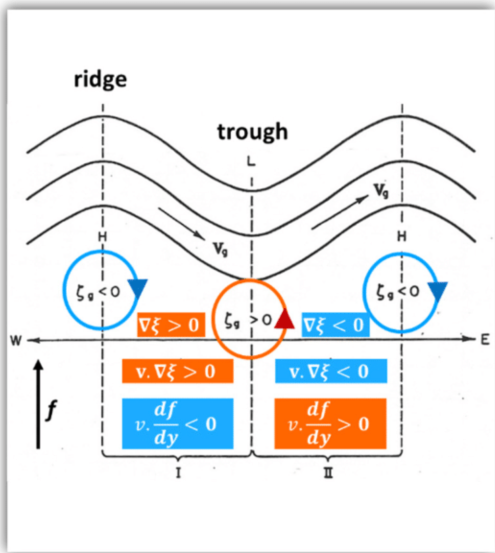
### *2.4.b) Advection of absolute vorticity*

⇨ Advection of absolute vorticity is proportional to: $\frac{\partial\psi}{\partial t} \propto \mathbf{v}.\nabla(\nabla^2\psi + f)$

We study here an eastward flow with wave-type structure (see below), such that:

$$\nabla^2\psi = -(l^2 + m^2)\psi$$

↳ In the advection of absolute vorticity, there are two terms: one associated with the relative vorticity and the other associated with the planetary vorticity.




- ***Advection of relative vorticity – short waves:***

In the ridge, the flow is clockwise and the relative vorticity is negative, while in the trough, the relative vorticity is positive. In region I: $v.\nabla\xi > 0 \Rightarrow \frac{\partial\psi}{\partial t} > 0$.

↳ With the flow going eastwards, the zonal advection of relative vorticity will **send troughs and ridges eastwards**. This is the case for **short waves** for which $\xi$ dominates.

- ***Advection of planetary vorticity*–long waves:**

As $\frac{df}{dy} > 0$, the meridional advection of planetary vorticity is controlled by the northward southward oscillation of the flow. It results in the opposite effect to the relative vorticity advection and will **send troughs and ridges west**. These are long Rossby waves (see **#GFD3**).

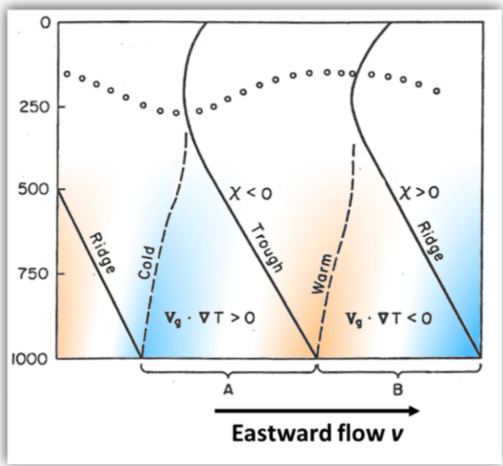
⇨ In conclusion, **there is a competition between the advection of planetary vorticity** (the Rossby wave term) **and the advection of relative vorticity** (the synoptic-scale term). Both will influence the way in which the pressure (weather) will develop.

#### *2.4.c) Vertical gradient of temperature advection*

⇨ The rate of change of the stream function, and by implication the pressure development, is proportional to the vertical gradient of the temperature advection:

$$\frac{\partial \psi}{\partial t} \propto \frac{\partial}{\partial z}\left(\mathbf{v}.\nabla \frac{\partial \psi}{\partial z}\right) \propto \frac{\partial}{\partial z}(\mathbf{v}.\nabla \theta) \qquad \left[\theta_0 = \frac{\partial p_0}{\partial z}\right]$$

The question is: “What is the vertical variation of the temperature advection?”




In the example on the side, we study an eastward flow going through the juxtaposition of cold and warm air masses in the $x$-direction. These air masses are associated with positive and negative zonal gradients of temperature which decrease with height. For instance:

In region A, there is cold advection at low-level $(v.\nabla\theta > 0)$, then $\frac{\partial(v.\nabla\theta)}{\partial z} < 0$ and a trough develops.

In region B, there is warm advection at low-level $(v.\nabla\theta < 0)$, then $\frac{\partial(v.\nabla\theta)}{\partial z} > 0$ and a ridge develops.

#### *2.4.d) Vertical velocity: quasi-geostrophic omega equation*

⇨ With the quasi-geostrophic set, it is also possible to make diagnostics for weather analysis, in particular, to diagnose the vertical velocity.

• We could deduce the vertical velocity by **integrating the continuity equation**:

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

✋ This is mathematically sound but it is ill-conditioned. $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are large terms which have cancellation between them (small differences between large terms). Such calculation for vertical velocity is not numerically accurate for real data sets.

• The quasi-geostrophic system to the rescue ☺. From the continually stratified version of the quasi-geostrophic theory (detailed in **#GFD2.3i**), the vertical velocity can be written:

$$w = -B_u\left(\frac{\partial}{\partial t}\frac{\partial p_o}{\partial z} + \mathbf{v}_0.\nabla\frac{\partial p_0}{\partial z}\right)$$

↳ We eliminate the tendency term by estimating the Laplacian of this formula and using the vertical gradient of the vorticity equation:

$$\nabla^2\left(\frac{\partial}{\partial z}\frac{\partial p_o}{\partial t}\right) = -\nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_0}{\partial z}\right) - B_u^{-1}\nabla^2 w_1$$

$$\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial t}(f+\xi_0) + \mathbf{v}_0.\nabla(f+\xi_0) = \frac{\partial w_1}{\partial z}\right\} \to \nabla^2\left(\frac{\partial}{\partial z}\frac{\partial p_0}{\partial t}\right) = \frac{\partial^2 w_1}{\partial z^2} - \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f+\xi_0))$$

⇨ Equating the two RHS yields a diagnostic equation for the vertical velocity in terms of the geostrophic stream function ($\xi_0 = \nabla^2 p_0$). It is called the **quasi-geostrophic omega equation**:

$$\left(B_u^{-1}\nabla^2 + \frac{\partial^2}{\partial z^2}\right) w_1 = \frac{\partial}{\partial z}(\mathbf{v}_0.\nabla(f + \xi_0)) - \nabla^2\left(\mathbf{v}_0.\nabla\frac{\partial p_o}{\partial z}\right)$$

↳ It is written as an elliptic operator on the vertical velocity, equal to the summed contribution of two terms.

- The first RHS term is related to the **vertical gradient of the absolute vorticity advection**.
- The second RHS term is the **Laplacian of the temperature advection**.

↳ This is the other way around to **#GFD2.4a**, in which we had vorticity advection and the vertical gradient of the temperature advection.

⇨ If we study a wave-type pattern, both elliptic operators can be represented by a simple change of sign. It follows that:

$$w \propto -\frac{\partial}{\partial z}\left(\mathbf{v}.\nabla(f + \xi)\right) - \mathbf{v}.\nabla\theta$$

⇨ Note that this time we have eliminated the tendency term (rather than the vertical velocity term) between the vorticity and thermodynamic equations and obtained a diagnostic equation for $w$ (rather than a prognostic equation for $\psi$). This equation is **usually derived in pressure coordinates**.

### *2.4.e) Application of the omega equation*

⇨ Go to: https://www.meted.ucar.edu/labs/synoptic/qgoe_sample/qgoe_widget.htm

↳ It all gets very complicated and you have to sit and scratch your head a long time looking at these equations, making sure you have got the sign right… because if you get the sign wrong you get it all completely wrong.

### *2.4.f) Recap*

⇨ Here is a summary for all these simplified quasi-geostrophic illustrations:

- The fall or rise of geopotential is proportional to:
  - positive or negative vorticity advection
  - the rate of decrease with height of the cold or warm advection
- For diagnosing the vertical velocity, rising or sinking motion is proportional to:
  - the rate of increase with height of the positive or negative vorticity advection
  - warm or cold advection

# CHAPER 3
# Rossby waves and instability

# CHAPTER 3
# Rossby waves and instability

**GFD3 Contents**

In this chapter, we stay in the **quasi-geostrophic framework** (see **#GFD2**) and focus on **Rossby waves**. We start with a general idea of what happens to a parcel of air or water if it is displaced on the planet where there is a variation in the Coriolis parameter, i.e. what are the consequences of **conserving the potential vorticity**.

We will derive the **dispersion relation** for Rossby waves by looking for **trigonometric/wave-like solutions**. We will overview different cases:

1) **Barotropic Rossby waves** (see **#GFD3.1**) and **topographic Rossby waves** (see **#GFD3.1g**),
2) **Baroclinic Rossby wave**, in a multi-layer model (see **#GFD3.2a**) and then in a continuously stratified fluid (see **#GFD3.2b**). We will decompose the variability in the vertical, i.e. extract independent **vertical modes**.

We will study the wave solution propagating through a non-uniform **background flow with shear**. Waves are solutions with trigonometric variations and imaginary exponentials, so the time variation is an oscillation and there is a propagation. What if the exponential becomes real?

3) It results in a perturbation that grows in time exponentially and becomes **unstable**. We will review the conditions required for this to happen in a barotropic (see **#GFD3.3**) and then baroclinic (see **#GFD3.4**) frameworks.

## GFD3.1: Barotropic Rossby waves

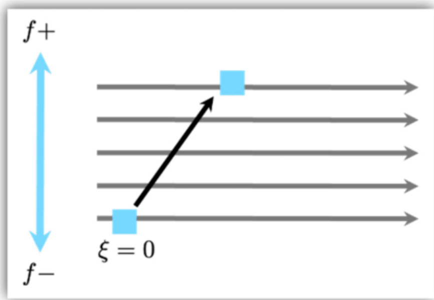
### *3.1.a) Parcel displacement in a vorticity gradient*

General idea

⇨ Let's consider as parcel of fluid:

▪ In a non-divergent barotropic framework, i.e. the ***absolute vorticity is conserved*** (see **#GFD1.3b**) following the parcel: $q = f + \xi$.

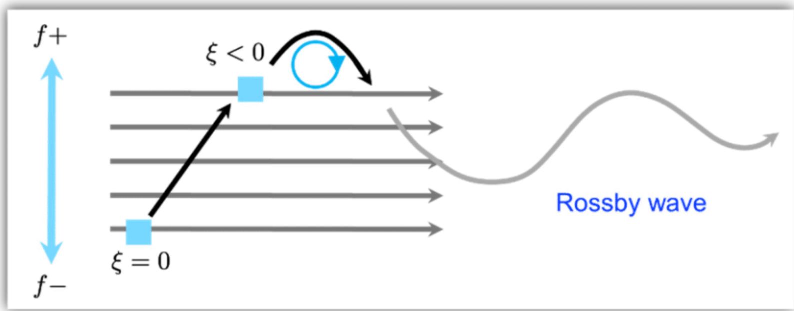
▪ On a planet with some curvature, i.e. the planetary vorticity (Coriolis parameter) $f$ varies with latitude (larger $f$ to the north - smaller $f$ to the south).




⇨ At the origin, this parcel of fluid has no relative vorticity ($\xi = 0$). Imagine, for some reason, there is a perturbation that displaces the parcel (a little bit) to the north, where $f$ gets larger.

⇨ In accord with the conservation of absolute vorticity ($f + \xi$), the relative vorticity of the flow will compensate for this increase in $f$ and $\xi$ must become negative ($\xi < 0$). Negative relative vorticity is associated with a **clockwise curvature of the flow**.

↳ So, the flow curves back down towards the south and the parcel will return to its latitude of origin. This is a *stable situation*, i.e. the solution oscillates such that the force that restores it to its position of origin is somehow proportional to the distance from the origin position.




⇨ You can imagine it overshooting and going back down south in which case it will come back north and it will produce a wave, a **Rossby wave**. A wave for which the restoring force is not just the Coriolis force, but the variation of the Coriolis force with the latitude.

☝ We need **variable $f$** for this to happen, so this **cannot work on a $f$-plane**.

### *3.1.b) The conservation of vorticity*

⇨ We will derive the **dispersion relation for Rossby waves**.

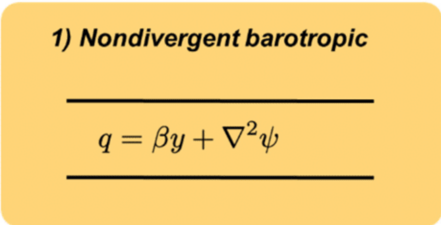
⇨ We remain in the **general framework of quasi-geostrophy** in which the potential vorticity is conserved following the flow, i.e. $\frac{Dq}{Dt} = 0$, with the material tendency given by the local tendency plus the advection terms.

↳ Here, the advection terms depend on:

- a background **zonal** flow $U$
- and the small perturbation flow $(u', v')$ associated with the wave.

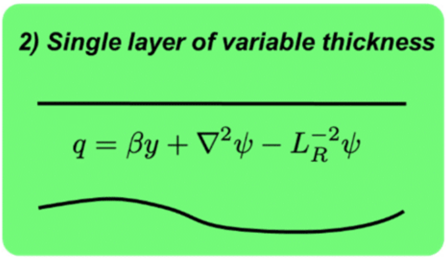
This gives: $\frac{D}{Dt} = \frac{\partial}{\partial t} + (U + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}$

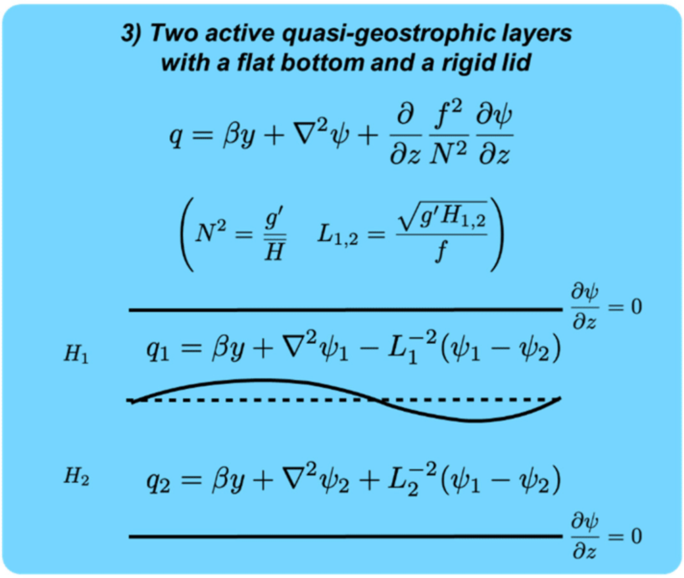
⇨ ***q* depends on the type of flow we consider**. We will study the Rossby wave dispersion relation in **three different contexts**:




**1)** ***Non-divergent barotropic flow*** (see **#GFD3.1c**) = One active layer bounded above and below by a rigid lid and a flat bottom. The flow is uniform in the vertical, i.e. it is barotropic. In this case, the potential vorticity is the absolute vorticity (see **#GFD2.3h**): $q = f + \xi = \beta y + \nabla^2\psi$. The stream function ($\psi$) is the ***perturbation of the stream function associated with the wave*** and the ***stream function of the background flow U***.

**2)** Then we will study the effect of variable layer thickness *on* a barotropic flow (see **#GFD3.1f**). In this case, you can generate vorticity by divergence and there is a vortex stretching term in the potential vorticity formulation (see **#GFD2.3h**). It is often called ***equivalent barotropic***, as there is only one active layer, the layer below is a motionless abyss.




**3)** We will finally consider the full baroclinic framework (see **#GFD3.2**) bounded above and below by a rigid lid and a flat bottom. We impose no vertical velocity at these boundaries, i.e. $\frac{\partial\psi}{\partial z} = 0$. In this framework, we consider that the fluid is Boussinesq, so the vortex stretching term in the continuously-stratified fluid potential vorticity formula is slightly simplified (see **#GFD2.3i**).

↳ If you **discretize** the vertical derivative and do a finite difference, you can easily derive the potential vorticity expressions for a discretized two-layer framework (see **#GFD3.2a**). You obtain simple differences between the stream functions. In the upper layer, it reduces to the inverse square of the Rossby radius times the difference between the stream function in the two layers. In the lower layer, a distinct Rossby radius (the thicknesses can be different in each layer) times by the difference between the stream function in the two layers.

↳ The formulae for the potential vorticity are coupled: $q_1$ depends on $\Psi_2$ and $q_2$ depends on $\Psi_1$.

Note that $q$ is conserved with the flow. $\psi = \psi_B + \psi$. With $\psi_B = -Uy$, $\nabla^2\psi_B = 0$. $\psi_B$ is thus crossed out from the PV equation

### *3.1.c) CASE 1: Non-divergent barotropic case*

Equation

$$q = \beta y + \nabla^2 \psi$$

⇨ We **develop** the substantial derivative in the potential vorticity conservation equation (see **#GFD2.3g**), using the characteristics of the background flow:

$$u = U + u' = U - \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = v' = \frac{\partial \psi}{\partial x}$$

We obtain: $\frac{\partial}{\partial t}(\beta y + \nabla^2\psi) + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}(\beta y + \nabla^2\psi) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\beta y + \nabla^2\psi) = 0$

We can cross out some of these terms:

- As $\beta y$ does not vary with time or $x$.
- We **linearize the equation** and consider that perturbations are small compared to the mean flow. Terms with a square of perturbation are neglected.

$$\frac{\partial}{\partial t}(\beta y + \nabla^2\psi) + \left(U - \frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}(\beta y + \nabla^2\psi) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\beta y + \nabla^2\psi) = 0$$

↳ The linear equation in perturbation $\psi$ can be written: $\left(\frac{\partial}{\partial t} + U\right)\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0$

Solutions

⇨ We are going to look for **wave-like solutions**, **plane-wave solutions**:

$$\psi = Re\,\tilde{\psi}e^{i(lx+my-\omega t)}$$

▪ They have the form of an amplitude coefficient times an imaginary exponential:

- $l$ is the **zonal wavenumber** ($2\pi$ divided by the $x$-wavelength),
- $m$ is the **meridional wavenumber** ($2\pi$ divided by the $y$-wavelength),
- $\omega$ is the **angular frequency** ($2\pi$ divided by the period).

Properties

▪ Taking a derivative of this trigonometric function yields the same function multiplied by some constant coefficients:
$$\frac{\partial}{\partial t} \to -i\omega \qquad \frac{\partial}{x} \to il \qquad \nabla^2 \to -(l^2 + m^2)$$

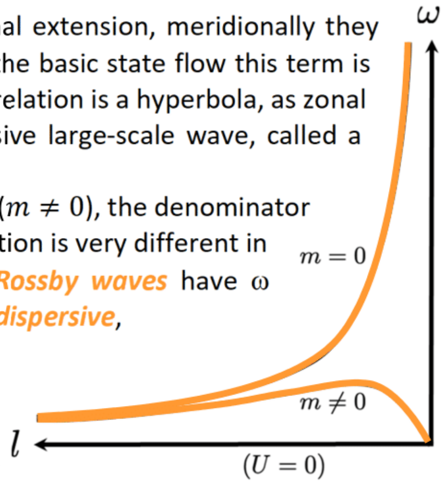
↳ Substituting the solution and its derivatives into the linear potential vorticity equation gives:
$$-i\omega\left(-(l^2+m^2)\right) + il\left(-(l^2+m^2)\right)U + \beta il = 0$$

Dispersion relation

⇨ It results in a relation between ω, $l$ and $m$ (with 2 other geophysical parameters $U$ and $\beta$). This is the **dispersion relation for barotropic Rossby waves**:
$$\omega = Ul - \frac{\beta l}{l^2 + m^2}$$

- The phase speed $c = \frac{\omega}{l}$ is equal to $U - \frac{\beta}{l^2+m^2}$. $l^2 + m^2$ and $\beta$ are always positive. So, the **Rossby waves always propagate westwards**, opposite to the background eastward flow $U$.

- With $m = 0$ (i.e. the waves have an infinite meridional extension, meridionally they cover the entire planet), ω is proportional to $-\beta/l$. Relative to the basic state flow this term is negative, so we plot it on the negative quadrant. The dispersion relation is a hyperbola, as zonal scales get bigger, frequencies get higher. This is a very dispersive large-scale wave, called a **Rossby Haurwitz wave**.




- As soon as you set a meridional scale to your structure ($m \neq 0$), the denominator does not disappear. When $l = 0$ then $\omega = 0$. The dispersion relation is very different in this case. For the meridionally-confined structures, the ***long Rossby waves*** have ω almost proportional to $l$, which means that they are almost ***non-dispersive***, until a certain point. The maximum $\omega$ is found for $l = m$, and then for the shorter waves (for larger $l$), they become very dispersive.

Non-dispersive waves: all the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape

Wave propagation speed

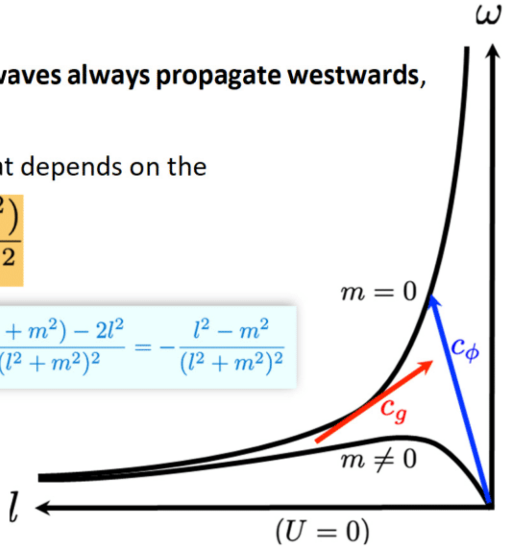
### *3.1.d) Rossby wave dispersion*

• The **phase speed**: $c = \frac{\omega}{l} = U - \frac{\beta}{l^2 + m^2}$

↳ With $l^2 + m^2$ and β always positive, the **Rossby waves always propagate westwards**, opposite to the background eastward flow $U$.

• The **group speed** on the other hand has a sign that depends on the sign of its numerator: $c_g = \frac{\partial\omega}{\partial l} = U + \frac{\beta(l^2 - m^2)}{(l^2 + m^2)^2}$

$$\frac{\partial\omega}{\partial l} = U - \beta\frac{\partial}{\partial l}(l(l^2+m^2)^{-1}) = \frac{1}{(l^2+m^2)} + l(-(l^2+m^2)^{-1} \times 2l) = \frac{(l^2+m^2) - 2l^2}{(l^2+m^2)^2} = -\frac{l^2 - m^2}{(l^2+m^2)^2}$$

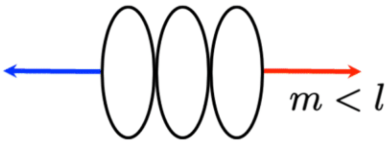



⇨ On this graph, the phase speed is the **arrow that points from the origin towards the curve** (whichever curve is used), while the **group speed is the tangent to the curve**.

↪ Relative to the background current, the direction of propagation of the energy of the wave **depends on the shape of the wave** (ratio of zonal to meridional scales):

➢ If $l = m$ , the group speed is zero.

➢ If $l > m$ , i.e. waves which have a larger meridional scale than their zonal scale, then the ratio term in the group speed formula is always positive. Relative to the eastward background flow ($U$), the phase speed of these waves (in blue here) will be to the west, while their group speed will be to the east.




➢ If $l < m$, the waves are elongated in the zonal direction and the ratio term is negative. The group speed and the phase speed are both to the west (relative to the eastward background flow ($U$),). These waves are more non-dispersive and are easier to observe because they will not lose their shape as they propagate westwards.


⇨ From the dispersion relation, it comes that:
- **Rossby waves are dispersive**. **Longer waves go faster**.
- **Waves closer to the equator go faster** ($\beta$ is maximum at the equator, zero at the poles)

### *3.1.e) Rossby wave propagation mechanism*

Westward propagation illustration

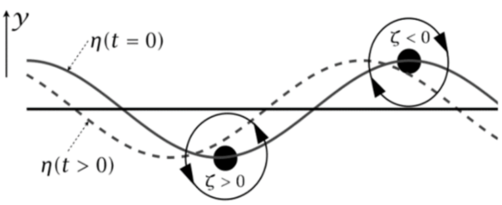
⇨ Why do Rossby waves propagate to the west?

• **Remember the parcel** which was displaced from its position of origin. To the north, it has acquired negative relative vorticity resulting in a clockwise circulation. To the south, positive relative vorticity has been induced, i.e. an anti-clockwise circulation.

• Imagine now a **streamline of potential vorticity** that follows the parcel. It has been moved to the north or to the south, portraying a wave.

⇨ **How would the stream line be displaced by this secondary circulation?**

It will be pushed away from the origin on the west, and towards it on the east. So, at a later time, the streamline will follow the dashed curve, effectively moving it to the left on the diagram. The Rossby wave is thus propagating to the west.




↳ ***The secondary circulation induced by the constraint of conserving the vorticity produces the westward propagation.***

### *3.1.f) CASE 2: Divergent case (variable layer thickness)*

$$q = \beta y + \nabla^2\psi - L_R^{-2}\psi$$

⇨ In the case of a variable layer thickness, the same advection operator is applied to a different definition of potential vorticity. The latter contains a **vortex stretching term** (see **#GFD2.3h**), which is $-L_R^{-2}\psi$.

☝ In the PV conservation equation, the stream function is the sum of contributions from:

- the stream function associated with the perturbation $\psi$
- and the **background flow stream function** $\psi_B = -Uy$

Equation

↳ $\nabla^2\psi_B = 0$, but for the divergent case, the contribution of the background flow $U$ remains in the ***vortex stretching term***, such that:

$$\left(\frac{\partial}{\partial t} + (U+u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\left(\beta y + \nabla^2\psi - L_R^{-2}(\psi - Uy)\right) = 0$$

⇨ As in the non-divergent case (**#GFD3.1c**):

- $U + u' = U - \frac{\partial \psi}{\partial y}$ and $v' = \frac{\partial \psi}{\partial x}$
- Terms associated with **time and $x$ variation** of $\beta y$ and $L_R^{-2}Uy$ can be crossed out.
- We **linearize the equation**, so terms with a square of perturbation are neglected.

↳ The resulting PV conservation equation can be sorted into **two terms**:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2\psi - L_R^{-2}\psi\right) + \left(\beta + L_R^{-2}U\right)\frac{\partial \psi}{\partial x} = 0$$

▪ The first LHS term is the material tendency of the perturbation relative vorticity and vortex stretching term, i.e. ***the mean flow advecting the perturbation***.

▪ The second term is the other way round, i.e. ***the perturbation flow affecting the mean***. This is the perturbation meridional flow ($v' = \frac{\partial \psi}{\partial x}$) advecting the potential vorticity associated with the background flow plus $\beta$.

Solutions and dispersion relation

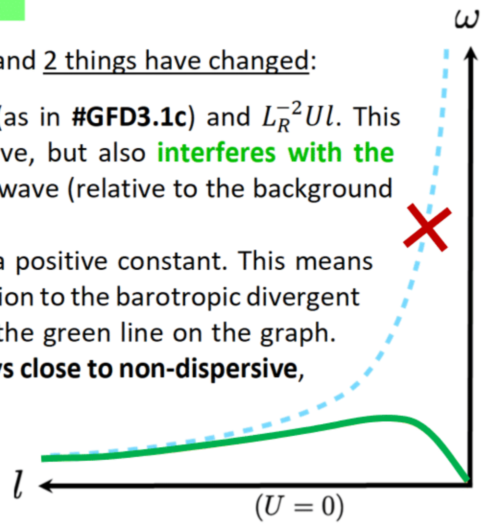
⇨ As in **#GFD3.1c**, we derive the **dispersion relation** by substituting ***plane-wave solutions*** ( $\tilde{\psi} e^{i(lx+my-\omega t)}$ ), and their derivative properties into the PV conservation equation. It follows that:

$$\omega = Ul - l\frac{\beta + L_R^{-2}U}{l^2 + m^2 + L_R^{-2}}$$

↳ The second term is more complicated than before and 2 things have changed:

• ***On the numerator***, there are now two terms: $\beta l$ (as in **#GFD3.1c**) and $L_R^{-2}Ul$. This means that the **background flow** not only displaces the wave, but also **interferes with the properties of the wave**. In particular, the ***phase speed*** of the wave (relative to the background flow) will be altered by the background flow.

• ***On the denominator***, there is an extra term $L_R^{-2}$ , a positive constant. This means that the very dispersive **Rossby Haurwitz waves** are not a solution to the barotropic divergent framework (see dotted line). The solution always resembles the green line on the graph. The phase speed is bounded and **long-Rossby waves are always close to non-dispersive**, with group speed to the west (even for $m = 0$).

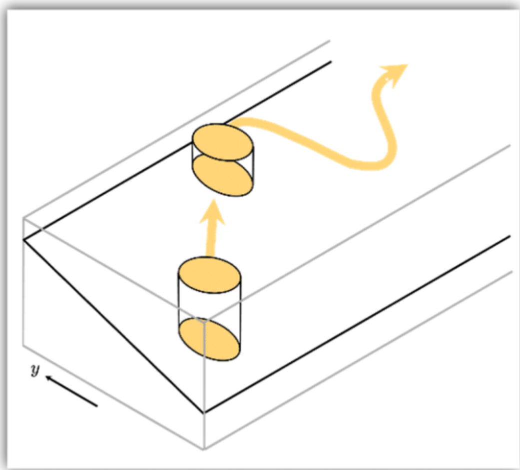


### *3.1.g) Topographic Rossby waves*

⇨ We focus now on a slightly different case in which potential vorticity can be changed by externally constraining the layer thickness. From **#GFD2.3h**, the quasi-geostrophic potential vorticity can be written as:

$$q = f_0 + \beta y + \xi - \frac{f_0}{H}\delta h$$

*Note that, in the example below, we can disregard the changes in the planetary vorticity* ($\beta y$).




⇨ Consider an ocean getting shallower towards the north (see illustration on the right).

↳ In this example, the thickness of the Ocean is proportional to the latitude $y$, $h_{OC} = \alpha y$ . Since the Ocean gets thinner to the north, $\alpha$ is negative.

This topographic effect will add a constrain on $\delta h$, so the quasi-geostrophic potential vorticity is:

$$q = f_0 + \beta y + \xi - \frac{f_0}{H}(\alpha y + \eta)$$

⇨ Imagine that a column of fluid is displaced northward.

↳ Because of the change in the topography, the column is squashed (it gets shallower).

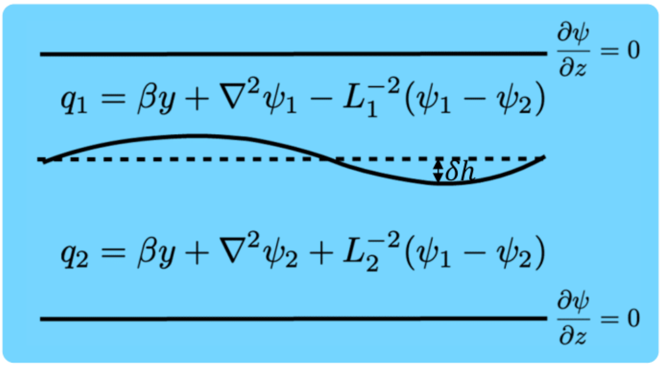
↳ Its potential vorticity has to be conserved, which triggers negative relative vorticity. This induces a clockwise flow, similar to **#GFD3.1a**. Just like before, the displacement can generate a Rossby wave, which lives on this slope.

↳ Mathematically, the external constraint term (topographic effect) $-\frac{f_0}{H}\alpha y$ is positive and identical to the β-effect. *In the northern hemisphere, an ocean floor that is shallowing to the north will have the same effect as* $\beta$*. In the southern hemisphere the ocean floor must shallow to the south.*

⇨ These waves are called topographic Rossby waves.

## GFD3.2: Baroclinic Rossby waves

### *3.2.a) CASE 3: Two active layers*




⇨ In this framework, we have two active layers in which the quasi-geostrophic potential vorticity is conserved (see **#GFD2.3h**):

$$q_1 = \frac{f + \xi_1}{h_1} \approx \frac{1}{H_1}\left(f + \xi_1 - \frac{f}{H_1}\delta h\right)$$
$$q_2 = \frac{f + \xi_2}{h_2} \approx \frac{1}{H_2}\left(f + \xi_2 + \frac{f}{H_2}\delta h\right)$$

⇨ We retrieve **geostrophic stream functions** for each layer (see **#GFD1.2e**):

$$\boldsymbol{f}_0 \times \boldsymbol{u}_1 = -\frac{1}{\rho_0}\nabla P_1 = -g\nabla(h_1 + h_2) \quad \text{and} \quad \boldsymbol{f}_0 \times \boldsymbol{u}_2 = -\frac{1}{\rho_0}\nabla P_2 = -g\nabla(h_1 + h_2) - g'\nabla h_2$$

$$\psi_1 = \frac{g}{f_0}(h_1 + h_2) \quad \text{and} \quad \psi_2 = \frac{g}{f_0}(h_1 + h_2) + \frac{g'}{f_0}h_2$$

↳ The interface displacements (from the rigid lid) are $\delta h = -h_2 = \frac{f_0}{g'}(\psi_1 - \psi_2)$. Therefore, the vortex stretching term is a coupled term defined in terms of the difference between the two stream functions.

⇨ As in **#GFD3.1f**, the conservation equation can be expanded by using the advection operators defined in terms of the stream function. ✋ For simplicity, the background flow ($U$) has been disregarded in this example. Simplification of time and $x$-invariant terms and linearization yields:

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 - L_1^{-2}(\psi_1-\psi_2)\right] + \beta\frac{\partial \psi_1}{\partial x} = 0$$
$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 + L_2^{-2}(\psi_1-\psi_2)\right] + \beta\frac{\partial \psi_2}{\partial x} = 0$$

$$L_1^{-2} = \frac{f_0^2}{g'H_1} \quad L_2^{-2} = \frac{f_0^2}{g'H_2}$$

↳ As in **#GFD1.2f**, the two equations for the potential vorticity are coupled (the top layer equation depends on $\psi_1$ and $\psi_2$, so does the bottom layer equation).



✋ The problem now is to **decouple these equations**. Similar to the shallow water equations (in **#GFD1.2**), we have to define new variables that are a linear combination of $\psi_1$ and $\psi_2$ which will provide <u>two independent equations</u>. For this example, we do not need to compute the eigenvalues and eigenvectors of the coupling matrix. The variable transformations are more straightforward.

1) We can first define a **barotropic mode**, noted $\bar{\psi}$, as the weighted sum of the stream functions by the layer thicknesses.

$$\bar{\psi} = \frac{H_1\psi_1 + H_2\psi_2}{H_1 + H_2}$$

↳ It can also be expressed in terms of the Rossby radius:

$$\bar{\psi} = \frac{L_2^{-2}\psi_1 + L_1^{-2}\psi_2}{L_1^{-2} + L_2^{-2}} \qquad L_R^{-2} = L_1^{-2} + L_2^{-2}$$

2) We can define a **baroclinic mode** $\hat{\psi}$ which is just the difference between the two layers:

$$\hat{\psi} = \psi_1 - \psi_2$$

⇨ If we manipulate the set of equations so the variables are $\bar{\psi}$ and $\hat{\psi}$, we obtain two independent equations, one for the barotropic mode and one for the baroclinic mode:

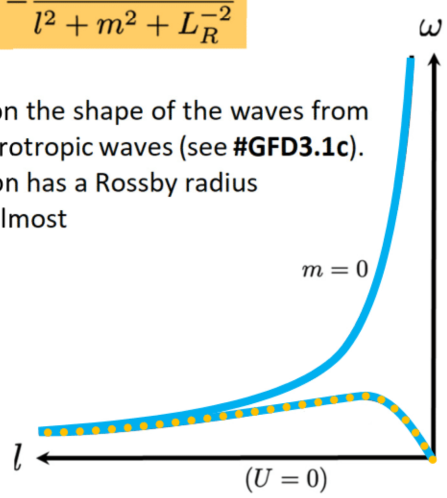
$$\frac{\partial}{\partial t}\nabla^2\bar{\psi} + \beta\frac{\partial \bar{\psi}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\left[(\nabla^2 - L_R^{-2})\hat{\psi}\right] + \beta\frac{\partial \hat{\psi}}{\partial x} = 0$$

↳ Naturally, the **barotropic mode** equation resembles the barotropic potential vorticity equation (see **#GFD3.1c**), while the **baroclinic mode** equation includes an extra stretching term.



⇨These two modes are associated with distinct dispersion relations:

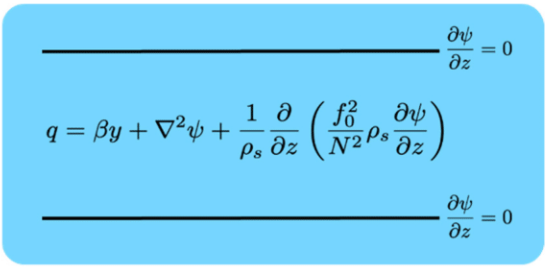
$$\omega = -\frac{\beta l}{l^2 + m^2} \quad \text{and} \quad \omega = -\frac{\beta l}{l^2 + m^2 + L_R^{-2}}$$




• The **barotropic mode** dispersion relation will depend on the shape of the waves from the extreme **Rossby Haurwitz wave** to the non-dispersive long barotropic waves (see **#GFD3.1c**).

• The **baroclinic mode** is slower and its dispersion relation has a Rossby radius term in the denominator $L_R^{-2} = L_1^{-2} + L_2^{-2}$ and always yields almost non-dispersive long Rossby waves.

### 3.2.b) CASE 4: Extension to the vertical continuum




⇨ Let us now consider the dynamics of linear waves in stratified quasi-geostrophic flow on a $\beta$-plane, in a domain confined between two **rigid surfaces** at $z=0$ and $z=-H$, with a **resting basic state**.

↳ The **quasi-geostrophic potential vorticity** for a continuously stratified fluid is conserved following with the flow (see **#GFD2.3i**).

⇨ The interior flow is governed by the quasi-geostrophic potential vorticity conservation (see **#GFD2.3i**):

$$\frac{\partial q}{\partial t}+J(\psi,q)=0$$

↳ As in **#GFD3.1f**, the conservation equation can be expanded and **linearized**, leading to:

$$\frac{\partial}{\partial t}\left[\nabla^2\psi+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\frac{{f_0}^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]+\beta\frac{\partial\psi}{\partial x}=0$$

↳ If the boundaries are flat, rigid, and slippery surfaces, then $w\ =\ 0$ at the boundaries. Also, if there is no surface buoyancy gradient, the linearized equation is:

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial z}\right)=0\text{ at }z=0\text{ and }z=-H$$

Vertical structure

⇨ As in the single-layer case (**#GFD3.1c**), we seek solutions of the form of ***plane-wave solutions***, $\psi=Re\,\tilde{\psi}(z)e^{i(lx+my-\omega t)}$, where $\tilde{\psi}(z)$ **determines the vertical structures of the waves**.

↳ We indeed have to account for the fact that **the wave amplitude might vary in the vertical**. For the two-layer case, we had two modes because we had two layers. For the vertically continuous case, ***we have functions of z***.

Vertical separation

⇨ **Substituting** the solution and its derivative into the linear potential vorticity equation does not yield a simple algebraic expression. It results in a **differential equation** for the wave coefficient $\tilde{\psi}(z)$:

$$\omega\left[-(l^2+m^2)\tilde{\psi}(z)+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\frac{{f_0}^2}{N^2}\frac{\partial\tilde{\psi}(z)}{\partial z}\right)\right]-\beta l\tilde{\psi}(z)=0$$

↳ To solve this equation, we make a separable dependence assumption, implying that **horizontal and vertical structures of the waves can be separated**, $\tilde{\psi}(z)$ satisfies:

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\frac{{f_0}^2}{N^2}\frac{\partial\tilde{\psi}(z)}{\partial z}\right)=-\Gamma\tilde{\psi}(z)\qquad(*)$$ ($\Gamma$ is the separation constant)

⇨ Then the equation of motion becomes:

$$-\omega[(l^2+m^2)+\Gamma]\tilde{\psi}-\beta l\tilde{\psi}=0$$

↳ And the dispersion relation follows:

$$\omega=-\frac{\beta l}{(l^2+m^2)+\Gamma}$$

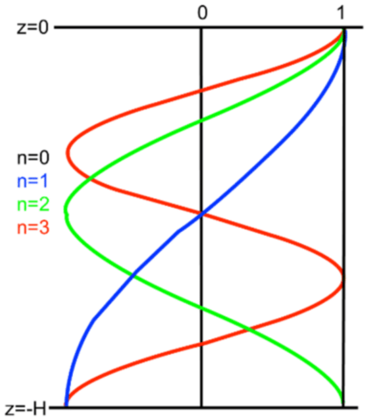
⇨ Equation $(*)$ constitutes an ***eigenvalue problem*** for the vertical structure, with boundary conditions $\frac{\partial\psi}{\partial z}=0$ at $z=0$ and $z=-H$. The resulting eigenvalues $\Gamma$ are proportional to the inverse of the squares of the deformation radii for the problem and the eigenfunctions are the vertical structure functions.

***What do these vertical structures look like?***

**A simple example**:

⇨ For simplification, we consider waves propagating in a **Boussinesq fluid**, with a **constant stratification**. The eigenproblem for the vertical structure (with previous boundary conditions) is:

$$\frac{{f_0}^2}{N^2}\frac{\partial^2\tilde{\psi}(z)}{\partial z^2} = -\Gamma\tilde{\psi}(z) \ (**)$$




⇘ There is a sequence of solutions to this equation, namely:

$$\tilde{\psi}_n(z) = \cos(n\pi z/H), \quad n = 1,2..., \quad k_v = n\pi/H$$

▪ The first solution ($n = 1$, blue line) is half a wave in the vertical, the second solution ($n = 2$, green line) is a full-wave in the vertical. The third mode is one and a half waves in the vertical, etc... These constitute $n$ baroclinic modes.

⇘ The structure of the baroclinic modes becomes more ***complex*** as the vertical wavenumber ***n increases***.

▪ Each mode has a different eigenvalue:

$$\Gamma_n = (n\pi)^2\left(\frac{f_0}{NH}\right)^2, \quad n = 1,2\ ...$$

▪ This equation can be used to define the deformation radii for this problem, namely:

$$\mathrm{L}_n \equiv \frac{1}{\sqrt{\Gamma_n}} = \frac{NH}{n\pi f_0} = \frac{N}{f_0 k_v}$$

▪ The phase speed of the Rossby waves is given by $\omega/l$.

▪ The dispersion relation is different for each mode:

$$\omega = -\frac{\beta l}{(l^2+m^2)+\frac{f^2}{N^2}k_v^2}$$

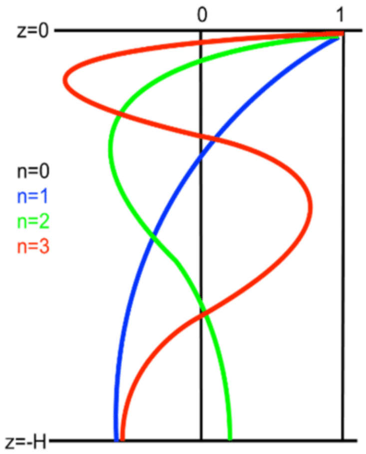
☝ For each different vertical structure, we have a different Rossby wave with different properties.

⇘ For each mode, Rossby waves have a different phase speed.

**A more realistic stratification**:

⇨ In a **Boussinesq fluid**, the eigenproblem for the vertical structure is more complex as the stratification depends on the vertical ($N^2(z)$):

$$\frac{\partial}{\partial z}\left(\frac{{f_0}^2}{N^2}\frac{\partial\tilde{\psi}(z)}{\partial z}\right) = -\Gamma\tilde{\psi}(z) \qquad (***)$$




⇘ For a stable stratification, this differential equation with its boundary conditions is reduced to a *Strum-Liouville* system which can be solved numerically.

▪ The structure of the baroclinic modes which depends on the structure of the stratification, becomes increasingly ***complex*** as the vertical wavenumber ***n increases***.

▪ The ***variability of the vertical structure*** is confined in the ***thermocline layer*** where the stratification is maximum.

☝ In addition to these baroclinic modes, the **barotropic mode** with $n = 0$, that is $\tilde{\psi}(z) = 1$, is also a solution of $(*)$ for any density profile (black line).

⇘ The dynamics of the barotropic mode is independent of height and independent of the stratification of the basic state, and so these Rossby waves are identical to the Rossby waves in a homogeneous fluid contained between two flat rigid surfaces (see **#GFD3.1c**).

### 3.2.c) Vertically propagating Rossby waves

⇨ Rossby waves propagate horizontally, as the restoring force is in the horizontal.

↳ But they have a **vertical component to their propagation** as well. For instance, Rossby waves can be out of phase in different layers, and they can thus effectively propagate with a vertical component to their propagation.

⇨ The **vertical wavenumber** for each mode $k_{vn} = n\pi/H$ can also be expressed in terms of $c_n$, the gravity wave phase speed for the $n^{\text{th}}$-mode, as $k_{vn} = N/c_n$. ✋ $c_n$ is not the Rossby wave speed!

⇨ The **dispersion relation for very long Rossby wave** ($l^2 + m^2 \ll k_v^2 f_0^2/N^2$) can be approximated, as:

$$\omega = -\frac{\beta l}{(l^2+m^2) + \frac{f_0^2}{N^2}k_v^2} \approx -\frac{\beta l N^2}{f_0^2 k_v^2} = \frac{\beta l c_n^2}{f_0^2}$$

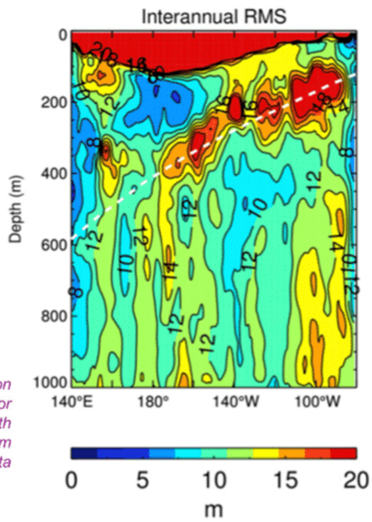
- The horizontal group speed is: $\dfrac{\partial \omega}{\partial l} = -\dfrac{\beta N^2}{f_0^2 k_v^2} = -\dfrac{\beta c_n^2}{f_0^2}$
- The vertical group speed is: $\dfrac{\partial \omega}{\partial k_v} = \dfrac{2\beta l N^2}{f_0^2 k_v^3} = \dfrac{2\beta l c_n^3}{f_0^2 N}$

↳ We can trace the signal path associated with the **vertical propagation** in the $x - z$ plane by calculating the **ratio between the two group speeds**:

$$\frac{dz}{dx} = \frac{c_g^z}{c_g^x} = -\frac{2lc_n}{N} = \frac{2f_0^2\omega}{\beta N c_n}$$

This provides the **slope** at which the energy propagates, allowing us to **trace the direction of propagation** of perturbations.

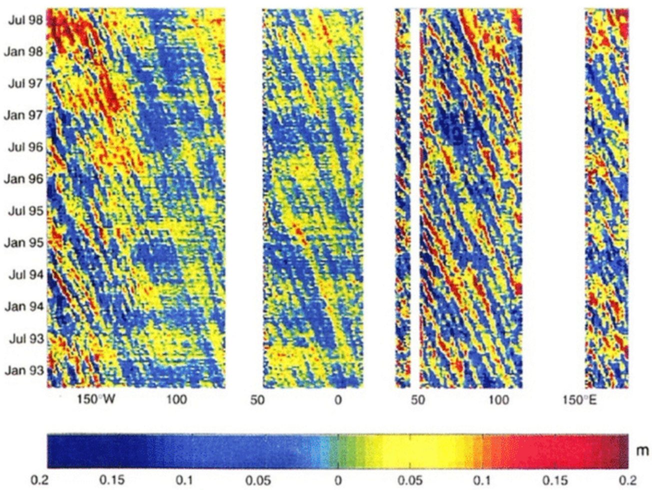
On the right is a figure by Vergara et al. (2017) highlighting a ray, showing the propagation of a perturbation to the thermocline depth. It gives observational evidence of vertically propagating Rossby waves.




*Propagation pathway for isotherm depth variability from ARGO data*

### 3.2.d) Observations

Below is a (quite old) global longitude-time representation of the Sea Level anomalies (perturbations) at 25°S from Topex/Poseidon altimetry data from 1993 to 1998. The longitude in degrees covers the 3 tropical Oceans: Pacific, Atlantic, and Indian.



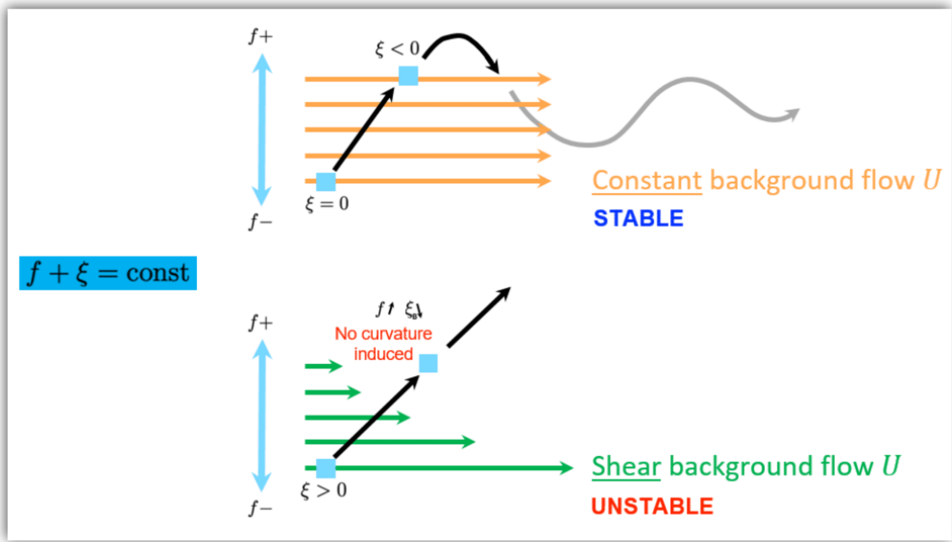

The **diagonal stripes** are the signature of westward propagation. It takes about **five years to cross the Pacific basin**. This cannot be the signature of an external/barotropic Rossby wave, because that would go too fast to be picked up by this altimeter time resolution (dt=10days). It could be the adjustment of the sea level to a perturbation on the thermocline, i.e. the trace of a (slower) baroclinic Rossby wave traversing the Pacific in a few years. However, it is not entirely sure whether this is exactly what it is or whether it has to do with non-linear phenomena like eddies.

# GFD3.3: Barotropic Instability

### *3.3.a) Growing Rossby waves?*

⇨ What happens when we allow **the basic state flow** to become more interesting/complicated?

• Up to now, we have assumed that the basic state flow is just **uniform** westerlies ($U = cst$). Our parcel was displaced around an equilibrium controlled by a horizontal restoring force and it created a Rossby wave (see **#GFD3.1a**).




General idea

⇨ What happens if the basic flow resembles a **sharp jet** in the westerlies, with strong **meridional shear**?

• At the origin, the relative vorticity ($\xi$) is not zero anymore. Imagine dropping a wheel into the flow, it will spin anti-clockwise (because of the shear), i.e. with positive relative vorticity ($\xi > 0$).

↳ - When the parcel is moved north, the planetary vorticity gets bigger ($f$ ↗).

- According to the conservation of potential vorticity (absolute vorticity $f + \xi$), we would expect the relative vorticity to get smaller ($\xi$ ↘).

- but the background flow was chosen such that its shear is smaller there than in the south, so no secondary circulation develops (no curvature is induced). The particle just goes north, as if it is allowed to just take off.

⇨ This is the beginning of the consideration of instability.

### *3.3.b) Perturbations on a parallel shear flow*

$$q = \beta y + \nabla^2 \psi$$

⇨ Let's consider a **barotropic non-divergent flow** on a β-plane ($f = f_0 + \beta y$), with a **geostrophic parallel shear flow in the background**:

$$u = \bar{u}(y) = \frac{1}{\rho f}\frac{d\bar{p}}{dy}$$

⇨ On top of the background flow, we have perturbations:

$$u = \bar{u}(y) + u' = \bar{u}(y) - \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = v' = \frac{\partial \psi}{\partial x}$$

↳ The **momentum and continuity equations** can be written:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

⇨ **Linearization** yields:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} - fv' = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$
$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + fu' = -\frac{1}{\rho}\frac{\partial p'}{\partial y}$$

⇨ The **vorticity equation** is derived by taking the curl of the momentum equations $(\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1))$ and simplifying. It follows: $\frac{\partial \xi}{\partial t} + \bar{u}\frac{\partial \xi}{\partial x} - v'\frac{\partial^2 \bar{u}}{\partial y^2} + v'\frac{\partial f}{\partial y} = 0$

↳ In terms of the stream function, **the linear vorticity equation** on a $\beta$-plane writes:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0$$

↳ We obtain a vorticity equation in which:
- The first term is the advection of perturbation vorticity by the background flow,
- The second term is the advection by the perturbation flow of the absolute vorticity associated with the basic state.

Dispersion relation

⇨ The difference compared to the uniform background flow case (**#GFD3.1c**) is that we have meridional variations (shear) in the basic state flow.

⇨ From the **linear vorticity equation in a parallel shear flow** (see **#GFD3.3b**), we derive a dispersion relation for the Barotropic Rossby waves by introducing solutions of ***plane-wave form***: $\psi = Re\,\tilde{\psi}e^{i(lx+my-\omega t)}$ (similar to **#GFD3.3b**):

$$\omega = Ul - \frac{(\beta - U_{yy})l}{l^2 + m^2}$$

↳ Note the presence of the ***relative vorticity of background flow ($U_{yy}$) in the inner term***.

### *3.3.c) Stationary Rossby Waves*

⇨ Atmospheric scientists are interested in stationary Rossby waves, i.e. Rossby waves which stay in place, with $\omega = 0$. The relationship between the properties of the wave ($l^2$ and $m^2$) and the background flow follows:

$$U(l^2 + m^2) = (\beta - U_{yy})$$

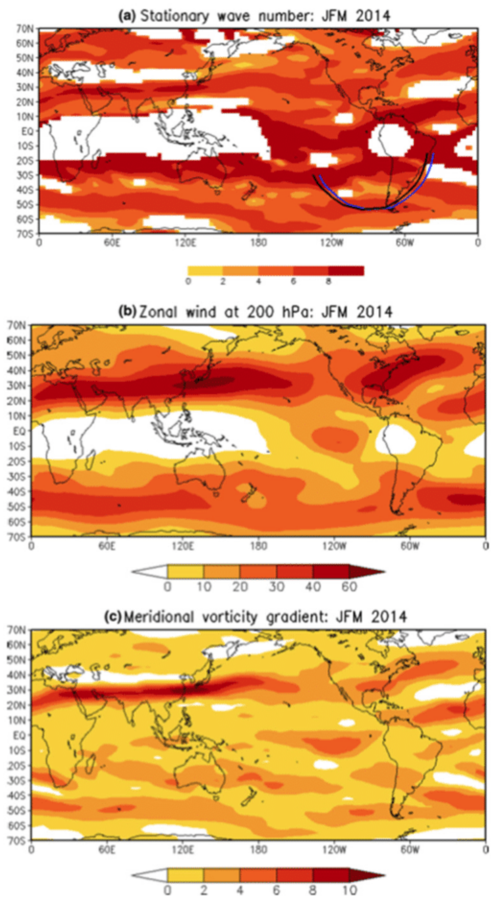
↳ There is a relation between the horizontal wavenumber $k$ ($k^2 = l^2 + m^2$) and the background flow:

The wavelength is $2\pi\sqrt{\frac{U}{\beta}}$

$$k = \sqrt{(\beta - U_{yy})/U}$$

⇨ For the stationary wave to exist, the wavenumber has to be real, i.e. $(\beta - U_{yy})/U$ has to be positive. This means that $\beta - U_{yy}$ must have the same sign as U (*which usually means both must be positive).*

↳ **Fig.c** (from Coelho et al., 2016) shows the meridional gradient term ($U_{yy}$), which is positive almost everywhere. This means that for stationary Rossby waves to exists, we must have easterlies. **Fig.b** shows the associated zonal wind component ($U$) and **Fig.a** represents the stationary wavenumber $k$. White areas denote regions in which $\beta - U_{yy}$ and $U$ have different signs and $k$ is



complex/imaginary. In these regions, stationary or very low-frequency Rossby waves cannot exist. Contrastingly, there is a large area in the eastern Pacific where stationary Rossby waves can exist, this is the pacific waveguide.

⇨ Ray paths of the stationary Rossby waves can also be calculated as before (see **#GFD3.2c**) from the ratio of the zonal group speed ($\frac{\partial\omega}{\partial l}$) and meridional group speed ($\frac{\partial\omega}{\partial m}$):

$$\mathbf{c}_g = \left(U + \frac{\beta_*(l^2 - m^2)}{k^4}, -\frac{2\beta_* lm}{k^4}\right) \qquad (\beta_* = \beta - U_{yy},\ k^2 = l^2 + m^2)$$

↳ Plotting the different components of the group speed provides the theoretical direction of the stationary Rossby wave, as illustrated in **Fig.a**.

### *3.3.d) Growing solutions*

⇨ Whenever we consider waves, the other side of the coin is instability. Let's get on to this idea of a solution that can grow. A gravity wave is stable, while a thunderstorm is unstable. A Rossby wave is stable and barotropic instability leads to rapid development. In **#GFD3.3a**, we considered a parcel of fluid that can take off in the meridional direction.

We are seeking ***unstable solutions*** for the **linear vorticity equation in a parallel shear flow** derived in **#GFD3.3b**:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial\psi}{\partial x} = 0$$

⇨ In the linear vorticity equation, the coefficients of the $x$-derivatives are not themselves functions of $x$. Thus, we may seek **solutions** that are harmonic functions (sines and cosines) in the $x$-direction, but the ***y-dependence must remain arbitrary at this stage*** and we seek solution such that:

$$\psi(x,y,t) = \phi(y)e^{i(lx-\omega t)}$$

↳ We **substitute** this solution into the vorticity equation (see **details of the calculous** on the following page) and it is very similar to what we did with the vertical dependence in **#GFD3.2b**. We obtain a differential equation for $\phi(y)$, namely:

Rayleigh equation

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta - d^2\overline{u}/dy^2}{\overline{u}(y) - c}\phi = 0$$

↳ This is the **linear vorticity equation for disturbances to parallel shear flow**, in which $c = \omega/l$, known as **Rayleigh's equation**. **We are not going to solve this equation**! We are going to analyze it for the possibility of growth.

⇨ The wave part of the solution is trigonometric with imaginary exponentials. But if what is inside the exponential has an ***imaginary part*** then you would get a real exponential.

- If ω is purely real then $c = \omega/l$ is the phase speed of the wave.
- If ω has a positive imaginary component ($\omega_i$) then the wave will grow exponentially and will thus be ***unstable***: $\omega = \omega_r + i\omega_i,\ \ \omega^* = \omega_r - i\omega_i$ is the complex conjugate.

↳ Supposing that $l$ is real, the phase speed $c = \omega/l$ can be complex too:

$$c = c_r + ic_i,\ \ c^* = c_r - ic_i$$

✋ $l$ could be complex but it would not add anything. It would just be more mathematics.

We are interested in whether it is possible for $c$ to have an imaginary part. Because if it does that means ω has an imaginary part, which means there is a possibility of instability. We are going to analyze **Rayleigh's equation** for the possibility of $c$ having an imaginary part.

⇨ ⇨ If we add channel boundary conditions ($\phi = 0$ at $y\ =\ 0, L$), in general, we get a set of solutions for $\phi$ associated with complex conjugate pairs of values for $c$ (or $\omega$). The imaginary part of the solution is associated with growth or decay. The growth rate is the imaginary part of $\omega$.

## *Details for the derivation of the Rayleigh's equation*

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0 \qquad \psi = \phi(y)e^{i(lx-\omega t)}$$

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = \frac{\partial}{\partial x}\left(\phi\, il\, e^{<>}\right) + \frac{\partial}{\partial y}\left(\phi_y e^{<>}\right) = \left(-\phi l^2 + \phi_{yy}\right)e^{<>}$$

$$\psi_x = \phi\, il\, e^{<>}$$

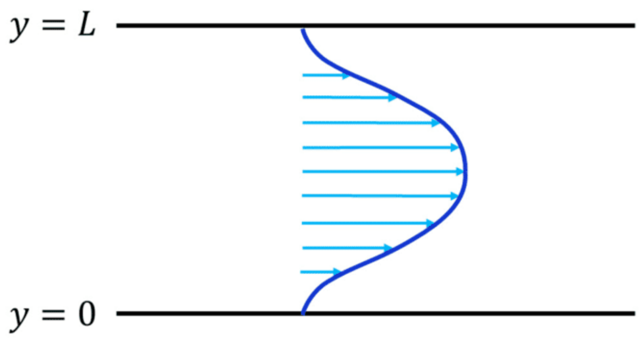
$$-i\omega(-\phi\, l^2 + \phi_{yy}) + il\,\overline{u}(-\phi\, l^2 + \phi_{yy}) + (\beta - \overline{u}_{yy})\phi\, il = 0$$

$$-\frac{\omega}{l}(\phi_{yy} - \phi\, l^2) + \overline{u}(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$-\left(\frac{\omega}{l} - \overline{u}\right)(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$(\overline{u} - c)(\phi_{yy} - \phi\, l^2) + (\beta - \overline{u}_{yy})\phi = 0$$

$$\phi_{yy} - l^2\phi + \left(\frac{\beta - \overline{u}_{yy}}{\overline{u} - c}\right)\phi = 0$$

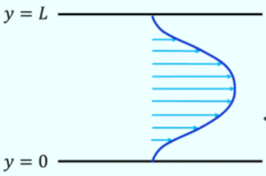


### 3.3.e) Conditions for growth: the Rayleigh criterion

⇨ ⇨ If we add channel boundary conditions ($\phi = 0$ at $y = 0, L$), in general, we get a set of solutions for $\phi$ associated with complex conjugate pairs of values for $c$ (or $\omega$). The imaginary part of the solution is associated with growth or decay. The growth rate is the imaginary part of $\omega$.

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta - d^2\overline{u}/dy^2}{\overline{u}(y) - c}\phi = 0$$

⇨ We **multiply** Rayleigh's equation by **the complex conjugates of $\phi$** and **integrate the equation across the domain from 0 to L**. The two first terms are integrated by parts.




$$\frac{d}{dy}(\phi\phi_y) = \phi_y^2 + \phi\phi_{yy} \implies d(\phi\phi_y) = \phi_y^2\,dy + \phi\phi_{yy}\,dy$$

$$\int \phi(\phi_{yy} - l^2\phi)\,dy = \int (\phi\phi_{yy} - l^2\phi^2)\,dy = \int_0^L d(\phi\phi_y) - \int_0^L (\phi_y)^2\,dy - \int_0^L l^2\phi^2\,dy$$

$$= [\phi\phi_y]_0^L - \int_0^L |\phi_y|^2 + l^2\phi^2\,dy$$

It gives: $$-\int_0^L \left(\left|\frac{d\phi}{dy}\right|^2 + l^2|\phi|^2\right)dy + \int_0^L \frac{\beta - d^2\overline{u}/dy^2}{\overline{u} - c}|\phi|^2 dy = 0$$

↳ ***The equation equates to zero if its real*** (see **#GFD3.3f**) ***and imaginary parts are both zero.***

↳ The first LHS term is positive definite and real. Therefore, if there is anything **imaginary** in this integral, it must be in the second term.

⇨ To get rid of the imaginary part in the denominator, we multiply top and bottom by $(\overline{u} - c)^*$. We get $\overline{u} - c$ on the outside of the integral. We do not care about $\overline{u}$ or the real part of $c$ because they are both real. We only care about the imaginary part of $c$ ($c_i$). This procedure allows us to isolate the imaginary part of the integral:

$$c_i \int_0^L \left(\beta - \frac{d^2\overline{u}}{dy^2}\right)\frac{|\phi|^2}{|\overline{u} - c|^2}dy = 0$$

↳ This quantity has to be equal to zero because it is the only imaginary bit of the whole equation.

↳ Either - $c_i = 0$, i.e. there is no imaginary part for the phase speed and the flow is **stable**

or - $c_i \neq 0$, in which case **the integral must be zero**.



⇨ How can this integral be zero? The ratio is real and positive, which means that $\beta - \overline{u}_{yy}$ is either zero everywhere or ***at the very least*** $\beta - \overline{u}_{yy}$ **must change sign somewhere in the domain** between (0 and $L$).

⇨ This term, which can be written $\frac{d}{dy}(f_0 + \beta y - \overline{u}_y)$, i.e. the meridional **gradient of the absolute vorticity must change sign** somewhere in the domain.

$\beta - \overline{u}_{yy}$ must change sign somewhere in the domain between (0 and L).
↳ If the Rayleigh criterion is satisfied, we might have an instability.

⇨ This is a necessary condition for the integral to be zero, which is

- a necessary condition for the phase speed to have an imaginary part, which is
- a necessary condition for **instability**

↳ This is the **Rayleigh criterion for instability**

⇨ The condition is to have an extremum (maximum or minimum) in the **absolute vorticity**, i.e. its gradient changes sign somewhere in the domain, i.e. the velocity profile has an inflection point.

⇨ This Rayleigh condition is a **necessary condition** for **barotropic instability**.

Fjørtoft criterion

### *3.3.f) More conditions for growth: the Fjørtoft criterion*

⇨ There is another necessary condition called the **Fjørtoft condition**. The Rayleigh condition dealt with the imaginary part of the linear vorticity equation in a parallel shear flow. But the **real part** must also be satisfied, so:

$$-\int_0^L \left(\left|\frac{d\phi}{dy}\right|^2 + l^2|\phi|^2\right)dy + \int_0^L (\bar{u}-c_r)\left(\beta - \frac{d^2\bar{u}}{dy^2}\right)\frac{|\phi|^2}{|\bar{u}-c|}dy = 0$$

The first LHS term is negative, which means that:

$$\int_0^L (\bar{u}-c_r)\left(\beta - \frac{d^2\bar{u}}{dy^2}\right)\frac{|\phi|^2}{|\bar{u}-c|}dy > 0$$

⇨ This inequality $\int A(u-c) > 0$ is similar to the imaginary part of the equation we dealt with deriving the **Rayleigh condition**, i.e. $\int A = 0$

**Fjørtoft logic**:
- It consists of decomposing $(u-c)$ into two terms: $(u-u_0)+(u_0-c)$:

$$\int A(u-u_0) = \int A(u-c) + \int A(c-u_0) > 0$$

- With $(c-u_0)$ being a constant and with $\int A = 0$, the last term cancels.

↳ It follows that $\int A(u-u_0) > 0$ **must be true for any value of $u_0$**

- With $\frac{|\phi|^2}{|\bar{u}-c|^2} > 0$, this means ***at the very least*** $(\bar{u}-u_0)(\beta-\bar{u}_{yy})$ **must be positive somewhere in the domain for any value of $u_0$**.

$\forall u_0$, $(\bar{u}-u_0)(\beta-\bar{u}_{yy})$ must be positive somewhere in the domain.
↳ If the Fjørtoft criterion is satisfied, we might have an instability.

⇨ ***Large values of $u_0$***:
For a very large positive value of $u_0$, $(u-u_0) < 0$ and $(\beta-\bar{u}_{yy})$ must be < 0 somewhere.
For a very large negative value of $u_0$, $(u-u_0) > 0$ and $(\beta-\bar{u}_{yy})$ must be > 0 somewhere.
↳ This a weaker version of saying that $(\beta-\bar{u}_{yy})$ must change sign. This is the **Rayleigh criterion**, saying that the gradient of the absolute vorticity has to change sign in the domain.
↳ So large values of $u_0$ add nothing to the Rayleigh criterion.

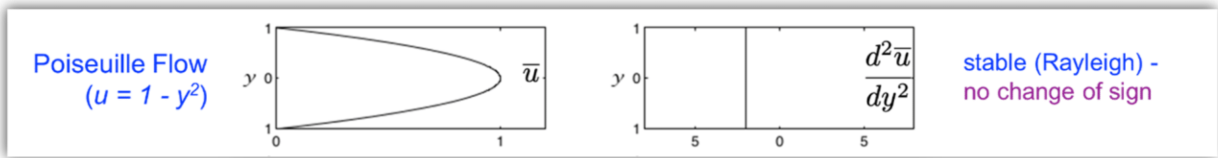
⇨ ***Medium values of $u_0$***:
It is most useful to choose **$u_0$ to be the value of $U(y)$ at which $(\beta-\bar{u}_{yy})$ vanishes**. This leads to the **Fjørtoft criterion**. Moderate values of $u_0$, such as $(u-u_0)$ also changes sign in the domain somewhere, adds an extra criterion which is more of a constraint than just the **Rayleigh criterion** (see example in **#GFD3.3g**). The **Fjørtoft criterion** is satisfied if **the magnitude of the absolute vorticity has an extremum inside the domain**, and not at the boundary or at infinity – the velocity profile must have an inflection point inside the flow. **Fjørtoft** necessary condition is **thus a stronger condition** than the **Rayleigh criterion**.

✋ Both **Rayleigh** and **Fjørtoft** criteria are just ***necessary conditions***. They are not sufficient conditions. This means that, when analyzing a potential vorticity map, if one of these conditions is satisfied, it does not mean that the flow is unstable, it means that ***it is possible for the flow to be unstable***.

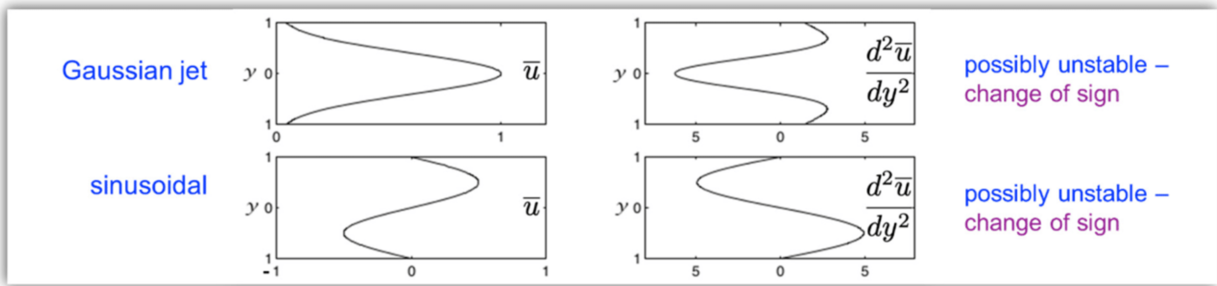
On the other hand, the non-satisfaction of a necessary condition is a sufficient condition, which means that ***if the Rayleigh or the Fjørtoft condition is not satisfied then the flow is stable***.

### *3.3.g) Stable and unstable profiles*

⇨ In the examples below, parallel shear-flow could induce instability. The left column displays the zonal component of the flow, while the right column shows the associated second derivative.



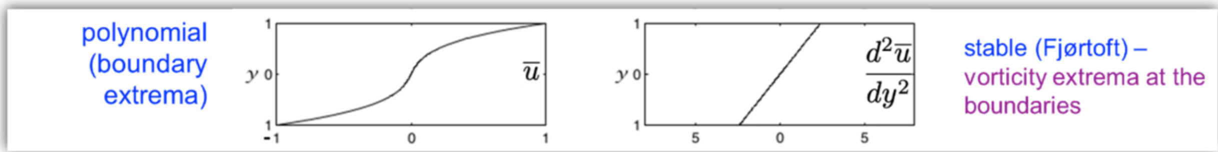

**1)** A Poiseuille flow corresponds to the quadratic form of a viscous fluid flowing in a pipe. The derivative of the vorticity is a constant that does not change sign in the domain. By **Rayleigh's criterion**, **it must therefore be stable**.




**2)** A Gaussian jet has an exponential form with extrema. It is therefore potentially **unstable**.

**3)** A sinusoidal profile has another sinusoidal function as its derivative. Likewise, this flow is potentially **unstable**.

The ***β-effect can be either stabilizing or destabilizing***. If the $\beta$-effect were present and large enough to have $(\beta - \bar{u}_{yy})$ one-signed, it would stabilize the Gaussian or sinusoidal jets.



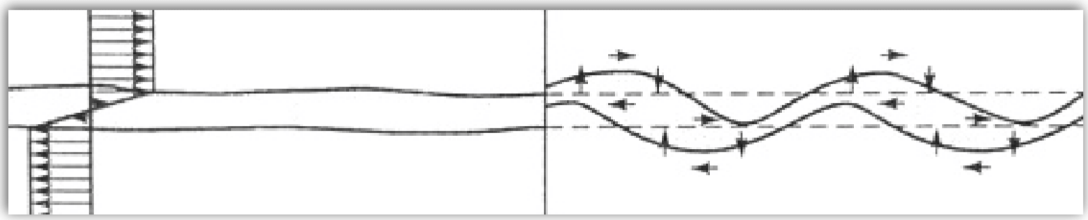

**4)** This third-order polynomial profile is **stable** by **Fjørtoft's criterion** (note that the vorticity extrema are at the boundaries).

- By the **Rayleigh** criterion, it could be unstable because the basic flow vorticity has extremes.

- The **Fjørtoft's** criterion dictates that $\bar{u}$ has to have the same sign as $-\bar{u}_{yy}$ somewhere in the domain. Here, they have opposite signs everywhere. It thus fails **Fjørtoft's** criterion. Fjørtoft's $u_0$ constant could shift $\bar{u}$ but the sign requirement must be true for all values of $u_0$. In this case, it fails for $u_0 = 0$. The polynomial profile is thus **stable**.

### *3.3.h) Physical mechanism*

How does the flow become barotropically unstable? Here is an example of a background flow:

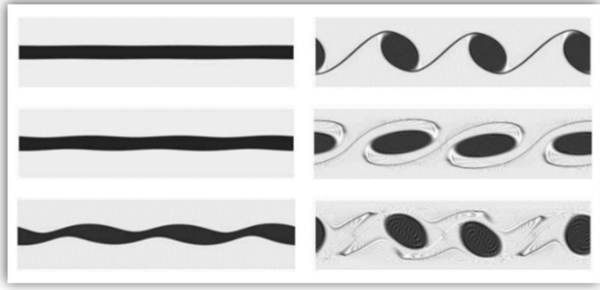


▪ To the **south**, we have uniform **easterlies**, and to the **north**, uniform **westerlies**. In these two regions, there is no background vorticity.

▪ In between, there is a **transition zone** with a strip of parallel shear flow, i.e. a strip of background negative vorticity (clockwise) – an extremum. The flow is potentially unstable.

Consider a small perturbation, such as a streamline is moved slightly to the north. It exports its vorticity into a region where there is none. At the same time, on the other side of the vorticity strip, but just out of phase, the same thing happens.

The secondary circulation is going to displace the vorticity contours, so that it deforms the vorticity strip and the situation amplifies and the deformation continues.



## GFD3.4: Baroclinic Instability

### *3.4.a) Baroclinic instability*

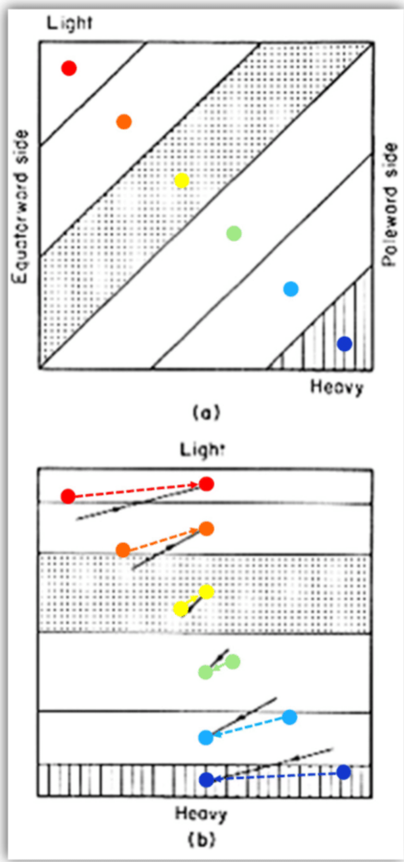
It is similar but for a baroclinic flow, in which there is a **vertical dependence in the background flow**. ***How would a perturbation grow on this sort of flow?***

Let's start by thinking about energy. Instabilities are growing perturbations, where do they get their energy from?

- We have seen in **#GFD3.3** that barotropic instabilities take their energy out of **the horizontal shear** of the background flow.
- Baroclinic instability also takes its energy from some property of the background flow.

Let's consider the ***configuration shown in the figure below***. Upward is at the top and northward to the right.

We define a very ***idealized geophysical situation*** on a rotating planet, in which there are tilted layers of different densities. To the north and at low levels we have heavy dense fluid (cold water in the ocean). Towards the equator and at upper levels the water gets warmer and lighter. In between, there are tilted homogeneous density layers. In each of these layers of different density, the ***colored dot is placed at the center of gravity*** of the layer. Note that if it is in the atmosphere, we need to take into consideration the compressibility of air and we must consider potential temperature.




***If we take all these layers and flatten them out, where would the center of gravity go?*** Imagine filling the same amount of space and laying out each layer horizontally. The densest layer is spread horizontally at the bottom, while the lightest layer becomes the surface layer of our ocean. The center of gravity of the denser layers has moved downwards and for the lighter layers it has moved upwards as the fluid rearranges. ***The center of gravity of the whole fluid would go down*** because heavy layers have more influence on the total center of gravity than the lighter layers.

**By rearranging the fluid, we have moved the center of gravity downwards**. This means **we have liberated potential energy** to supply kinetic energy. Release of instability can be considered as a transfer of energy from a basic state to a flow.

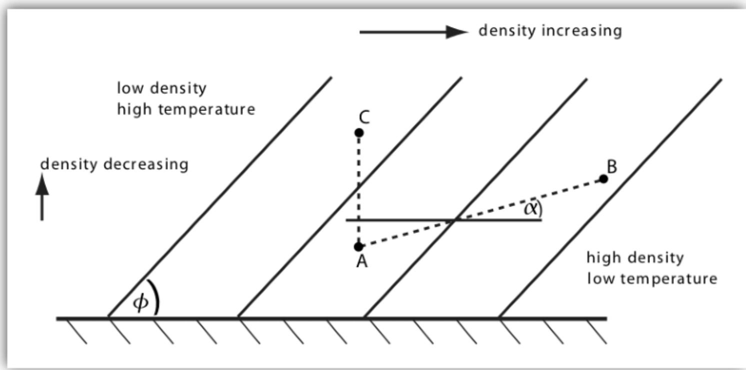
### *3.4.b) Sloping convection*

***Sloping convection*** is another way of thinking about baroclinic instability.
Is the following structure stable to perturbations?

- We consider a fluid that is **barotropically stable**, i.e. there is no reversal of the barotropic vorticity gradient (Rayleigh criterion, see **#GFD3.3e**), a parcel of fluid is not going to take off.
- The fluid is also **statically stable**, i.e. we do not have vertical gravity instability (cold air is at the bottom and warm air is at the top).
- Density contours are tilted, so that (potentially) cold air remains in the down/north region, while potentially warm air is up and south. In a rotating system, we can imagine a steady basic state with inclined density contours (we need rotation to balance the pressure gradient forces).

***Displacement A-C***: A parcel of fluid A is displaced vertically into position C. As it moves into this lighter layer, the parcel will be heavier than its surroundings and it is going to drop back down. This is static ***stability***.

***Displacement A-B***: A parcel of fluid A is displaced northwards into position B, along a slope. It moves into denser air and it is lighter than its surroundings. It can thus keep on going up and north. This a potential ***instability***/baroclinic instability called ***Sloping convection***. The energy stored in the density structure is released.




### *3.4.c) Optimal scales for growth*

⇨ At what kind of <u>scales</u> does this happen? One of the things that is important to understand is that the process of baroclinic instability depends on some sort of communication between different levels, and there are certain scales on which that happens.

⇨ Let's go back to the definition of the quasi-geostrophic potential vorticity equation ($f$-plane Boussinesq, see **#GFD2.3i**) and do a basic scale analysis:

- The <u>relative vorticity</u> ($\nabla^2\psi$) is of the order of $\sim\psi/L^2$, with $L$ the typical length scale at which we have vorticity gradients.
- The <u>vortex stretching term</u> is of the order of $\sim \dfrac{f^2\Psi}{N^2H^2} = \dfrac{\Psi}{{L_R}^2}$

↳ This means that these two terms are of comparable magnitude when $L$ is comparable to $L_R$. On length scales comparable to the Rossby radius, both of these terms will be important and this is what we need to amplify perturbations:

▪ if $L \gg L_R$, the relative vorticity term will be small and vertical coupling will dominate. There will not be much difference between the top and bottom layers. At these scales, the fluid will essentially be **barotropic**.

▪ if $L \ll L_R$, then the relative vorticity will dominate and there will not be any coupling between the layers. The fluid would behave just like uncoupled/**independent layers**.

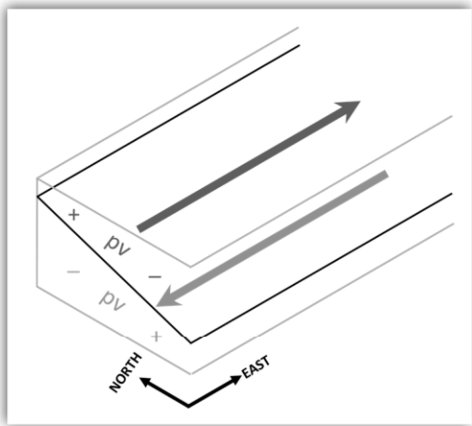
▪ **On the particular length scale of the Rossby radius**, there is some interplay between these two terms and this will allow **the liberation of potential energy stored in the background state** (horizontal variations of density or vertical shear of the wind).

⇨ If we are right that the Rossby radius scale is the scale on which perturbations can grow, then we should see these scales naturally in a geophysical fluid, just like ***Darwinian selection***. We observe the scales that amplify. If the mechanism of amplification depends on it being a certain scale then this is the scale that should be seen on weather maps or diagnostics of ocean variability.

↳ This is the scale we indeed observe: when looking at weather maps (see **#GFDintro**), we see cyclones and anticyclones. Altimetric sea surface height show ocean eddies - all on the Rossby radius scales $\left(L_R = \frac{NH}{f}\right)$.

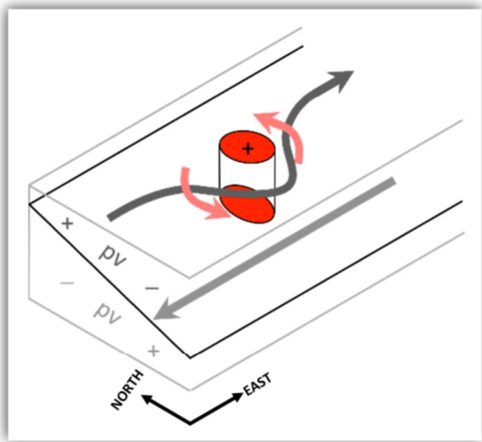
### *3.4.d) Physical mechanism*

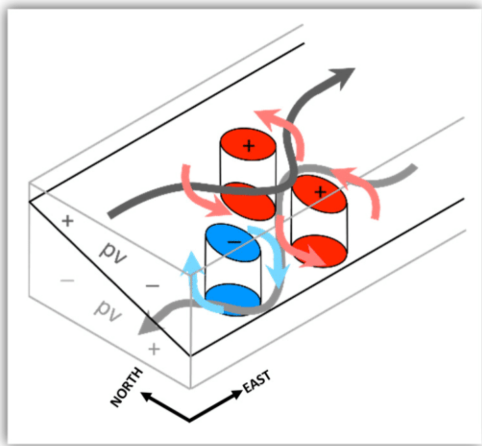
⇨ ***How does it work?*** Here is a schematic to explain the physical mechanism.




1) Consider a ***two-layered shear flow in thermal wind balance***. There are two layers with a slope between them. In the upper layer, the current is flowing eastwards, while in the lower layer, it is flowing westwards. The slope means that the layer thickness varies from north to south and accordingly the potential vorticity for the upper layer increases towards the north. In the lower layer, it is the other way around.

2) We introduce a positive potential vorticity **perturbation** (PV+) into the top layer with an associated cyclonic flow that diverts the upper-layer eastward jet.




3) Positive vorticity is associated with positive layer thickness that will **squeeze the layer below** and **drive a circulation in the same way**. In the lower layer, west of the upper-layer perturbation, this circulation will advect PV- southward, and east of the upper-layer perturbation, it will advect PV+ northwards creating **a perturbation dipole in the lower layer**. This will also divert the lower-layer westward jet.

4) In the center of the dipole, there is a southward component in the lower-layer flow, which in turn will **impact the upper layer dynamics**. This induces southward advection of more positive potential vorticity in the layer above, **amplifying the original perturbation**, which will grow.

↳ If there is the right phase relation between perturbations, they can mutually amplify and grow. In this configuration, there is **a slope of the dynamical perturbation towards the west with height** and which is consistent with the extraction of energy from the basic state sloping density surfaces to produce a circulation anomaly which can grow exponentially.

✋ At the same time, due to the upper-level potential vorticity gradient and the gradient of $f$, the entire structure propagates westwards (relative to the mean flow) as a Rossby wave.

### *3.4.e) Modal solutions*

⇨ Here is an overview of the **theoretical framework** in order to analyze under which ***conditions baroclinic instability is possible***.

↳ Here is a reduced version of the linear perturbation PV equation, in which $q'$ is a perturbation potential vorticity and $Q$ is the potential vorticity associated with the background state.

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial Q}{\partial y} = 0$$

⇨ We seek wave-like solutions in $x$ and the amplitude coefficients as a function of $y$ (as in **#GFD3.3d**) and also of $z$ because of the presence of a vertical component in the variations of the background flow, $\psi' = \tilde{\psi}(y,z)e^{i(lx-\omega t)}$. We substitute this solution into the linear vorticity equation:

$$(U-c)(\tilde{\psi}_{yy} + \frac{\partial}{\partial z}\frac{{f_0}^2}{N^2}\tilde{\psi}_z - l^2\tilde{\psi}) + Q_y\tilde{\psi} = 0$$

With the boundary conditions (equivalent to $w = 0$) at top and bottom: $(U-c)\tilde{\psi}_z - U_z\tilde{\psi} = 0$

↳ This is the equivalent of our Rayleigh equation in the barotropic case (see **#GFD3.3d**) but this equation contains both horizontal and vertical derivatives.

### *3.4.f) Conditions for growth*

⇨ In order to analyze the conditions for growth, as in **#GFD3.3e**, we multiply the equation by the complex conjugates of $\tilde{\psi}$ and integrate the equation across the whole domain, i.e. in the north-south direction (from 0 to L) and also in height (from 0 to H). It leads to:

$$\int_0^L\int_0^H |\Psi_y|^2 + f_0^2/N^2|\tilde{\psi}_z|^2 + l^2|\tilde{\psi}|^2 dzdy - \int_0^L\left\{\int_0^H \frac{Q_y}{U-c}|\tilde{\psi}|^2dz + \left[\frac{f_0^2/N^2U_z|\tilde{\psi}|^2}{U-c}\right]_0^H\right\}dy = 0$$

↳ The first LHS term is ***positive definite and real***. Therefore, if there is anything imaginary in this integral, it must be in the second term.

⇨ We analyze this term for the ***possibility of it having an imaginary part*** for the phase speed. To get rid of the imaginary part in the denominator, we multiply top and bottom by $(\bar{u}-c)^*$, and isolate the imaginary part:

$$-c_i\int_0^L\left\{\int_0^H \frac{Q_y}{|U-c|^2}|\tilde{\psi}|^2dz + \left[\frac{f_0^2/N^2U_z|\tilde{\psi}|^2}{|U-c|^2}\right]_0^H\right\}dy = 0$$

↳ If $c_i \neq 0$ then the integral must be zero. Instead of just having one criterion, the Rayleigh criterion (see **#GFD3.3e**), we need to think about all the circumstances in which this integral could be zero:

- Disregarding the second term (no vertical shear of the background flow, $U_z = 0$), there is the same condition as for the barotropic case, i.e. the basic state potential vorticity gradient ($Q_y$) could change sign somewhere in the domain.
- Disregarding the first term (no horizontal gradient of background PV, $Q_y = 0$), the vertical shear ($U_z$) has to have the same sign at the top ($z = H$) and bottom ($z = 0$).

*Then there is an interplay between the two terms. If these two terms have the opposite sign, it means:*

- The gradient of potential vorticity ($Q_y$) has to have the opposite sign to the vertical shear ($U_z$) at the top level ($z = H$), or
- $Q_y$ has the same sign as a vertical shear ($U_z$) at the bottom level ($z = 0$).

⇨ There are **4 possibilities**, called the **Charney-Stern-Pedlosky criteria**. These are necessary conditions but not sufficient conditions: ***if at least one of these four criteria is satisfied then we might have an instability***. If this the case, then waves can grow either in the interior of the fluid if we have a PV extremum for instance or on the boundaries if we have temperature gradients on the boundaries.

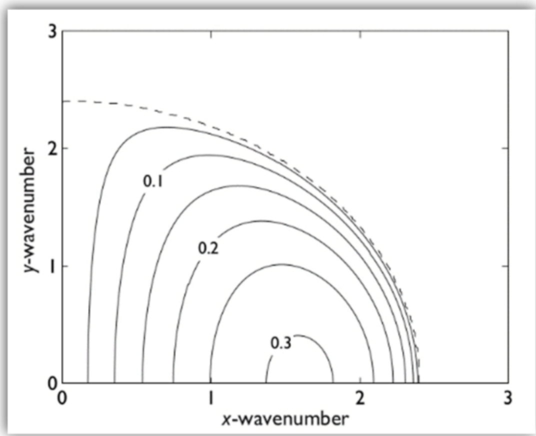
#### *3.4.g) The Eady problem*

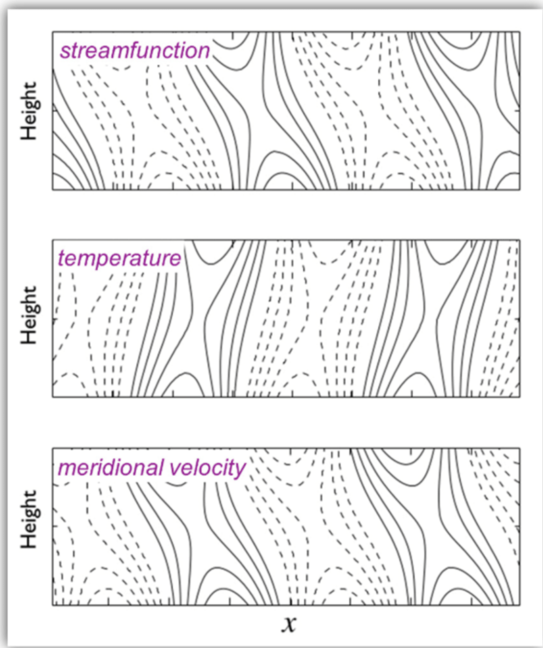
The analytical solution can be derived for a simple configuration. It is the Eady problem, in which:

- The motion is on an $f$-plane ($\beta = 0$).
- The fluid is uniformly stratified ($N^2$ is constant).
- The basic state has uniform shear: $U(z) = Uz/H$.
- The motion is contained between two rigid, flat horizontal surfaces.

↳ Constant vertical shear implies that the ***basic state PV is zero*** (Q = 0), which makes the Eady problem a special case that can be ***solved analytically***. ***Solutions have modes that grow on the boundaries***.

The non-dimensionalized **growth rate** as a function of the zonal and meridional wavenumbers (non-dimensionalized by the Rossby radius: figure on the right) shows stable conditions for short-waves and for any given zonal wavenumber the most unstable wavenumber is that with the gravest meridional scale. This figure also highlights the scale of the maximum exponential growth, close to the Rossby radius.




▪ The maximum growth rate is

$$\sim 0.31\,\frac{U}{L_R}$$

▪ Wavenumber and wavelength at which the instability is the greatest are:

$$k_m = \frac{1.6}{L_R} \qquad \lambda_m = \frac{2\pi}{k_m} = \frac{2\pi}{1.6} L_R$$

▪ The structures of these modes for the most unstable Eady mode are tilted with height towards the west.

*More details can be found in Vallis (2017)*

#### *3.4.h) What we learn from the Eady problem*

- The maximum growth rate is $0.31U/L_R$ and there is a length scale associated with the maximum instability, close to the Rossby radius scale (a factor of 3.9).
- There is a short-wave cutoff – short-waves are not unstable.
- The circulation (meridional current, stream-function) must slope westwards with height in westerly shear to extract energy from the basic state.

✋ In the Eady problem, the instability relies on an interaction between waves at the upper and lower boundaries. If either boundary is removed, the instability dies.

↳ To get a qualitative sense of the nature of the instability, we choose some typical parameters:

⇨ ***For the ocean***, we choose:

$$H \sim 1km, \qquad U \sim 0.1m.s^{-1}, \quad N \sim 10^{-2}s^{-1}$$

We then obtain:

$$\text{Rossby radius } L_R = \frac{NH}{f} \approx \frac{10^{-2} \times 1000}{10^{-4}} \approx 100km$$

$$\text{Instability scales } \sim 3.9 \times L_R \approx 400km$$

$$\text{Growth rate } \sim 0.3 \frac{U}{L_R} \approx \frac{0.3 \times 10}{10^5} \approx 0.026\ day^{-1} \quad (\text{Period} \approx 40\ days)$$

⇨ ***For the atmosphere***:

$$H \sim 10km, \qquad U \sim 10m.s^{-1}, \quad N \sim 10^{-2}s^{-1}$$

We then obtain:

$$\text{Rossby radius } L_R = \frac{NH}{f} \approx \frac{10^{-2} \times 10^4}{10^{-4}} \approx 1000km$$

$$\text{Instability scales } \sim 3.9 \times L_R \approx 4000km$$

$$\text{Growth rate } \sim 0.3 \frac{U}{L_R} \approx \frac{0.3 \times 10}{10^6} \approx 0.26\ day^{-1} \quad (\text{Period} \approx 4\ days)$$

↳ The time scale is a few days for a ***weather system***

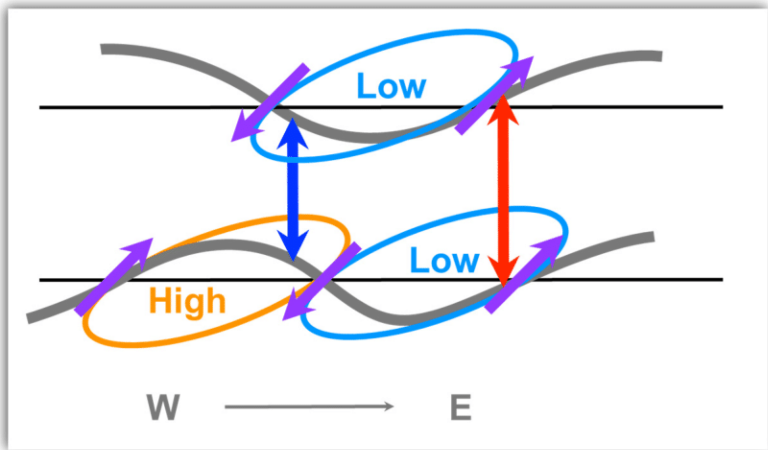
### *3.4.i) Heat transport in a baroclinic system*

⇨ Baroclinic instabilities are important for the **climatic system**.

1) Consider how **radiative forcing** heats the equatorial region and cools at the poles. There is a **zonal jet** flowing horizontally between the two regions, consistent with **thermal wind balance**.

2) If this radiative forcing persists, the **jet will get stronger** as the equator gets warmer and the pole will get cooler.

3) At some point, the jet will **break out into eddies**. The origin of the growth is the **unstable profile** either in the horizontal or in the vertical direction (or both) - baroclinically unstable. The perturbations that grow will have **this shape**:



▪ A **low-pressure anomaly in the upper layer** with the associated pressure surface dipping downwards. In the lower layer, there is **a dipole of high and low-pressure**, slightly shifted.

▪ The distance/**thickness** between surface pressure level is indicative of the **temperature** ($\frac{\partial\psi}{\partial z}$):

- East of the upper level Low, they are far apart: the air is warm.
- West of the upper level Low, they are close together: the air is cold.

▪ The perturbation flow advects the **warm air towards the north**, while cold air is advected to the south. Such a weather system transports warm air upward and poleward and cold air downward and equatorward.

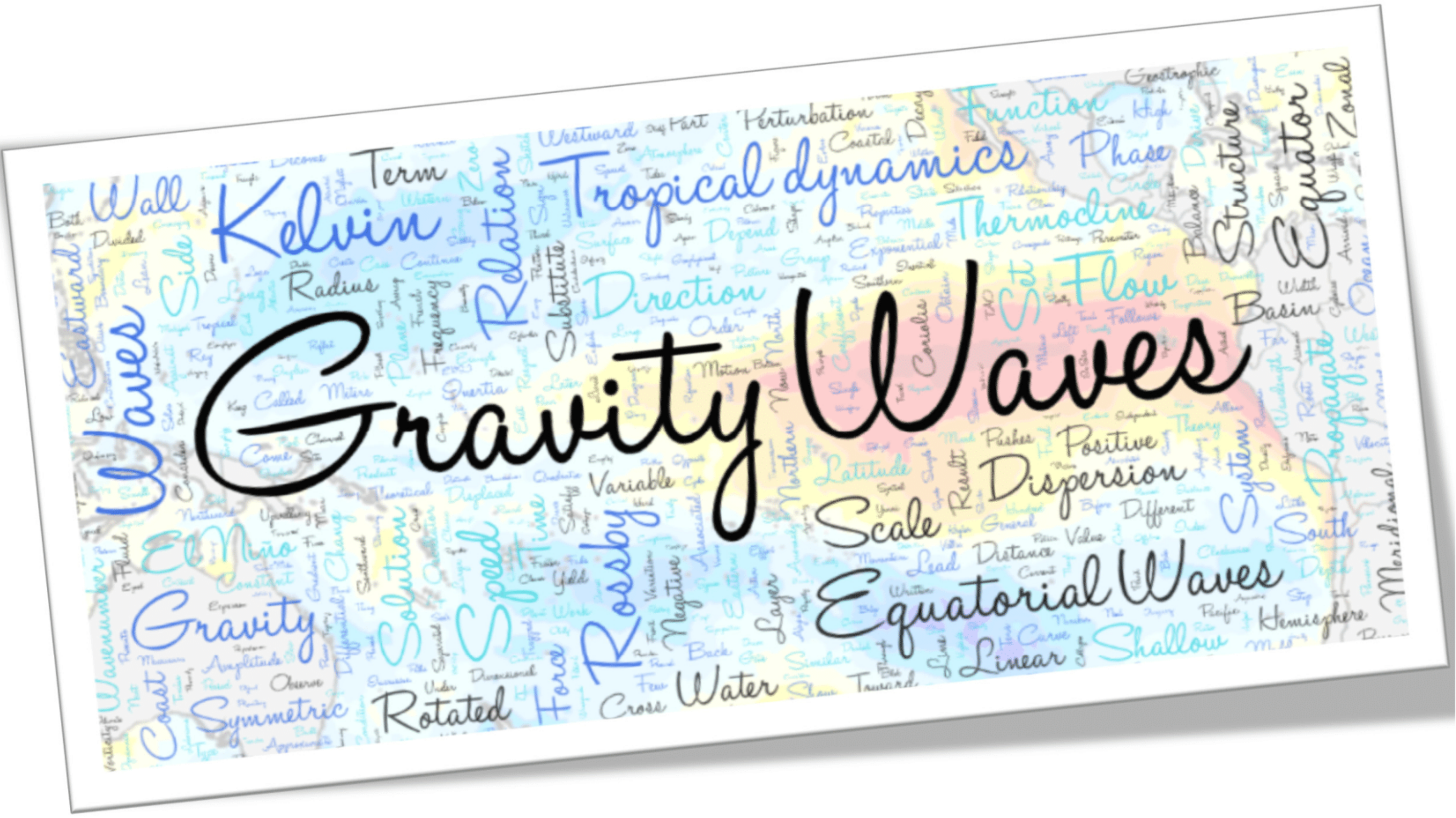
⇨ On the one hand, this configuration is the configuration that **perturbations need to exist** and grow. A configuration that has this **westward slope with height** leads to the **extraction of energy from the background state**.

⇨ On the other hand, this configuration is the configuration required to **transfer heat to the north** and thus **reduce the temperature gradient** between the equator and the pole, releasing the instability, flattening the isentropic slopes that are continually built up by the radiative forcing, and **dissipating the background jet**. This is an example of scale interaction (see **#GFD5**).

### *3.4.j) Baroclinic instability: summary*

1) There is clear evidence of a preferred scale for turbulent motion in the ocean and the atmosphere.
2) Simple scaling arguments and more sophisticated stability analyses show that there is a preferred scale for growth to occur.
3) If this growth depends on extracting energy from sloping density surfaces (or equivalently vertical wind shear or horizontal temperature gradients) then there must be an interplay between vortex stretching and relative vorticity terms in the conservation of potential vorticity.
4) This naturally select structures around the Rossby radius scale.
5) These structures can grow exponentially provided certain criteria are met, notably if there are extrema (maxima and minima) in the potential vorticity of the basic state.

# CHAPER 4
# Gravity waves and tropical dynamics



# CHAPTER 4
# Gravity Waves and Tropical Dynamics

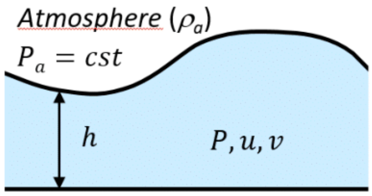
**GFD4 Contents**

This chapter focuses on gravity waves, which leads to the study of coastal Kelvin waves (see **#GFD4.2**) and extends the theory to equatorial Kelvin waves (see **#GFD4.3** and **#GFD4.4**). Then, we continue by discussing equatorial wave dynamics and its effects on tropical ocean variability (see **#GFD4.5**).

*QG theory filters out fast gravity waves (see **#GFD2.3d**), so we come back to the one-layer shallow-water equations derived in **#GFD1.2***

## GFD4.1: Gravity Waves in a Rotating Fluid

### *4.1.a) Gravity waves in shallow water*




⇨ Here is the **one-layer shallow water system** ($x$, $y$-momentum, and continuity equations, see **#GFD1.2f**):

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

($H$)

*Waves are the linear solutions of the equations*

⇨ Let's start with something simple: a **one-dimensional non-rotating linear system**. The shallow water system can be simplified as follows:

- A **one-dimensional system** in the $x$-direction, so we ***cross out the y-direction terms*** (no $v$ and no variations in $y$) and ***the y-momentum equation***.
- A **non-rotating system** means we ***cross-out the two Coriolis terms***.
- A **linear system**, so we can ***eliminate all the term which are quadratic in state variables*** (i.e. the advection terms).

↳ There is a little subtlety in the continuity equation, as we do not completely remove $h\frac{\partial u}{\partial x}$. We consider a **constant** average layer thickness $H$, such that $h = H + \eta$ and linearize this quadratic term by eliminating the product between the two state-variables ($u$ and the variations in layer thickness $\eta$), so $h\frac{\partial u}{\partial x} \approx H\frac{\partial u}{\partial x}$.

⇨ This results in two equations: $\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$ $\quad \frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}$

↳ We then differentiate the $x$-momentum equation with respect to $t$ and differentiate the continuity equation with respect to $x$, thus eliminating $\eta$. This leads to the following **second-order ordinary differential equation** for $u$:

$$\frac{\partial^2 u}{\partial t^2} = gH\frac{\partial^2 u}{\partial x^2}$$

*Trigonometric function with imaginary exponential*

⇨ The solution for this wave equation writes: $u = Re\,\tilde{u}\,e^{i(lx-\omega t)}$

▪ It is the real part of some amplitude coefficient $\tilde{u}$ times the classic imaginary exponential propagation part:

- $l$ is the **zonal wavenumber** ($2\pi$ divided by the $x$-wavelength),
- ω is the **angular frequency** ($2\pi$ divided by the period).

This is a wave that propagates in the positive $x$-direction when $l > 0$.

*Sine propagating in the $x > 0$ direction*

▪ Taking a derivative of this trigonometric function yields the same function multiplied by some constant coefficients:

$$\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \frac{\partial}{x} \rightarrow il$$

↳ Substituting the solution and its derivative into the wave equation results in **a relation between frequency ω and wavenumber $l$** (with two other geophysical parameters: gravity $g$ and average layer thickness $H$): $\omega^2 = gHl^2$ . This is the simple **dispersion relation** of a gravity wave, for which the **phase speed** is constant:

$$c = \frac{\omega}{l} = \pm\sqrt{gH}$$

*All the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape along its propagation*

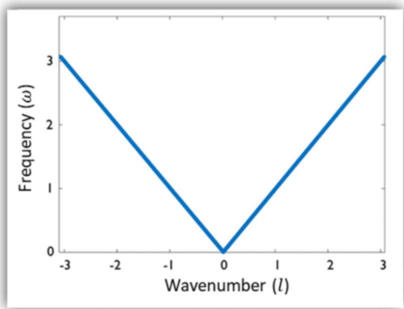
2d-order Equation

Solution and Properties

Dispersion relation

▪ The phase speed does not depend on wavelength or frequency. Waves with different wavelengths (or structures made up of a collection of different wavelengths) travel at the same speed and will propagate without losing their shape. We say that these waves are **non-dispersive**.

▪ Their group speed $\frac{d\omega}{dl}$ remains the same as the phase speed because it is just a linear relationship between $\omega$ and $l$.




#### *4.1.b) Adding rotation*

⇨ The next step is to put the rotation back into the linear system. We **put the Coriolis terms back**. As the Coriolis force pushes perpendicular to the direction of movement, we have to go back to a ***two-dimensional situation*** with ***3 equations*** again. These are the single-layer linear shallow water equations on a flat bottom and an $f$-plane with linear perturbations in $u$, $v$ and $\eta$:

$f = f_0$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

**METHOD1**: To solve this system, we can use the same method we just used, i.e. judiciously differentiate the equations in order to eliminate two of these three variables. This will lead to one high-order differential equation for one variable ($u$, $v$, or $\eta$). *You might have done this before...*

1) Derive the vorticity equation ($d(2)/dx - d(1)/dy$)
2) Derive the divergence equation ($d(1)/dx + d(2)/dy$)
3) Substitute the vorticity equation into the continuity equation
4) Substitute the divergence equation into the resulting equation
5) Differentiate with time and substitute with equation from step (3)

↳ With appropriate initial condition at $t = 0$, departures from geostrophic equilibrium follow:

$$\eta_{tt} - gH\nabla^2\eta + f^2\eta = 0$$

↳ Searching for plane-wave solutions ($\eta = \tilde{\eta}e^{i(lx+my-\omega t)}$) yields the dispersion relation:

$$\omega = \pm\sqrt{f^2 + gHk^2}$$

**METHOD2**: We can employ a more general (clever) method for finding wave solutions.

↳ It consists of substituting the plane-wave solutions $(u, v, \eta) = (\tilde{u}, \tilde{v}, \tilde{\eta})e^{i(lx+my-\omega t)}$ into the 3 equations separately.

▪ Solutions have the form of an amplitude coefficient times an imaginary exponential:

- $l$ is the **zonal wavenumber** ($2\pi$ divided by the $x$-wavelength),
- $m$ is the **meridional wavenumber** ($2\pi$ divided by the $y$-wavelength)
- $\omega$ is the **angular frequency** ($2\pi$ divided by the period).

▪ The derivatives become coefficients: $\frac{\partial}{\partial x} \rightarrow il\,\times$ $\quad \frac{\partial}{\partial y} \rightarrow im\,\times$ $\quad \frac{\partial}{\partial t} \rightarrow -i\omega\times$

$$-i\omega\tilde{u} - f\tilde{v} = -igl\tilde{\eta}$$

$$-i\omega\tilde{v} + f\tilde{u} = -igm\tilde{\eta}$$

$$-i\omega\tilde{\eta} + H(il\tilde{u} + im\tilde{v}) = 0$$

↳ Substituting the solution and its derivatives into the linear system results in a set of three algebraic equations, in which the three unknowns are the coefficients of amplitude ($\tilde{u}, \tilde{v}$, and $\tilde{\eta}$).

↳ The parameters are the wave properties ($l, m$, and $\omega$) and the geophysical constants ($f, g$, and $H$).

### *4.1.c) Inertia-gravity (Poincaré) waves*

⇨ We can write this set of equations in matrix form, resulting in an algebraic system:

$$\begin{pmatrix} -i\omega & -f & igl \\ f & -i\omega & igm \\ ilH & imH & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

▪ This equation is trivially satisfied if there is no wave-like perturbation ($\tilde{u} = \tilde{v} = \tilde{\eta} = 0$).

▪ The condition for the system to be satisfied and for the wave to have some amplitude is that ***the determinant of the matrix must be zero***.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \bullet \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \bullet \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \bullet \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

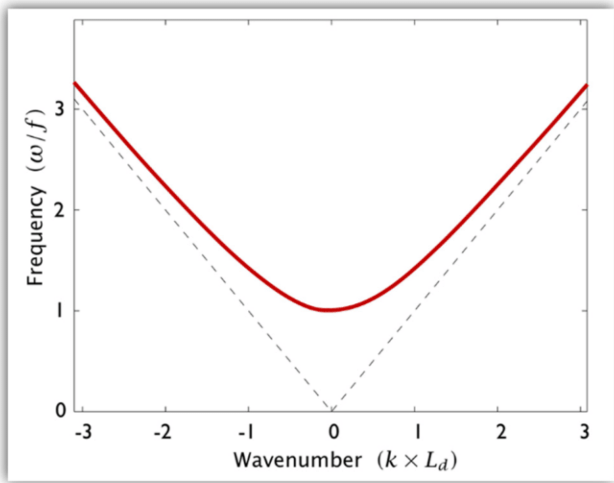
↳ It results in the dispersion relation for gravity waves in a rotating fluid:

$$\omega\left[\omega^2 - f^2 - gH(l^2 + m^2)\right] = 0$$

*k is the horizontal wave number: $k^2 = l^2 + m^2$*

⇨ The **solutions** are either:

- a steady geostrophic flow ($\omega = 0$, no oscillation in time ≡ a fixed stationary wave)
- a propagating wave that satisfies: $\omega = \pm\sqrt{f^2 + gHk^2}$




➢ If we set $f = 0$ we recover the dispersion relation for gravity waves in a non-rotating fluid (see **#GFD 4.1a**), i.e. non-dispersive waves with a constant phase speed $c = \pm\sqrt{gH}$.

➢ The additional $f^2$ under the square root means that the relationship between $\omega$ and $l$ is **not linear anymore**.

↳ It is shown in the figure to the left: frequency as a function of wavenumber. When $k > 0$, the wave propagates in the positive $x$-direction, and when $k < 0$, it propagates in the opposite direction.

▪ Dashed lines are **non-dispersive gravity waves without rotation** (see **#GFD4.1a**).

▪ The red curve shows the system with rotation, i.e. adding $f^2$ under the square root in the dispersion relation. These are ***inertia-gravity or Poincaré waves***.

➢ For short-waves (large values of the wavenumber $l$) rotation does not make much difference to the way the waves propagate. They behave like ordinary, non-dispersive gravity waves.

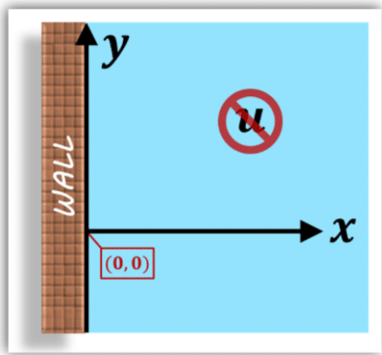
➢ For larger scales (wavelength much longer than the Rossby radius $\sqrt{f^2/gH}$), the **curve flattens out**, so the frequency has a **lower limit** of $f$, and the waves become **very dispersive**. At very small wavenumbers, the wave starts to behave rather oddly. As the horizontal scale of the wave becomes larger, the phase speed becomes faster. The slope of a line joining the origin to the curve gets steeper (see **#GFD3.1d**). But the group speed (the tangent to the curve, see **#GFD3.1d**) is equal to the phase speed for short waves and then for larger scales, it disappears. So, there is **no transmission of information from one position to another**, even though the **oscillations that are separated in space are perfectly coherent**. This is not really a wave anymore. It is **coherent oscillations in space separated by some distance**. In fact, it is just motion in **inertial circles**. This is why the waves are called inertia-gravity waves.

↳ It is a bit of a negative result. For large-scales, we have waves that basically collapse to inertial motion. ***We are left wondering if there is a way in which we can have large-scale propagating geophysical waves in a rotating planet***. The answer is yes, and they are called Kelvin waves (see **#GFD4.2** and **#GFD4.3**).

## GFD4.2: Boundary Kelvin Waves

### *4.2.a) Adding a wall*

⇨ We need to introduce a constraint to the equations to add a lateral boundary to the problem.




$$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0$$

We put a north-south wall on the western side of the Ocean and the flow cannot cross the wall, i.e. no flow perpendicular to the wall

⇨ In the $x$-direction, there is no flow through the wall (at $x = 0$), which suggests that we look for a solution with **$u = 0$ everywhere**.

↳ This results in **geostrophic balance** (equilibrium between pressure **$F_P$** and Coriolis force **$F_C$**) in the $x$-direction.

↳ In the $y$-direction, we recover the equations for non-rotating shallow-water gravity waves (see **#GFD4.1a**). So, in the $y$-direction, we have non-dispersive gravity waves propagating northwards or southwards with a fixed phase speed, independent of horizontal scales ($|c| = \sqrt{gH}$):

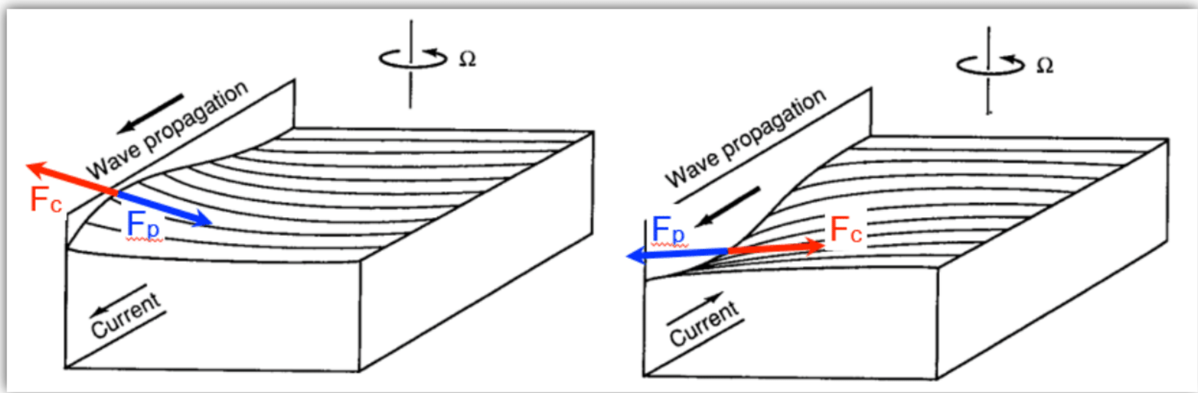
$$fv = g\frac{\partial \eta}{\partial x}$$

*Diagnostic equation:* ***Geostrophic balance***

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial v}{\partial y} = 0$$

*Prognostic equations:* ***Non-dispersive waves***



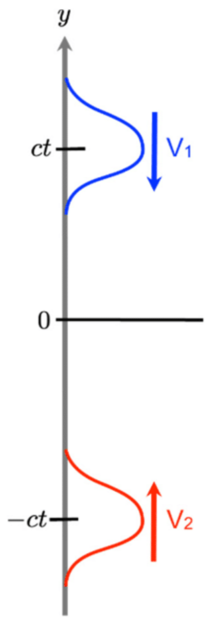

▪ If the fluid is heaped up against the wall, the pressure force will be pushing out into the fluid. The pressure gradient force and the Coriolis force will balance and in the northern hemisphere ($f > 0$) we will have southward flow.

▪ If there is a dip against the wall pressure, gradient force is pushing towards the wall and the Coriolis force balances this, so in the northern hemisphere ($f > 0$) the flow is northwards.

↳ We will have **oscillations** between **northward** and **southward** flow alternating with the crests and troughs of the wave, and the whole thing is propagating like a gravity wave along the wall. These waves are ***coastal Kelvin waves***.

⇨ **On the diagram**, it is mentioned that the wave is **propagating southwards**. We have not proved that yet and to do so we need to make a closer examination of the geostrophic balance equation (see **#GFD4.2b**).

### *4.2.b) Geostrophic balance*




⇨ Since the wave is ***non-dispersive***, all signals must travel at the same speed $c = \sqrt{gH}$. The solution for $v$ at $y = 0$ and time $t$ must be **the superposition of two independent waves traveling in opposite directions:**

- A wave coming from the north $V_1(x, y + ct)$
- A wave coming from the south $V_2(x, y - ct)$

All the wavelengths propagate at the same speed. A wave pattern will propagate at this speed

↳ Anything at $y = 0$ and time $t$ must consist of the sum of everything that was at a distance $c \times t$ either to the north or to the south. ***Anything else has either gone too far*** or ***has not arrived yet*** because there is only one speed that these waves can propagate at.

⇨ The corresponding surface displacement is $\eta = \sqrt{H/g}(-V_1 + V_2)$.

↳ This can be shown by substitution into the $y$-momentum equation:

$$\frac{\partial}{\partial t}(V_1 + V_2) = -\sqrt{gH}\frac{\partial}{\partial y}(-V_1 + V_2) \qquad \rightarrow \frac{\partial V_1}{\partial t} = c\frac{\partial V_1}{\partial y}$$

$$\frac{\partial}{\partial t}(-V_1 + V_2) = -\sqrt{gH}\frac{\partial}{\partial y}(V_1 + V_2) \qquad \frac{\partial V_2}{\partial t} = -c\frac{\partial V_2}{\partial y}$$

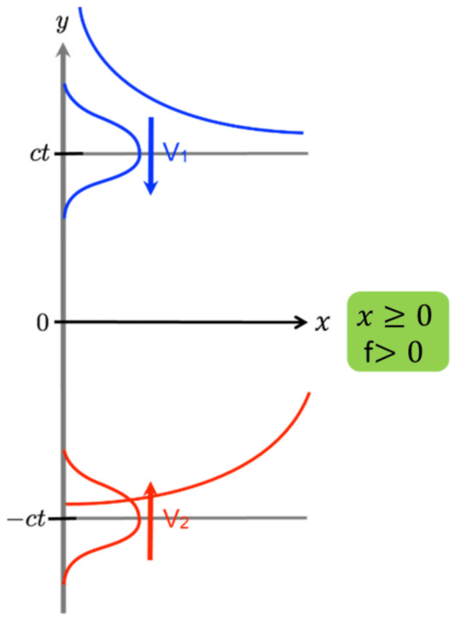
⇨ To obtain the $x$-dependence of these functions, ***we use the diagnostic equation***, geostrophic balance, which gives:

$$\frac{\partial V_1}{\partial x} = -\frac{f}{\sqrt{gH}}V_1 \qquad \frac{\partial V_2}{\partial x} = \frac{f}{\sqrt{gH}}V_2$$

↳These relations have ***exponential solutions*** with $x$: $V_1(x, ct)e^{-x/L_R}$ and $V_2 = V_2(x, ct)e^{x/L_R}$ respectively, with a scale distance of the Rossby radius $L_R = c/f$.

With $x \geq 0$ and $f > 0$:

- $\boldsymbol{V_1}$ has a **decaying** exponential solution in $x$ with boundary layer width $L_R$.
- $\boldsymbol{V_2}$ **grows exponentially** away from the wall, and so fails to satisfy the condition of boundedness at infinity. This solution thus must be eliminated (for physical reasons).



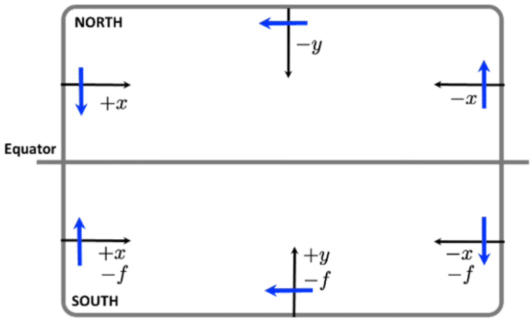

↳ We thus retain the solution $\boldsymbol{V_1}$ that implies that **coastal Kelvin waves must propagate southwards** (negative in $y$ direction) along a **wall on the western side** of the basin ($x$ positive offshore) **in the northern hemisphere** ($f > 0$).

The scale of the costal Kelvin wave is the Rossby radius $L_R = c/f$ (see **#GFD1.2a**)

### *4.2.c) Properties of Kelvin waves*

⇨ So far, we have considered Kelvin waves on the **western side** of an ocean basin in the northern hemisphere:
- our $x$-coordinate was positive towards the ocean basin center ($x \geq 0$)
- The planetary vorticity was positive ($f > 0$).

↳ The only admissible solution has a zonal structure decaying exponentially offshore $e^{-x/L_R}$



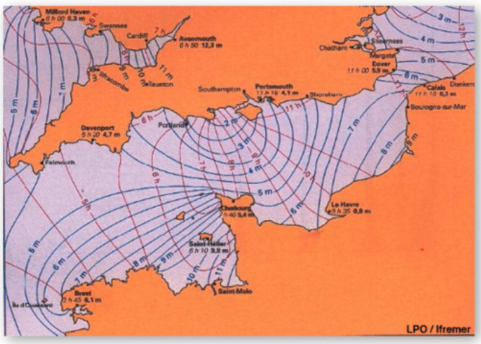

⇨ On the **eastern side** of the ocean basin, the $x$-coordinate is negative towards the center of the basin. Following the same logic, with $V_1$ and $V_2$, this change of sign leads to the conclusion that along a wall on the eastern side of the basin the wave must propagate northwards.

↳ In the **northern hemisphere** a Kelvin wave **keeps the coast to its right** as it is pushed against it by the Coriolis force.

⇨ As $f$-plane dynamics are isotropic in $x$ and $y$, if we have geostrophic balance in the north-south direction, Kelvin waves propagating in the zonal direction, along the northern wall of the basin (negative $y$-coordinate) will propagate westwards. See *Vallis* (2017, #3.7) for the rotated equation system.

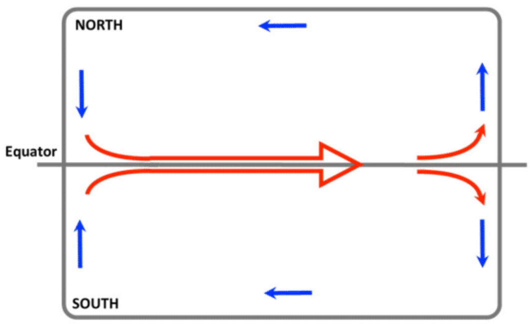
↳ **In the northern hemisphere Kelvin waves lean against the coast with the coast on their right as they propagate.**

As an illustration, this picture shows the English Channel where the tides come from the Atlantic Ocean through the channel. They can be described as coastal Kelvin waves, leaning against the French coast. The amplitude of the tidal variation is much higher on the French side than on the English side: up to 11 meters near St Malo, and only about 2 or 3 meters near Southampton. This explains why a tidal power plant was built on the French side.




⇨ In the **southern hemisphere** $f$ changes sign, so all these considerations are reversed, and Kelvin waves propagate with the coast to their left.

↳ Kelvin waves propagate around the basin anti-clockwise in the northern hemisphere and clockwise in the southern hemisphere. We are left wondering what happens when **coastal Kelvin waves meet at the equator**, along the Brazilian coasts for instance. In fact, they can carry on along the equator as **equatorial Kelvin waves**, propagating eastwards along the equatorial waveguide. Then, at the eastern boundary (African coast), the equatorial Kelvin waves will continue poleward in each hemisphere.




🕮 ***Imagine*** that you put a wall along the equator. It would be a southern boundary in the northern hemisphere and a northern boundary in the southern hemisphere. In both hemispheres, ***Kelvin waves can lean on this wall and propagate eastward***. Suddenly, the wall collapses, and the Kelvin waves in each hemisphere can just ***lean against each other*** as they travel eastwards along the equator. ***This can only happen on the equator where the sign of $f$ changes***. See **#GFD4.3** for theoretical details.

# GFD4.3: Equatorial Scaling and Kelvin Wave Solution

### *4.3.a) Scales of motion near the Equator*

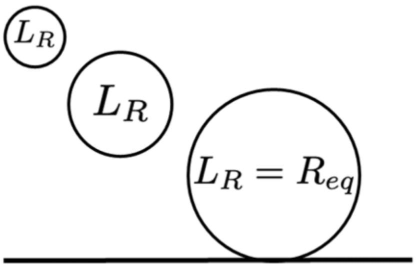
⇨ Let's focus on the **equatorial region**. At the equator:

- The latitude is zero ($\phi = 0$)
- The planetary vorticity is also zero ($f = 0$).
- The variation of the Coriolis parameter with distance $y$ in the north-south direction

is maximum at the equator: $\beta = \frac{\partial f}{\partial y} = \frac{2\Omega}{a} cos\phi = 2.28 \times 10^{-11}\ m^{-1}.s^{-1}$.

- We can use the $\beta$ approximation. Unlike in the extra-equatorial regions where $f = f_0 + \beta y$, at the equator $f_0 = 0$ and thus the $\beta$-approximation is reduced to $f = \beta y$ .

⇨ **The Rossby radius** is the typical scale of motion: $L_R = \frac{NH}{f} = \frac{\sqrt{g'H}}{f} = \frac{c}{f}$. In the mid-latitudes, this works fine. Close to the equator, it does not work as a useful scale because $f = 0$. We thus have to consider another way of imagining what the relevant scale is at the equator.




⇨ On the left is a sketch showing **circles** that describe the **Rossby radius** as a function of latitude. As $f$ decreases approaching the equator, the Rossby radius gets larger.

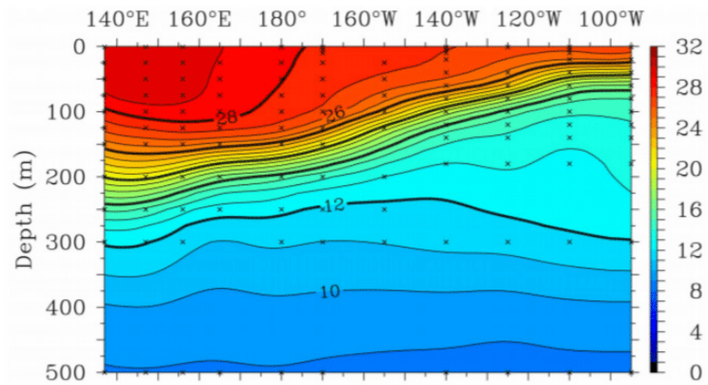
↳ There is a critical latitude $y = R_{eq}$, where the Rossby circle first touches the equator and the center of the circle is exactly the Rossby radius away from the equator. This distance $R_{eq}$ can be defined as the **equatorial radius**.

↳ Using the $\beta$-approximation at $y = R_{eq}$ ($f = \beta R_{eq}$) and $R_{eq} = L_R = \frac{c}{\beta R_{eq}}$, it follows that:

$$R_{eq} = \sqrt{\frac{c}{\beta}}.$$

⇨ To **evaluate the amplitude** of this scale of motion at the equator, we have to quantify the phase speed of the gravity waves: $c = \sqrt{g'H}$.

↳ The diagram on the right presents the temperature as a function of depth and longitude in the equatorial Pacific, from in-situ TAO data. The region of strong vertical gradients is called the thermocline. In a conceptual one-layer shallow water model, it separates the active layer from the resting abyss layer.



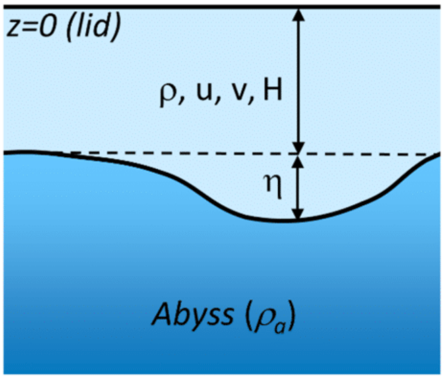

- The change in density across the thermocline is $\sim 2kg/m^3$ for an average density $\sim 1000kg/m^3$. This gives a **reduced gravity** of the order of $g' = g\frac{\Delta\rho}{\rho} \sim 0.02m.s^{-2}$.
- The typical **depth of the thermocline** in the equatorial Pacific changes quite a lot with longitude. It ranges from a couple of hundred meters in the west to a few tens of meters in the eastern basin. We approximate $H \sim 100m$.

↳ Internal gravity waves propagating on the thermocline have a **phase speed**

$$c = \sqrt{g'H} \sim \sqrt{2} = 1.4\ m.s^{-1}$$

- The equatorial **radius of deformation** is $R_{eq} = \sqrt{\frac{c}{\beta}} \sim 250\text{km} \sim 2.2^0$
- The time $T_{eq}$ for a wave to travel distance $R_{eq}$ is $T_{eq} = \frac{1}{\sqrt{\beta c}} \sim 2$ days

### *4.3.b) Linear equatorial shallow water model*




⇨ Let's analyze the **shallow water equations** for a conceptual equatorial model consisting of one active layer with a rigid lid overlying a motionless abyss.

• We use the $\boldsymbol{\beta}$ **approximation** at the equator: $f = \beta y$. The shallow water equations (see **#GFD1.2g**) are:

$$\frac{\partial u}{\partial t} - \beta y v = -g' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \beta y u = -g' \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

⇨ This slight change, going from an $\boldsymbol{f}$**-plane** with a constant coefficient $f$ (see **#GFD4.1b**) to an equatorial $\beta$-plane in which there is a $y$ tin the Coriolis term will yield some slightly different results.

### *4.3.c) The equatorial Kelvin wave solution*

⇨ We start with **a special case** in which there is no flow across the equator. This is the **Kelvin wave framework**, just like when we had a wall (see **#GFD4.2a**). The flow is along the equator ($u$) and <u>$\boldsymbol{v = 0}$ everywhere</u>.

⇨ The equations simplify to:

$$\cancel{\frac{\partial v}{\partial t}} + \beta y u = -g' \frac{\partial \eta}{\partial y}$$

*Cross-equatorial Geostrophic balance*

$$\frac{\partial u}{\partial t} - \cancel{\beta y v} = -g' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} \right) = 0$$

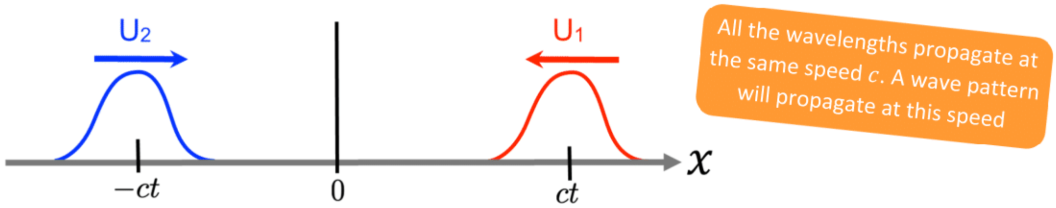
*Non-dispersive waves*
$c = \sqrt{g'H}$

⇨ We obtain a set of equations similar to the one we had for the coastal Kelvin waves (see **#GFD4.2a**), i.e. <u>2 prognostic equations</u> consistent with **non-dispersive waves** propagating along the equatorial wave-guide with a phase speed $c = \sqrt{g'H}$ (✋) and a <u>diagnostic equation</u> for **cross-equatorial geostrophic balance**, that will determine the meridional structure of the waves.

⇨ Using the same logic as before (see **#GFD4.2b**), the solution for $u$ at $x = 0$ and time $t$ can consist only of **the superposition of two independent waves traveling in opposite directions:**

- A wave propagating ***westwards*** $U_1(x + ct, y)$
- A wave propagating ***eastwards*** $U_2(x - ct, y)$

⇨ Anything at $x = 0$ and time $t$ must consist of the sum of what was at a distance $c \times t$ either to the east or to the west. Anything else has either gone too far or has not arrived yet because there is only one speed ($c$) at which these waves can propagate.



⇨ The solution for $\eta$ can be written in terms of $\boldsymbol{U_1}$ and $\boldsymbol{U_2}$, as $\eta = \sqrt{\frac{H}{g'}}(-U_1 + U_2)$

This can be verified by substituting the solution into the prognostic equations.

⇨ As in **#GFD4.2b**, the **cross-equatorial geostrophic balance** informs us about the **meridional structure of the wave** and we will consequently discard one solution.

↳ To proceed, we substitute the solutions for $u(x, y, t)$ and $\eta(x, y, t)$ into the diagnostic equation. The following expressions must be satisfied separately:

$$\beta y U_1 = c\frac{\partial U_1}{\partial y} \qquad\qquad \beta y U_2 = -c\frac{\partial U_2}{\partial y}$$

⇨ These relations have exponential solution with $y$: $U_1 \sim e^{\frac{\beta}{2c}y^2}$ and $U_2 \sim e^{-\frac{\beta}{2c}y^2}$ respectively.

↳ Only the solution $\boldsymbol{U_2}$, i.e. the ***wave propagating eastward***, is ***exponentially decaying*** in $y^2$.

✋ Note the difference with coastal waves that depended on non-zero $f$, and thus, $y$. Now we have a $y^2$ dependence (because of the extra $y$ in the equation set, see **#GFD4.3b**). The decay works both to the north and south with the same propagation direction.

⇨ The scale distance of these waves is $\sqrt{\beta/2c} = \sqrt{1/2R_{eq}^2}$ (see **#GFD4.3a**) and it is symmetric about the equator. Kelvin waves decay away from the equator regardless of whether $y$ is negative or positive. Equatorial Kelvin waves are ***equatorially trapped***.

#### *4.3.d) Equatorial Kelvin wave properties*

⇨ The **equatorial Kelvin wave** solutions for the three variables $(u, v, \eta)$ can be written as a function of $\psi$, a dimensionless waveform that propagates in the $x$-direction:

$$u = c\,\psi(x - ct)\; e^{-y^2/2R_{eq}^2}$$
$$\boxed{v = 0}$$
$$\eta = H\,\psi(x - ct)\; e^{-y^2/2R_{eq}^2}$$

⇨ **Kelvin waves are a special solution for equatorial waves** for which $v = 0$.

They are:
- non-dispersive waves
- trapped at the equator
- propagate eastwards at a phase speed $c = \sqrt{g'H}$

## GFD4.4: Equatorial Waves – General Solution

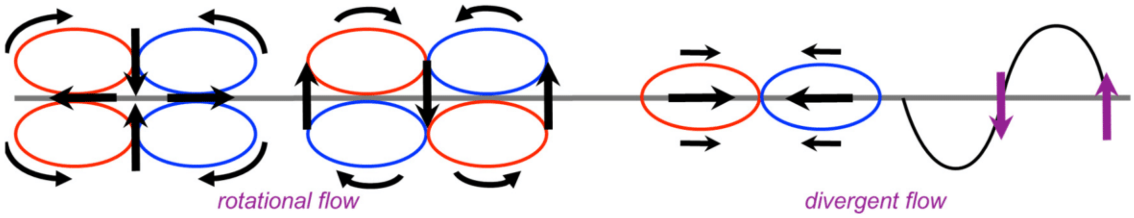
#### *4.4.a) The general solution*

⇨ To derive the general solutions for equatorial waves, we substitute **wave-like solutions** into the **equatorial shallow water equations** (see **#GFD4.3b**):

$$\frac{\partial u}{\partial t} - \beta y v = -g'\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + \beta y u = -g'\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Expected wave structures

⇨ We first consider **different types of wave structures** that we might encounter. Below, colored circles represent the geostrophic stream function ($\eta$, the depth of the thermocline in the Ocean or surface pressure in the Atmosphere). Positive $\eta$ anomalies are in red, negative in blue, associated with anti-cyclonic and cyclonic circulation respectively in the northern hemisphere.




• The first example (left sketch) is symmetric about the equator, i.e. positive/negative anomalies in $\eta$ north and south of the equator. The rotational flow around a positive anomaly is clockwise in the northern hemisphere and anti-clockwise in the southern hemisphere. It is the other way around for the negative anomalies.

↳ $\eta$ is symmetric and so is the zonal current ($u$). But the meridional current ($v$) is anti-symmetric about the equator, so $v$ is out of phase by a quarter of a wavelength – in **quadrature**.

• The second scenario (middle sketch) shows anti-symmetric thermocline perturbations associated with symmetric $v$. Extrema in $v$ are **displaced by a quarter of the wavelength**.

• Not all the solutions have <u>rotational</u> type flow. Let's picture a <u>divergent</u> type flow. The third sketch exemplifies the equatorial Kelvin wave solution. $v = 0$ and there is a convergence/divergence between equatorial positive and negative $\eta$ anomalies. It is associated with downward/upward movement of the stream function. This configuration is consistent with eastward propagation.

Wave-like solutions

⇨ We thus look for **wave-like solutions** that ***propagate along the equator*** ($x$-direction, positive or negative). For $v$, we have added this $\frac{\pi}{2}$ as a phase displacement for the solution, so:

$$u = \tilde{u}(y)e^{i(lx-wt)}$$
$$v = \tilde{v}(y)e^{i\left(lx-wt+\frac{\pi}{2}\right)}$$
$$\eta = \tilde{\eta}(y)e^{i(lx-wt)}$$

$l$ is the zonal wavenumber

✋ The amplitude coefficient depends on $y$. This is going to complicate the resolution.

Harmonic (wave) equation

↳ We substitute these solutions into the **equatorial shallow water equations** and (after a laborious process, see details on the next pages) we obtain a ***harmonic (second-order differential) equation*** for the amplitude $\tilde{v}(y)$:

$$\frac{d^2\tilde{v}}{dy^2} + \frac{\beta^2}{c^2}(Y^2 - y^2)\tilde{v} = 0 \qquad Y^2 = \left(\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega}\right)\frac{c^2}{\beta^2}$$

$c = \sqrt{g'H}$ is a constant

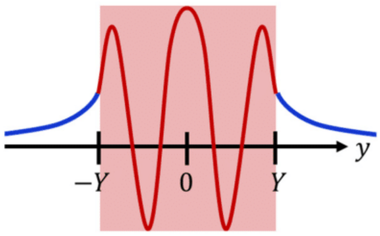
✋ It is not an algebraic linear equation because of the $y$-dependence in of $\tilde{v}$.

⇨ In this equation, there is a coefficient $\frac{\beta^2}{c^2}(Y^2 - y^2)$ in front of $\tilde{v}$. It can be positive or negative depending on the sign of $Y^2 - y^2$.

▪ **<u>In the vicinity of the equator</u>**, $y^2 < Y^2$, the coefficient is positive and the spatial structure of the solution will be oscillating in $y$.

▪ **<u>Outside the $Y$-bound</u>** ($y^2 > Y^2$), the coefficient is negative and the equation admits exponentially decaying solutions.

↳ This portrays a picture of the zonally-propagating wave where near the equator the spatial structure is an oscillation and at a certain distance, it just decays away to zero. The **wave is trapped at the equator** and $Y$ is the **width of the equatorial waveguide**. As the expression of $Y$ depends on various properties of the wave ($\omega, \beta, c = \sqrt{g'H}, l$), different waves will have different widths but it basically scales the wave structure in a similar way to the equatorial radius ($R_{eq}$) for the Kelvin waves.



## *Derivation of the equatorial wave equation for v*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta y v = -g' \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial v}{\partial t} + \beta y u = -g' \dfrac{\partial \eta}{\partial y} \\ \dfrac{\partial \eta}{\partial t} + H\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0 \end{cases}$$

$$u = \tilde{u}(y)e^{i(kx-\omega t)} \qquad \eta = \tilde{\eta}(y)e^{i(kx-\omega t)}$$

$$v = \tilde{v}(y)e^{i(kx-\omega t \pm \pi/2)} = \tilde{v}(y)e^{i(kx-\omega t)}e^{\pm i\pi/2} = \pm i\tilde{v}(y)e^{i(kx-\omega t)}$$

$v$ in quadrature with $u$,
\+ or - makes no difference, we choose +

$$\begin{cases} -i\omega\tilde{u} - i\beta y\tilde{v} + ig'k\tilde{\eta} = 0 \\ \omega\tilde{v} + \beta y\tilde{u} + g'\dfrac{\partial \tilde{\eta}}{\partial y} = 0 \\ -i\omega\tilde{\eta} + H\left(ik\tilde{u} + i\dfrac{\partial \tilde{v}}{\partial y}\right) = 0 \end{cases}$$

**We want to eliminate $u$ and $\eta$ to get an equation for $v$.**

We drop tildes and prime on $g$, and we use subscript notation for derivatives. The linear system can be written:

$$\begin{cases} \omega u + \beta y v - gk\eta = 0 & (1) \\ \omega v + \beta y u + g\dfrac{\partial \eta}{\partial y} = 0 & (2) \\ -\omega\eta + Hku + H\dfrac{\partial v}{\partial y} = 0 & (3) \end{cases}$$

$$\begin{cases} \partial/\partial y(1) + k \times (2) \rightarrow \omega u_y + \beta v + \beta y v_y + \omega k v + \beta y k u = 0 & (A) \\ \omega \times (2) + g \times \partial/\partial y(3) \rightarrow \omega^2 v + \beta y \omega u + gHku_y + gHv_{yy} = 0 & (B) \\ \omega \times (1) - gk \times (3) \rightarrow \omega^2 u + \beta y \omega v - gk^2 Hu - gkHv_y = 0 & (C) \end{cases}$$

⇨ $gHk \times (A) - \omega \times (B) \rightarrow$

$$gHk(\beta v + \beta y v_y + \omega k v) + gHk^2\beta y u - \omega^3 v - \beta y \omega^2 u - gH\omega v_{yy} = 0$$

$$-gH\omega v_{yy} + gHk\beta y v_y + (gHk\beta + gH\omega k^2 - \omega^3)v + (gHk^2\beta y - \beta y\omega^2)u = 0 \qquad (D)$$

⇨ $(D) + \beta y \times (C) \rightarrow$

$$-gH\omega v_{yy} + gHk\beta y v_y + (gHk\beta + gH\omega k^2 - \omega^3)v + \beta^2 y^2 \omega v - \beta y gHk v_y = 0$$

⇨ $\div -gH\omega \rightarrow \dfrac{d^2\tilde{v}}{dy^2} + \left[\dfrac{\omega^2}{g'H} - k^2 - \dfrac{k\beta}{\omega} - \dfrac{\beta^2}{g'H}y^2\right]\tilde{v} = 0$

or $\dfrac{d^2\tilde{v}}{dy^2} + \dfrac{\beta^2}{c^2}(Y^2 - y^2)\tilde{v} = 0$ where $\begin{cases} c \text{ is the gravity wave speed)} \\ Y^2 = \dfrac{g'H}{\beta^2}\left[\dfrac{\omega^2}{g'H} - k^2 - \dfrac{k\beta}{\omega}\right] \end{cases}$

*4.4.b) Meridional structure*

⇨ The **general solutions** of the $\tilde{v}$ harmonic equation have the form of a discrete set of meridional structures:

$$\tilde{v} \propto H_n(y')e^{-\frac{y'^2}{2}} \qquad (y' = y/R_{eq})$$

$$R_{eq} = \sqrt{c/\beta}$$

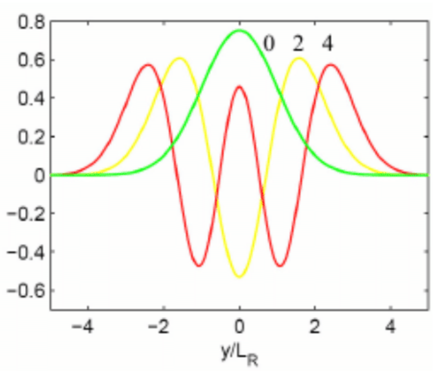
↳ $\tilde{v}$ is the product of a ***Hermite polynomial*** ($H_n$) and a ***decaying exponential***. This is similar to what we had for the meridional structure of equatorially-trapped Kelvin waves (see **#GFD4.3c**), except that now it is multiplied by the Hermit polynomial. The product of these two functions is called a **parabolic cylinder functions**.

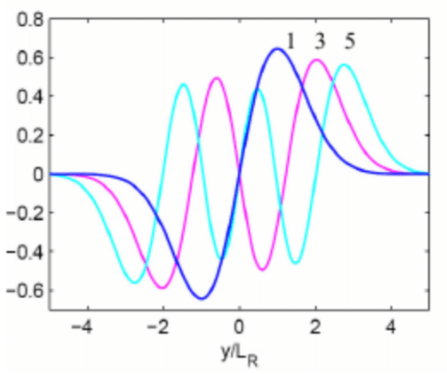
⇨ The **Hermit polynomials** are defined as:

$$H_0(y') = 1$$
$$H_1(y') = 2y'$$
$$H_2(y') = 4y'^2 - 2$$
$$H_3(y') = 8y'^3 - 12y'$$
$$H_4(y') = 16y'^4 - 48y'^2 - 12 \quad ...$$

↳ They have mathematically-useful properties such as $H_n$ can be written in term of the previous and the next polynomial and there is a simple expression of the derivative:

$$y'H_n = nH_{n-1} + \frac{1}{2}H_{n+1} \qquad\qquad \frac{dH_n}{dy'} = 2nH_{n-1}$$




• **For even numbers $n$**, $H_n(y')$ is symmetric about the equator and so is $\tilde{v}$. The symmetric meridional structures are illustrated on the left for $n = 0, 2, 4$. For $n = 0$, $\tilde{v} \propto e^{-y'^2/2}$, for $n = 2$ there is a single wiggle in the middle, for $n = 4$ there are two wiggles in the middle, etc…

↳ These structures are associated with a ***cross-equatorial flow and create thermocline displacements asymmetric with respect to the equator*** (i.e. a maximum in the thermocline displacement on one side of the equator, and a minimum on the other side).

• **For odd numbers $n$** ($n = 1, 3, 5, ...$), the meridional structures of the equatorial wave are anti-symmetric in $v$ and changes sign across the equator. As $n$ increases, the number of wiggles increases too.

↳ These structures are associated with ***no cross-equatorial flow (convergence/divergence) and symmetric thermocline displacements***.

✋ The ***Kelvin wave solution*** is not in this set of solutions, because it is associated with $v = 0$. You have to consider $n = -1$ and in that case, $u$ and $\eta$ will be symmetric.

### 4.4.c) The dispersion relations

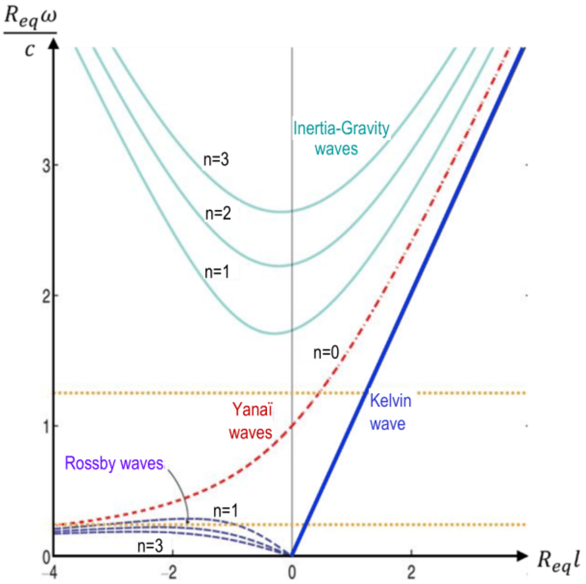
⇨ To derive the propagation properties, we substitute the general parabolic cylinder function solutions into the differential equation (see details on the following page). This leads to a ***set of dispersion relations***:

$$\frac{\omega^2}{c^2} - l^2 - \frac{\beta l}{\omega} = (2n+1)\frac{\beta}{c} = \frac{(2n+1)}{{R_{eq}}^2}$$

Here, $c = \sqrt{g'H}$ is a constant
$\frac{\omega}{l}$ is the phase speed of the wave

These are the ***theoretical dispersion relations*** for shallow water modes on the equatorial $\beta$-plane. This is a **family of dispersion relations** $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{l}, \boldsymbol{n})$ for distinct tropical waves (including equatorial Kelvin waves) associated with discreet values of $n$ (index for the Hermite polynomials).

⇨ As the dispersion relations are cubic in $\omega$, there are **3 mathematical roots** for each value of $n \geq 1$. Positive roots are represented below: $\omega$ as a function of the zonal wavenumber $l$. For $l > 0$, equatorial waves propagate eastwards, while for $l < 0$, they propagate westwards.




• For high frequencies ($\omega \gg 1$), we can neglect $\frac{\beta l}{\omega}$ to obtain $\omega^2 = (2n+1)\beta c + c^2 l^2$. This is a dispersion relation for ***equatorial inertia-gravity waves***. They propagate in either direction and are similar to inertia-gravity waves in mid-latitudes (see **#GFD4.1c**). The dispersion curve has a minimum frequency that depends on $n$. It is slightly off-centered (not completely symmetric) because there is the $\beta$-effect in the equations.

• For low frequencies, we can neglect $\frac{\omega^2}{c^2}$ to obtain $\omega = -\frac{\beta l}{l^2 + (2n+1)R_{eq}^{-2}}$. These are ***equatorial Rossby waves*** similar in principle to their counterparts in mid-latitudes (see **#GFD3.1f**) that critically depend on the $\beta$-effect. They propagate westwards, but their group speed can be eastwards for high wavenumbers.
There is a mathematical solution for a negative $\omega$.

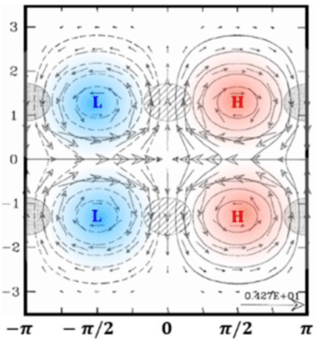
• Between inertia-gravity and Rossby wave solutions, there is a hybrid solution for $n = 0$. At low frequencies, this wave propagates westwards and behaves like a Rossby wave. And at higher frequencies, it propagates eastwards and behaves like a gravity wave. This wave is called a ***mixed Rossby gravity wave*** or Yanaï wave.

• The blue straight line corresponds to non-dispersive waves that propagate eastwards along the equator. These are the ***equatorial Kelvin waves*** (see **#GFD4.3**) which behaves like a gravity wave in absence of rotation (see **#GFD4.1a**).

### 4.4.d) Waves properties

⇨ Let's have a look at the two-dimensional structure of some of these tropical waves.

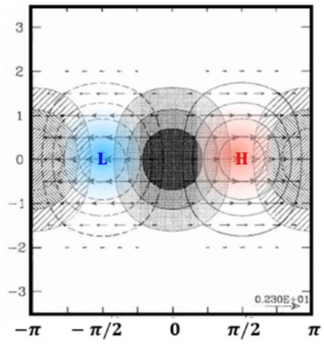
$n = 1,\ l^* = 1$
**Equatorial Rossby wave**




• First is the **Rossby wave structure** for $n = 1$. Contours shows the north-south **symmetric $\eta$** (the stream function, or dips in the thermocline or high-pressure areas in the atmosphere). The geostrophic flow circulates around $\eta$ perturbations and arrows depict the **anti-symmetric meridional flow**. This wave propagates towards the west.

• Putting $n = -1$ into the dispersion relation yields an eastward-propagating non-dispersive Kelvin wave with $v = 0$, associated with alternate **convergent/divergent** zonal flow.

$n = -1, l^* = 1$
**Equatorial Kelvin wave**



## *Derivation of the dispersion relations*

$$\frac{d^2v}{dy^2} + \frac{1}{R_{eq}^4}(Y^2 - y^2)v = 0$$

$$R_{eq} = \sqrt{\frac{c}{\beta}}, \quad Y^2 = \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right) R_{eq}^4 \quad \{= (2n+1)R_{eq}^2\}$$

$$y' = y/R_{eq}, \quad Y' = Y/R_{eq} \rightarrow \quad \frac{1}{R_{eq}^2}\frac{d^2v}{dy'^2} + \frac{1}{R_{eq}^4}(Y'^2 - y'^2)vR_{eq}^2 = 0$$

dropping primes $\dfrac{d^2v}{dy^2} + (Y^2 - y^2)v = 0$ solution $v = H_n e^{-y^2/2}$

should lead to non-dimensional dispersion relation $Y^2 = 2n+1$

using $\dfrac{dH_n}{dy} = 2nH_{n-1}$ and $yH_n = nH_{n-1} + \dfrac{H_{n+1}}{2}$

$$\frac{dv}{dy} = \frac{dH_n}{dy}e^{-y^2/2} - yH_n e^{-y^2/2} = \left[\frac{dH_n}{dy} - yH_n\right] e^{-y^2/2}$$

$$\frac{dv}{dy} = \left[2nH_{n-1} - \left(nH_{n-1} + \frac{H_{n+1}}{2}\right)\right] e^{-y^2/2} = \left[nH_{n-1} - \frac{H_{n+1}}{2}\right] e^{-y^2/2} = [yH_n - H_{n+1}]\, e^{-y^2/2}$$

$$\frac{d^2v}{dy^2} = \left[H_n + y\frac{dH_n}{dy} - \frac{dH_{n+1}}{dy} - y\,(yH_n - H_{n+1})\right] e^{-y^2/2}$$

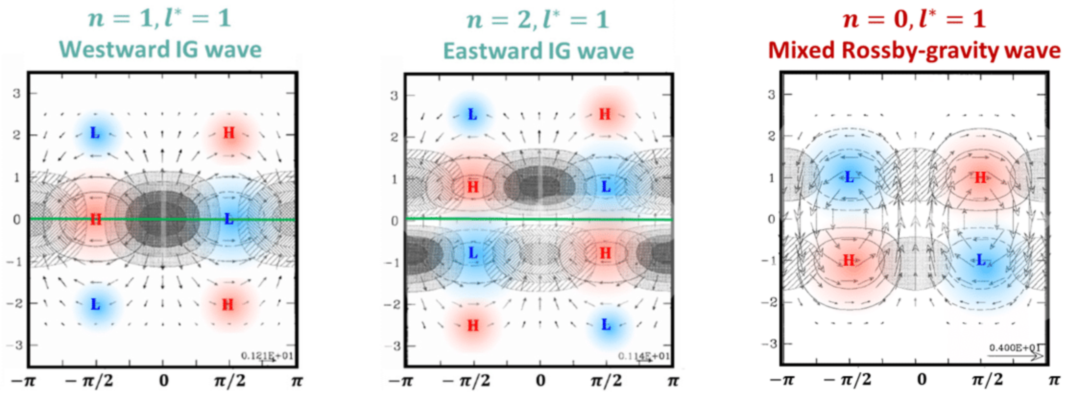
$$= [H_n + 2ynH_{n-1} - 2(n+1)H_n - y^2H_n + yH_{n+1}]\, e^{-y^2/2}$$

$$= [H_n + y\,(2nH_{n-1} - yH_n + H_{n+1}) - 2(n+1)H_n]\, e^{-y^2/2}$$

$$= [-H_n - 2nH_n + y^2H_n]\, e^{-y^2/2}$$

so $[y^2 - (2n+1)]\, H_n e^{-y^2/2} + (Y^2 - y^2)H_n e^{-y^2/2} = 0$

thus $Y^2 = 2n+1$




• **Inertia-gravity waves** have divergent structures and they can propagate either eastwards or westward. Eastward/westward structures look fairly similar but they are not exactly the same because there is a slight asymmetry between the two directions.

• **Mixed Rossby gravity waves** have anti-symmetric structures in $\eta$ and a mixture of rotational and divergent flow. They can propagate either eastwards or westwards.

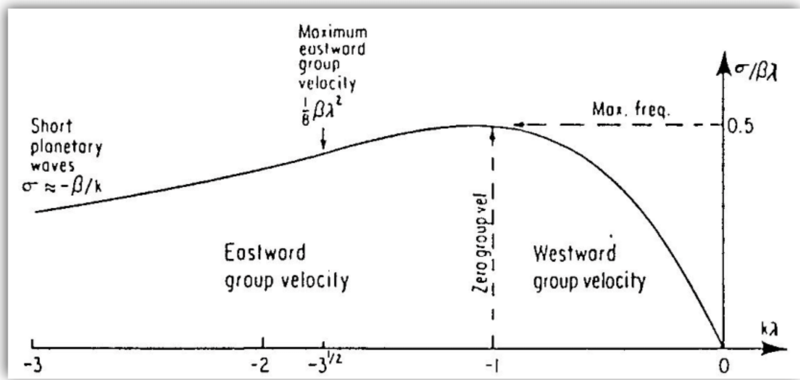
## GFD4.5: Equatorial Waves – Special Cases and Examples

### *4.5.a) Equatorial Rossby waves*

⇨ At low frequencies (long-wave approximation), close to the origin point (where $\omega = 0$), $\omega \ll f$. In the theoretical dispersion relationships for shallow water modes on the equatorial $\beta$ plane ($\omega^2/c^2 - l^2 - \beta l/\omega = (2n+1)R_{eq}^{-2}$, see **#GFD4.4c**) we can **neglect the first term in $\omega^2$** and it rearranges to:

$$\omega = -\frac{\beta l}{l^2 + (2n+1)R_{eq}^{-2}}$$

↳ This is the dispersion relation for long equatorial Rossby waves. It is very similar to mid-latitude barotropic Rossby waves (see **#GFD3.1**). We now have $(2n+1)R_{eq}^{-2}$ in the denominator where we had $L_R^2$ (the Rossby radius) for the mid-latitude Rossby waves in a one-layer model (see **#GFD3.1f**).




▪ For shorter waves the phase speed remains westward, but the group speed becomes eastward. These short-waves are of little importance because they are very dispersive. They are hard to observe because they are slow and dissipate.

▪ As in **#GFD3.1f**, for small wavenumber ($l \ll 1$, **long-waves**) equatorial Rossby waves are **almost non-dispersive** ($\omega = -\frac{cl}{2n+1}$) and can be detected on equatorial Rossby rays.

Non-dispersive waves: all the wavelengths propagate at the same speed. A wave pattern (sum of different wavelengths) will not change shape

▪ The dispersion relation is fairly straight at the origin and then curves round to a maximum frequency. The latter corresponds to zero group speed.

#### 4.5.b) Equatorial Rossby rays

⇨ It is important to recall that as the wave propagates its dispersion relation changes. This is because the wave may change latitude, and $f$ enters into the dispersion relation.

↳ The propagation of Rossby waves can be traced along the equatorial Rossby rays by computing the ratio of the group speeds (as in **#GFD3.2c** or **#GFD3.3c**). This provides the trajectory $\frac{dx}{dy}$. We consider here that $f$ is "slowly varying". For long Rossby waves, the direction of the group speed is given by:

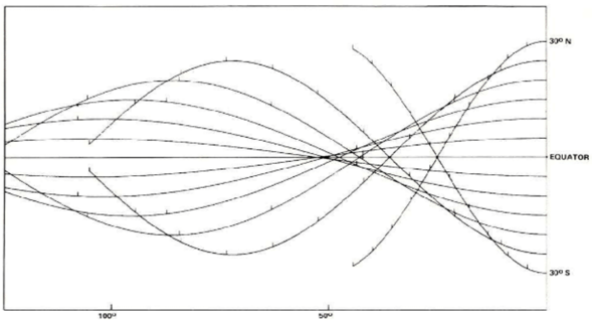
$$\frac{dx}{dy} = \frac{\partial\omega/\partial n}{\partial\omega/\partial l} = -\frac{2\omega}{\beta}\left(-\frac{\beta l}{\omega} - \frac{\beta^2 y^2}{c^2}\right)^{1/2}$$

⇨ The integral yields an equation for the latitude: $y = -\left(\frac{c^2 l}{\omega\beta}\right)^{\frac{1}{2}} cos\left(\frac{2\omega}{c}x + \theta_0\right)$

↳ Rossby waves of constant frequency and zonal wavenumber will **change their meridional wavenumber** and thus their direction of propagation.

• They end up oscillating about the equator by **refraction**: just like if the equator had a high-refractive-index and waves were attracted towards regions of high-refractive-index. It is another way to show that equatorial Rossby waves are **equatorially trapped**.

• This behavior is modified by the presence of a mean flow (current, winds).



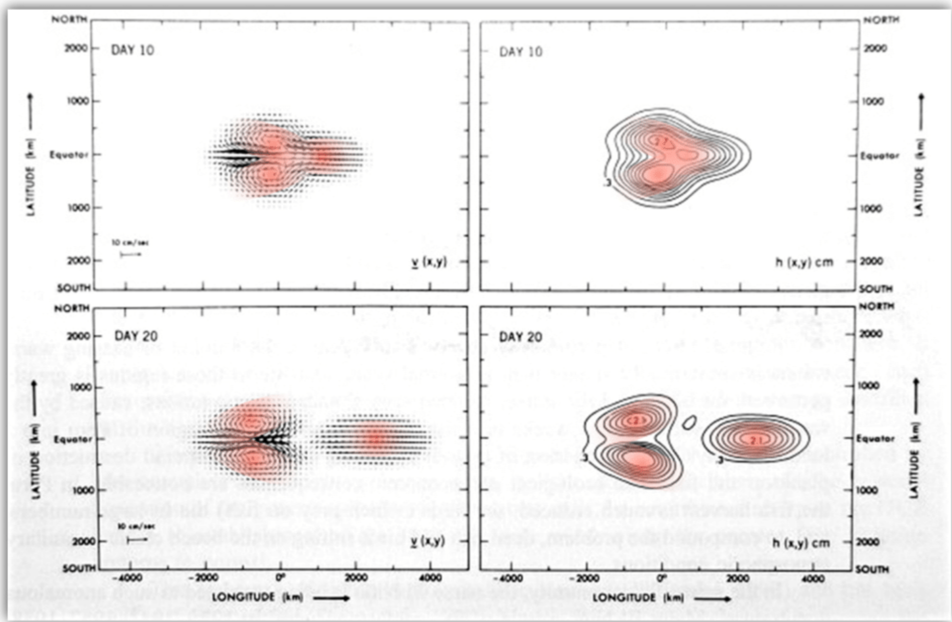

#### 4.5.c) Oceanic adjustment

⇨ Here is a ***practical example*** illustrating how the Ocean will adjust to an initial thermocline perturbation. Below are snapshots from an ocean shallow water model simulation.

• In this experiment, at $t = 0$, a bell-shaped perturbation to the thermocline is allowed to dissipate. Initially, the thermocline has been artificially pushed down (downwelling) in a symmetric round-blob like structure, resembling a Gaussian or cosine squared.

• 10 days later, the induced flow field (a) and thermocline displacement (b) emerge. We observe a symmetric single bulge Kelvin wave ($n = -1$) propagating eastwards, while a symmetric dipole associated with a double bulge Rossby wave ($n = 1$) propagates westwards.

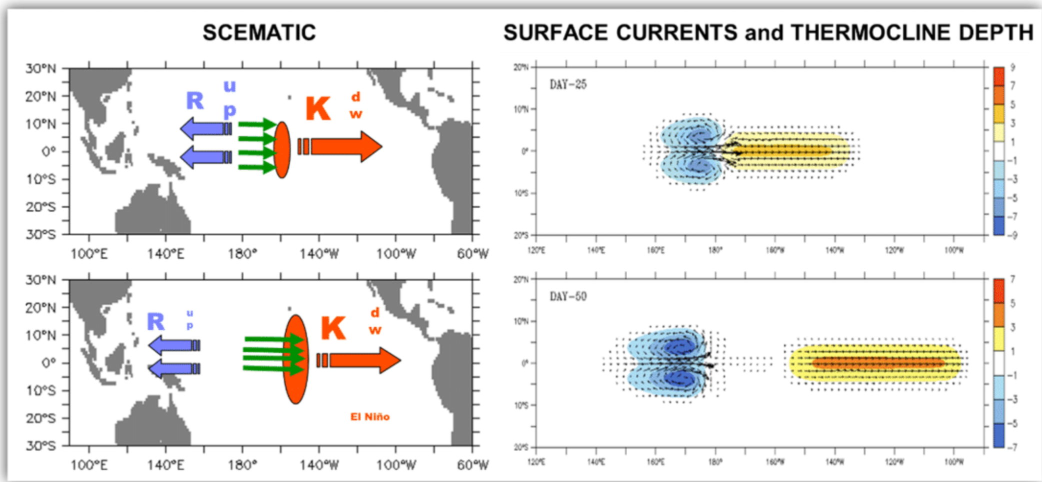
• At $t = 20$ days, the two structures are completely separated. The equatorial Kelvin wave propagate faster than the equatorial Rossby wave. This is consistent with the slope of the dispersion relation for long-wave approximation (✋ see **#GFD4.4c**).



*4.5.d) ENSO theories: the delayed oscillator*

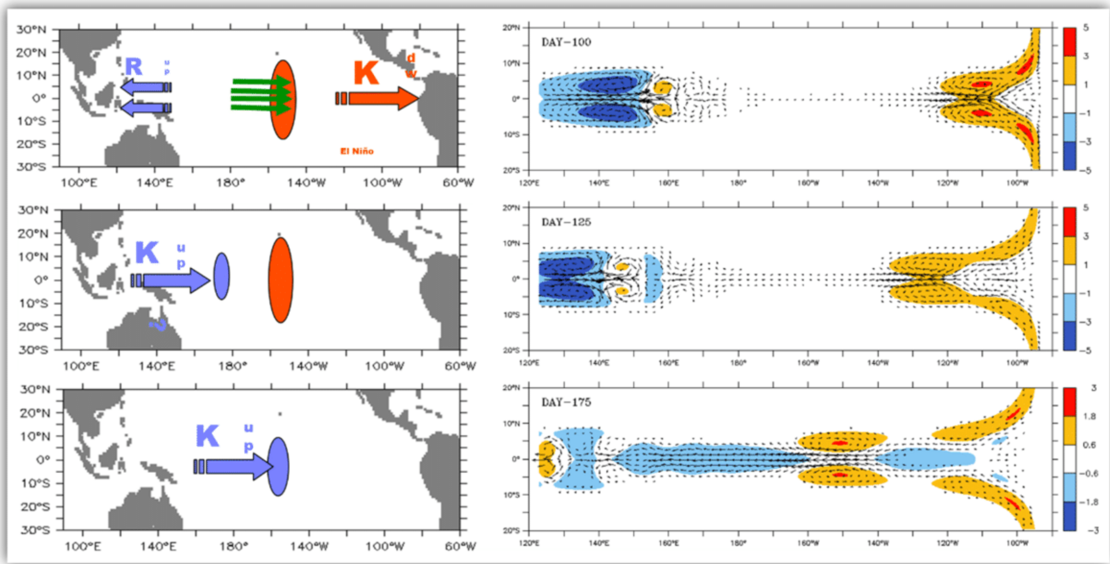
⇨ This was put forward as a candidate for the explanation for the **life-cycle of El Niño**.

↳ **El Niño** is a perturbation to the depth of the thermocline and the sea surface temperature in the eastern equatorial Pacific Ocean. **El Niño** starts very often with an **abrupt change in the surface wind-stress** forcing, a westerly wind anomaly in the equatorial western basin. The thermocline is pushed down (***downwelling***) ahead of the wind anomaly and pulled up (***upwelling***) behind it. The thermocline perturbations propagate eastwards as a ***downwelling Kelvin wave*** and westwards as an ***upwelling Rossby wave***. The combined effects of the two waves will tend to **flatten out the thermocline**, resulting in a **warming** of the eastern Pacific temperatures.




▪ A ~couple of months later, the ***downwelling Kelvin wave*** arrives at the South-American coast. Part of its energy is transmitted along the coast as coastal Kelvin waves, but a significant part reflects into a **downwelling Rossby wave**. This amplifies the deepening of the thermocline in the eastern equatorial basin and increases the sea surface warming.

▪ When the ***wind-forced upwelling Rossby wave*** arrives at the other coast, it reflects and transforms itself into an eastward-propagating ***upwelling Kelvin wave***. (In its third life, it will also reflect into an upwelling Rossby wave.) Along their propagation, these ***upwelling waves*** will raise the thermocline depth and thus cancel the original wind-forced perturbation.



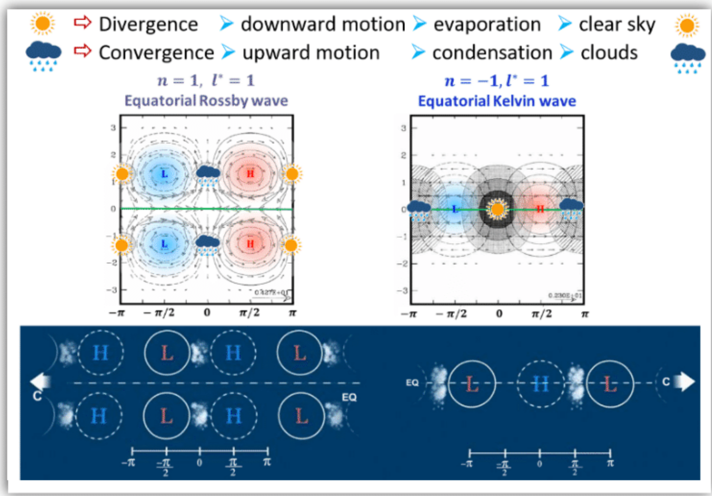

↳ This theory is called the **delayed oscillator theory**: **The initial El Niño perturbation sows the seeds of its own destruction** a few months later after these waves have crossed the Pacific Ocean and come back again.

⇨ It is **one of the theories** to explain the **El Niño cycle** but it is not the only one.

### *4.5.e) Tropical convection in the atmosphere*

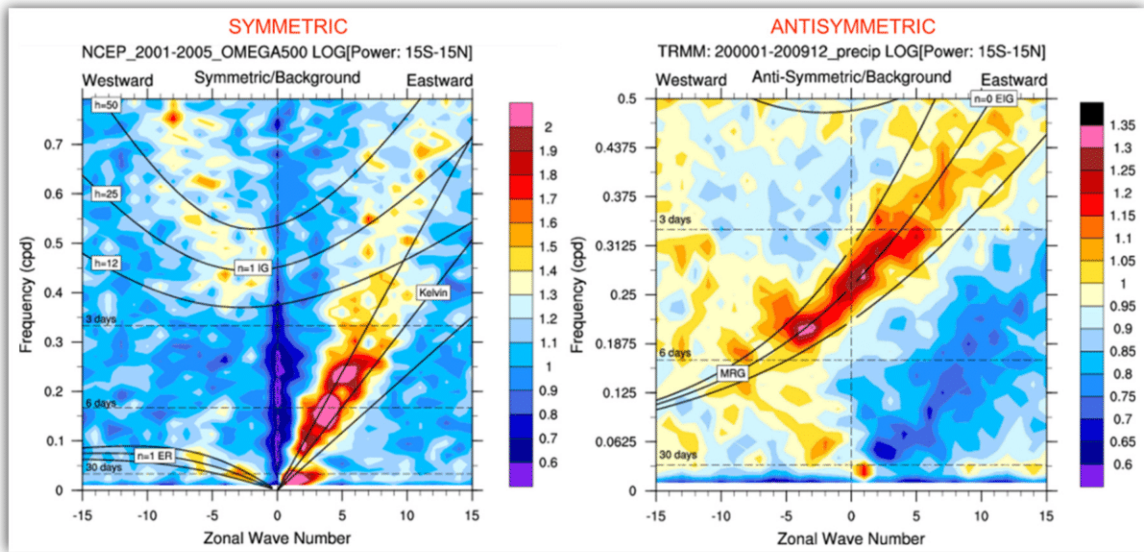
⇨ Here is an example for the **atmosphere**. The equatorial theory developed in this chapter was presented more from an oceanographic perspective, but it works just as well in the atmosphere.

↳ The main difference is that the **equatorial radius is much wider**: the geophysical parameters are such that the equatorial zone in the atmosphere is not just a few hundred kilometers (see **#GFD4.3a**) but is ~ the tropical band, namely 20°S-20°N (the width of a Hadley cell).




⇨ The figure below is the ***Wheeler-Kiladis diagram*** (from Wheeler and Kiladis, 1999). Observed atmospheric tropical variability is plotted as a function of its zonal and temporal scale. The red noise background is first filtered-out (this removes substantial part of the variance that does not have much structure) and the variable is separated into its meridionally-symmetric and anti-symmetric components with respect to the equator.

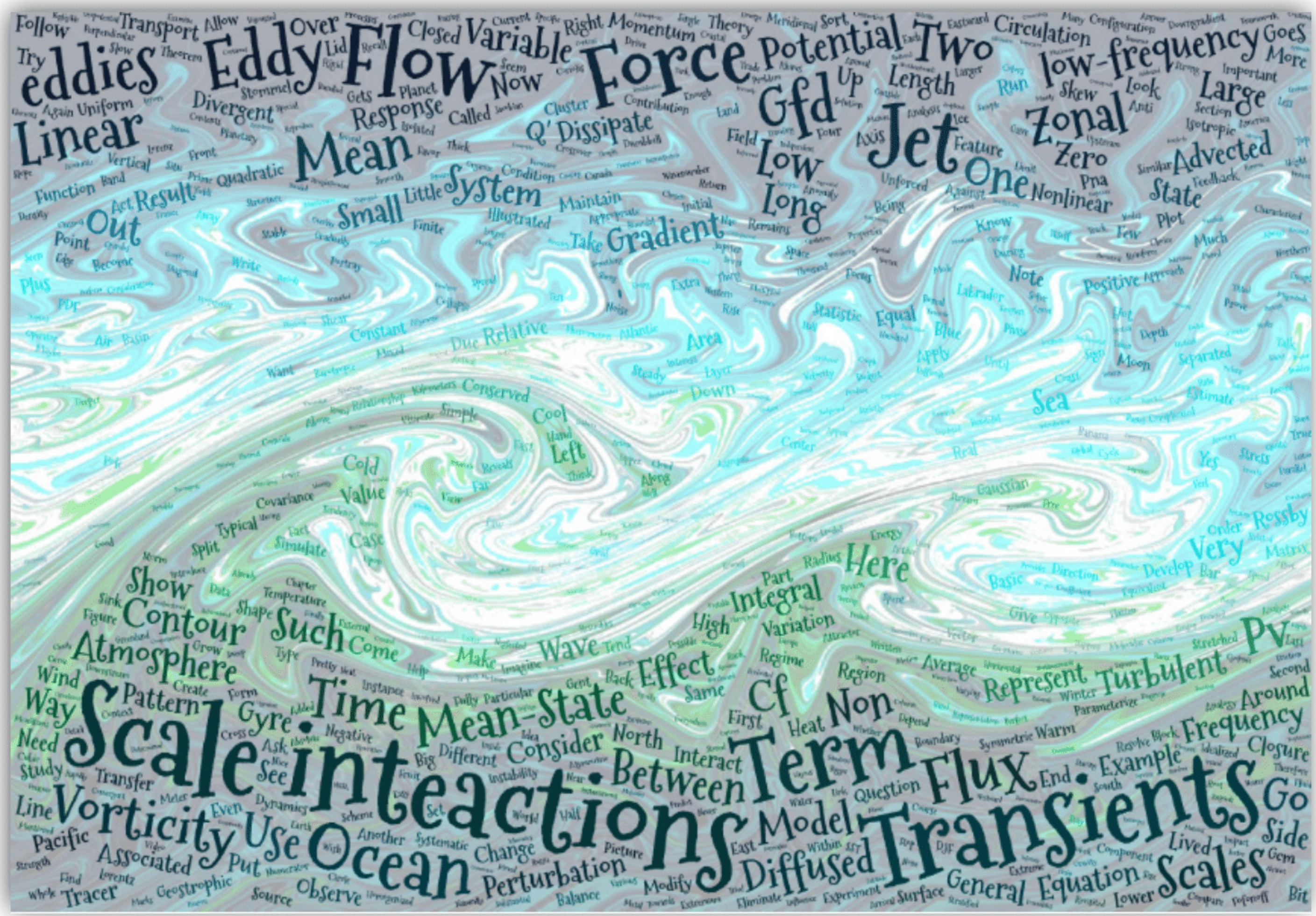
↳ For instance, the symmetric component of vertical velocity (left panel of the figure below) shows variability patterns that line-up nicely against the theoretical dispersion curves: Kelvin ($n = -1$), Rossby ($n = 1$), and Inertia-gravity waves ($n = 1$). The anti-symmetric component of the precipitation (right panel of the figure below) is consistent with the mixed Rossby gravity wave ($n = 0$).




✋ The **equivalent of $\boldsymbol{H}$** (previously the thermocline depth), used to compute the theoretical curves is some height in the atmosphere, but it is not the whole depth of the troposphere. It is a bit more complicated because $H$ is modified by convection. But if you pick the right value, you can find a curve that lines up with the observed variability.

**In the next chapter**, we will discuss scale interactions. So far, we have primarily focused on linear dynamics. We will now go full **nonlinear** and investigate **scale interactions** and **turbulence**.

# CHAPER 5
# Scale interactions
# in the Atmosphere and Ocean



# CHAPTER 5
# Scale interactions in the Atmosphere and Ocean

**GFD5 Contents**

⇨ So far, we have focused on **linear dynamics**. We considered **perturbations** to the flow and these perturbations remained **small**, so quadratic terms in the perturbation could be neglected. This is equivalent to making a <u>separation</u> between a basic flow and the perturbation. In **#GFD3.3** and **#GFD3.4**, we discussed **instabilities** and found conditions under which these perturbations can grow exponentially. We are left wondering ***what happens when the perturbations become big enough*** that they can no longer be considered small relative to the magnitude of the background flow or its gradients? How does the ***perturbation interact with the mean flow***?

↳ So, in this chapter, we will introduce the idea of **scale interactions** in the atmosphere and ocean. We will see ***how these transient systems can modify or interact with the mean flow***.

▪ We will see how transient systems can be involved in ***large-scale forcing and transport***, i.e. how perturbations can transport properties and contribute to lower-frequency heat and momentum fluxes and potential vorticity (see **#GFD5.1**).

▪ As we cannot represent every single little transient system, we will look for a systematic way of representing their ***aggregate effect***, their statistical effect, on the average flow (see **#GFD5.2**). Can their effects be represented in terms of lower-frequency variations or average flow? This is ***closure***. One simple approach to closure is to consider these transient systems as a form of ***diffusion***. Barotropic or baroclinic gradients in the mean flow can create instabilities (see **#GFD3.3** and **#GFD3.4**), and these transient systems can in return eliminate the gradients by diffusion.

▪ In this way, transient systems can modify ***large-scale potential vorticity***. We will look at some examples in which transient systems affect large scale Ocean circulation (see **#GFD5.3**).

▪ We will also review some ***atmospheric examples*** of how transients interact with long-lived features to influence low-frequency variability (see **#GFD5.4**).

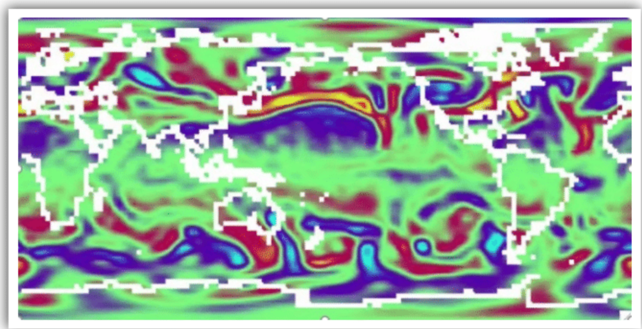
▪ Then, we will study the ***atmospheric response*** to other types of forcing anomalies. For example, we will see how the atmosphere responds to a change in the sea surface temperature, and how this basic response might be modified by the response of the transients (see **#GFD5.4b**).

▪ Finally, we will see how the ***flow on rotating planets tend to organize itself into zonal Jets*** (see **#GFD5.5**).

This chapter will also serve as an introduction to turbulent dynamics.

## GFD5.1: Scale Interactions and Transient Forcing

### *5.1.a) Atmospheric illustration: 250mb relative vorticity*



⇨ The video shows the **relative vorticity** in the atmosphere at 250 mb (from ECMWF ERA-Interim reanalysis) during boreal winter time (DJF).

↳ The patterns are very turbulent, portraying eddies propagating eastward in the extra-tropics.

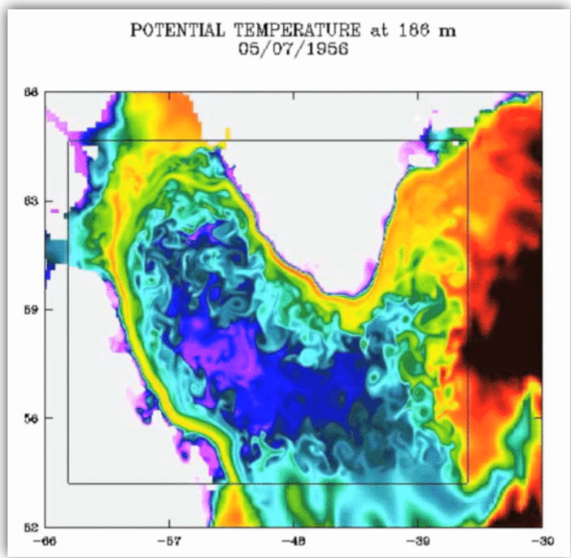
⇨ If you stare at these transient systems for long enough you can pick out some features:

• The variability is more active over the ***oceans*** than over the land and the northern Atlantic and Pacific Ocean basins are ***storm-track regions*** (see **#GFD1.1e**).

• Focusing on the western Pacific variability - the upstream part of the Pacific jet – we observe that features appear to be stretched out in the ***zonal direction***, while towards the east, turbulent patterns seem to be stretched out more in the ***meridional direction***.

↳ This is pretty systematic and as a result, ***there are consequences for how these transient systems interact with the Jet*** which they are traveling on (see **#GFD5.1d**).

### *5.1.b) Ocean illustration: Re-stratification of the Labrador Sea*




⇨ Here is an example of ***heat transfer by transient systems influencing the mean state***.

The figure shows the ***potential temperature*** in the Labrador Sea at 165-meter depth from a model simulation, between Greenland and Canada.

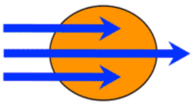
↳ We see a ***very strong annual cycle***. A ***sudden cooling*** at the surface (blue) marks the arrival of winter. It is associated with cold winds coming off Labrador. This ***cooling is convectively unstable and it is therefore mixed in the vertical*** very rapidly. As a result, the whole water column is cooled down to the bottom of the ocean.

⇨ ***After the winter, how is the stratified state reestablished?***

↳ ***Heat is transferred by turbulent eddies***. The "warm(er)" coastal current flowing around the Labrador Sea, along the Greenland and Canada coasts, is associated with strong gradients between the coast and the center of the basin. They favor the development of ***geostrophic eddies*** which transfer heat into the center of the basin and gradually re-establish the stratified state for the following summer.

### *5.1.c) Example: Momentum transport in zonal jets*

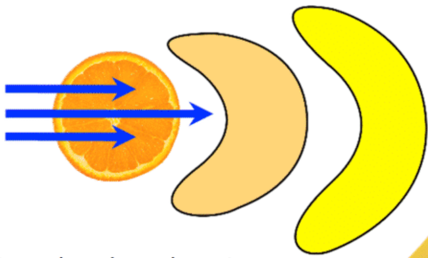
Here is ***an example in which transient eddy feedback maintains the mean flow***. Let's think about zonal jets and momentum transport in zonal jets.



⇨ We put a **cyclonic eddy in a zonal jet**. The figure shows a typical eddy (a closed contour) advected by a **sheared zonal jet**, maximum at the center.

***What will be the effect of the jet on the shape of this eddy?***

↳ It will shear it out. It will gradually change its shape as it goes downstream. This inspires a fresh fruit analogy, i.e. **turning an orange into a banana**.



⇨ And the fact that the eddy ends up looking like a banana is important for the general circulation. The variations of the jet follow this equation:

$$u_t + uu_x + vu_y = f(v - v_g) - \mathcal{D}$$

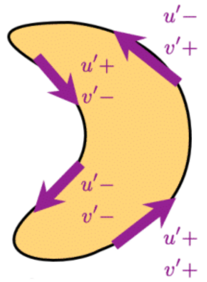
↳ There is a balance between the time variation plus the advection terms, and $f(v - v_g)$ (Coriolis and pressure gradient forces bundled into one term using the geostrophic wind) plus the dissipation.

⇨ Taking the time average of this equation shows that **the jet is diffused by dissipation and powered by the momentum fluxes**, i.e. mean dissipation is balanced by the quadratic advection terms:

$$\overline{u'u'_x} + \overline{v'u'_y} = -\mathcal{D}$$

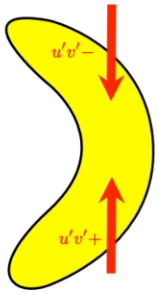
↳ The fluxes of momentum can be reformulated as the divergence of a flux, i.e.: $(\overline{u'u'})_x + (\overline{u'v'})_y = -\mathcal{D} + \overline{u'd'}$ . The term $(\overline{u'v'})_y$ - a covariance between $u'$ and $v'$ - is the most important term in this equation.

⇨ We can estimate its contribution by looking at the flow as it goes around one of these banana-shaped eddies.




• In the northern half of this eddy, the perturbation flow is going northwestwards so there is a negative covariance between $u'$ and $v'$. On the way back, $u'$ is positive while $v'$ is negative, i.e. a negative covariance between $u'$ and $v'$.

• In the southern half of the eddy, the flow is either southwestward or northeastwards, with a positive covariance between $u'$ and $v'$.




↳ In the north, we observe a southward flux of eastward momentum, while in the south there is a northward flux of eastward momentum. This gives rise to a ***convergence of momentum*** flux which will ***accelerate the jet towards the east*** and help to ***maintain the jet against the dissipation*** term on the RHS. This is how the jet is maintained by mature finite amplitude synoptic systems.

***An eddy that gets stretched out and deformed by a jet, will produce a convergent momentum flux that maintains the jet against dissipation***.

### *5.1.d) General consideration for tracer transport*

⇨ Let's formalize this by considering this generic non-linear system. The **tendency equation** with advection and forcing of a tracer $q$ (potential vorticity for instance) can be written:

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{F} - \mathcal{D}$$

↳ $\frac{\partial q}{\partial t}$ plus the advection term equal sources and sinks, i.e. forcing and dissipation.

• The forcing could be the wind stress for instance and dissipative sink could be diffusion.

• The advection term can be written as a Jacobian of $\psi$ and $q$ ($J(\psi, q)$) (see **#GFD2.3h**):

$$\frac{\partial q}{\partial t} + J(\psi, q) = \mathcal{F} - \mathcal{D}$$

$J(\psi, q)$ is the advection of $q$ by this non-divergent flow associated with the stream function $\psi$, such that $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$

⇨ We now split up the flow (of which the potential vorticity $q$ is a diagnostic) into two components, the **average flow** (noted with a bar, $\bar{q}$ and $\bar{\psi}$) and the **perturbation flow** which is varying in time (noted with a prime): $q = \bar{q} + q'$ and $\psi = \bar{\psi} + \psi'$. The tendency equation can be written as follows:

By definition $\frac{d\bar{q}}{dt} = 0$

$$\frac{\partial q'}{\partial t} + \underbrace{J(\bar{\psi}, \bar{q})}_{\text{mean flow advection}} + \underbrace{J(\bar{\psi}, q') + J(\psi', \bar{q})}_{\text{linear waves}} + \underbrace{J(\psi', q')}_{\text{turbulence}} = \mathcal{F} - \mathcal{D}$$

↳ The non-linear quadratic advection term is split into four terms:

1) ***mean flow advection***: the mean potential vorticity being transported by the mean flow.
2) ***2 linear terms***: perturbation PV being transported by the mean flow and the perturbation flow transporting the mean PV. These terms gave us waves and instabilities.
3) a quadratic term $J(\psi', q')$.

↳ In chapters 2, 3 and 4, we neglected the contribution of the quadratic term because it is quadratic in the perturbation and the perturbation was small. If the perturbation is not small anymore, this term is not negligible and we need to study what this term does.

⇨ If we are interested in the **systematic effect of this quadratic term**, its non-zero time mean, we can form the time mean of the tendency equation, leading to a ***budget equation*** for $q$:

$$J(\overline{\psi}, \overline{q}) = -\overline{J(\psi', q')} + \overline{\mathcal{F}} - \overline{\mathcal{D}}$$

transient "forcing"

↳ The advection of the mean tracer by the mean flow will be balanced by the mean of the forcing, the mean of the dissipation, and the mean of this transient forcing term. So, we now put this term on the RHS of the budget equation and consider it as a forcing term, a forcing by the transient fluxes. We discussed this in **#GFD1.1e**.

⇨ Just in passing, we note that for the special case of time-independent unforced flow (no time variation and no forcing/dissipation) there is ***a time-independent conservation law*** so that advection is equal to zero:

$$J(\psi, q) = 0$$

↳ $q$ would be strictly a function of $\psi$ ($q = q(\psi)$) meaning that contours of $q$ would overlay contours of $\psi$. This describes a closed circulation - $q$ contours coincide with $\psi$ contours. We will come back to this no-advection state in which nonlinearity is associated with closed circulations in **#GFD5.3c**.

↳ The figure shows some turbulence. Closed contours for which $J(\psi, q) = 0$ can be considered either for little turbulent eddies or for something much bigger like ocean gyres.

## GFD5.2: Effect of Transients on the Mean Flow: Closure & Diffusion

### *5.2.a) Forcing due to transients: Closure*

⇨ Imagine we wish to simulate or predict the slow, large-scale flow. Because the ***system is nonlinear*** the fast, small-scale component (maybe unresolved) will affect the slow, large scale variability.

↳ **Closure** is the systematic study of how we can represent the feedback of the **transients** on the lower-frequency flow variation.

⇨ Consider a non-linear development of a zonal wind $u$ according to the following abstract non-linear equation:

$$\frac{du}{dt} + uu + ru = 0$$

↳The term $uu$ is quadratic and $ru$ is linear. It is an idealized generic equation.

⇨ Let's examine this equation in term of low-frequency variations by taking the time average or the low-frequency component:

$$\frac{d\overline{u}}{dt} + \overline{uu} + r\overline{u} = 0$$

☝ ***We want to solve this equation for $\overline{u}$***. The problem is that we don't know $\overline{uu}$:

$$\overline{uu} \neq \overline{u}\,\overline{u} \quad \text{, it is } \quad \overline{uu} = \overline{u}\,\overline{u} + \overline{u'u'}$$

↳ In $\overline{uu}$, there is the contribution of the transients that need to be addressed.

⇨ If we try to write an equation for the quadratic term $\overline{uu}$ (by multiplying the non-linear abstract equation by $u$), we end up with an equation in which there is a cubic term $\overline{uuu}$, which is of no help:

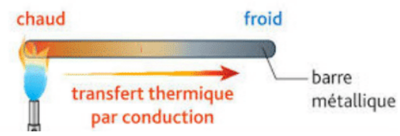
$$\frac{1}{2}\frac{d}{dt}\overline{uu}+\overline{uuu}+r\overline{uu}=0$$

⇨ We can do it again as many times as we want but at some point, we will need to represent the $(n\ +\ 1)^{\text{th}}$-order term in terms of the $n^{\text{th}}$ order term.

↳ To keep it simple, ***we will need to represent the quadratic term*** $\overline{u'u'}$ ***in terms of the mean flow***. And to do so, we must make additional physical assumptions. This is **turbulent closure**.

### *5.2.b) Diffusion and diffusivity*

⇨ There are various approaches to closure and one we have already mentioned is ***diffusion***. We can use diffusion to represent the systematic effect of transients in terms of the mean flow. We make the **analogy** that the effect of the transients is similar to ***molecular diffusion***.




↳ In a ***metal bar*** which is hot at one end and cold at the other, molecular diffusion will transport the heat from the hot end to the cold end and gradually the temperature will become uniform. This is because heat is transported downgradient.

↳ Here, ***we assume that geostrophic eddies act in a similar way***. If there is a gradient in some larger-scale field, the geostrophic eddies will tend to smooth out this gradient.

⇨ Let's go back to our tracer equation and consider a diffusive representation for the flux of the tracer $q$. For the moment, we ignore other forms of forcing and dissipation. We consider advection by a non-divergent flow:

$$\frac{\partial q}{\partial t}+\nabla.\mathbf{v}q=0$$

↳ We split this advection term into the advection by the time mean and the transient eddy's:

$$\frac{\partial \overline{q}}{\partial t}+\nabla.\overline{\mathbf{v}}\,\overline{q}=-\nabla.\overline{\mathbf{v}'q'}$$

⇨ Let's represent the eddy covariance term through analogy with molecular diffusion, i.e. transport down the mean gradient, so:

$$\overline{\mathbf{v}'q'}=-K\nabla\overline{q}$$

↳ This way, the transient forcing term will transfer properties down gradient and will smooth out gradients. We can substitute it into the non-divergent flow equation in which the substantial derivative of $\overline{q}$ is represented in term of $\overline{q}$, as:

$$\frac{D\overline{q}}{Dt}=\nabla.(K\nabla\overline{q})\quad(=\nabla.F)$$

*This is our assumption for how the transients are going to impact the mean flow.*
*It is a parameterization/closure*

⇨ In general, $\boldsymbol{K}$ is a matrix, a ***second rank tensor***. Diffusion is usually not ***isotropic*** for large-scale flows, meaning that diffusion in some directions might be stronger than in other directions. For example, the flux $\overline{\boldsymbol{v}'q'}$ is represented by a coefficient $(-\kappa^{vy})$ times the meridional gradient and another coefficient $(-\kappa^{vz})$ for the vertical gradient:

$$\overline{\mathbf{v}'q'}=-\kappa^{vy}\frac{\partial\overline{q}}{\partial y}-\kappa^{vz}\frac{\partial\overline{q}}{\partial z}$$

↳ These coefficients come from turbulence theory. We can estimate them by using some scaling arguments: $\kappa^{vy}\sim v'l'$ where $v'$ is a typical eddy velocity and $l'$ is a "mixing length".

### *5.2.c) Symmetric and asymmetric diffusion*

⇨ We can decompose $\boldsymbol{K}$ into symmetric and antisymmetric parts $\boldsymbol{K} = \boldsymbol{S} + \boldsymbol{A}$.

• The simplest specification is ***isotropic downgradient diffusion***.

$$K = S = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

↳ The $3 \times 3$ $\boldsymbol{K}$ matrix is just a diagonal matrix with the same constant down the diagonal and the diffusive flux is downgradient:

$$F = -\kappa \nabla \bar{q}$$

• In general, a downgradient flux is associated with symmetric matrices $\boldsymbol{S}$.

⇨ But this matrix can have an anti-symmetric part $\boldsymbol{A}$, such that: $F = -A\nabla q$, and:

$$F.\nabla \bar{q} = -(A\nabla \bar{q}).\nabla \bar{q} = 0$$

↳ $\boldsymbol{A}$ will consist of off-diagonal elements of opposite sign. As a result, the result of this matrix multiplied by a vector will be perpendicular to that vector ($\boldsymbol{A}x \perp x$). This means that this diffusive flux is neither upgradient nor downgradient, but it is parallel to the contours of the mean state. This flux is called a "skew flux".

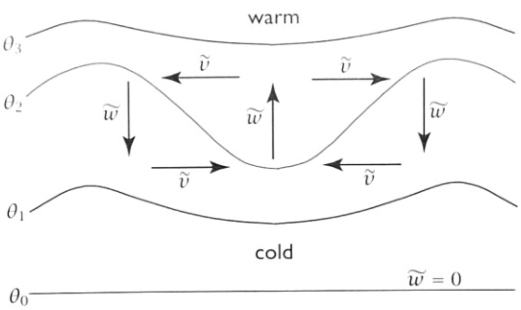
A **skew flux** is thus equivalent to ***advection*** by a non-divergent flow. Its velocity can be represented by a stream function: $\tilde{\mathbf{v}} = \nabla_{\wedge}\psi$. Therefore, ***the skew flux does not change the gradient, it goes along the gradient***.

⇨ *Whether or not it is appropriate to use straightforward isotropic downgradient diffusion or have some anti-symmetric terms in the matrix depends on the time-scale we analyze and on the tracer variable (conserved or not) for which we are trying to represent the effect of* **transients**.

### *5.2.d) Parameterization*

⇨ Here is an example of a parameterization that is often used in ocean models.

📖 It is more difficult to model the ocean than the atmosphere because the Rossby radius is significantly smaller, a few hundred kilometers vs. a thousand kilometers. Most atmospheric models now have no trouble resolving these scales. To resolve geostrophic **eddies** in the ocean, one needs to use a substantially higher resolution which is quite expensive to run on a computer, especially for long simulations. There is a trade-off between the length of the simulation and how much you can resolve.




↳ Imagine that the ***geostrophic eddy-scales are not fully resolved*** in a chosen model configuration. We thus have to represent their effect on the larger scales in some other way. The *Gent and McWilliams* parameterization is one approach to doing this. It is illustrated in the figure showing density surfaces near the thermocline.

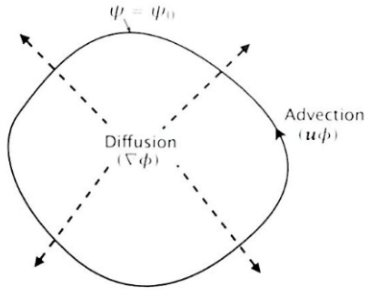
▪ Isotropic diffusion would simply flatten the density gradients in the vertical.

▪ Another way to do this, in agreement with the ***mechanism of baroclinic instability*** we studied in **#GFD3.4**, is to flatten-out the tilted density contours by ***advective flow*** causing a transfer of energy between the potential energy stored in the slope of isentropes and kinetic energy of the growing systems. The *Gent and McWilliams* scheme formulates the eddy closure in terms of asymmetric diffusion of thickness, i.e. the circulation associated with this scheme is represented by a skew flux in the diffusion scheme.

## GFD5.3: Systematic Effects of Transient Eddies on Ocean Gyres

### *5.3.a) Potential vorticity homogenization*

⇨ We discuss now ***the systematic effects of eddies on ocean gyres***. We study **a simple case** in which the effect of transients is represented as **an isotropic diffusion** (see **#GFD5.2b**) and we will discuss the diffusion of potential vorticity (PV).




↳ We are going to look at very large-scales in the Ocean and ask ourselves what is the effect of adding some diffusion of PV on the mean-field of PV? It is useful to study potential vorticity in this context for two reasons.

1) Because PV is conserved following the motion, so it is meaningful to talk about its diffusion.

2) Knowing the potential vorticity implies knowing the flow. There is an intimate connection between the large-scale flow and the large-scale potential vorticity.

⇨ Here is the generic advection-forcing-dissipation equation for the conservation of the potential vorticity:

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nabla.(\kappa \nabla q) + \mathcal{S}$$

- The effect of transients on PV are represented as an isotropic diffusion (see **#GFD5.2b**)
- $\mathcal{S}$ is a source of potential vorticity

↳ On the large-scales, we consider a model of **steady non-divergent** flow, **isolated from any sources** of PV.

- The flow being steady means that time variations can be crossed-out.
- We are in a region that is sheltered from wind stress forcing, i.e. away from the surface of the Ocean - in the deeper ocean where the flow does not feel the forcing effect $\mathcal{S}$.
- Non-divergent flow implies that $\boldsymbol{v}.\nabla q = \nabla.(\boldsymbol{v}q)$

↳ The PV conservation is written $\nabla.(\mathbf{v}q) = \nabla.(\kappa \nabla q)$

⇨ Let's consider a **closed contour** of the flow and **estimate the integral** of this equality over the area delimited by this contour.

$$\iint_A \nabla.(\mathbf{v}q)\, dA = \iint_A \nabla.(\kappa \nabla q)\, dA$$

• The left-hand side integrates to zero.

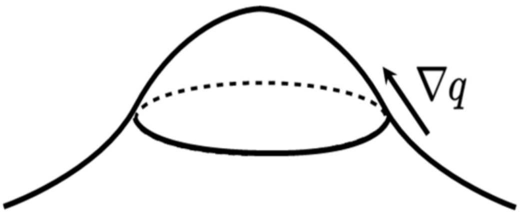
$$\iint_A \nabla.(\mathbf{v}q)\, dA = \oint (\mathbf{v}q).\hat{\mathbf{n}}\, dl = q \oint \mathbf{v}.\hat{\mathbf{n}}\, dl = q \iint_A \nabla.\mathbf{v}\, dA = 0$$

1) Following the divergence theorem (see **#GFD1.3a**), the area integral of a divergence is the line integral around that contour of the flux of $q$ perpendicular to the contour.
2) Since $q$ is constant on this contour (unforced flow) then $q$ can come out of the integral which is now the line integral of the flow perpendicular to the contour.
3) Following the divergence theorem again, the line integral can be rewritten in terms of the area integral of the divergence of the flow.
4) Since we imposed the flow to be non-divergent then the LHS is equal to zero within a flow-contour (which since we have steady free flow is also a $q$-contour).

• The right-hand side must also be zero within the area.
→ Using the divergence theorem, the area integral of the divergence of $\kappa\nabla q$ must be equal to $\kappa$ times the line integral of the component of gradient of $q$ that is perpendicular to the boundary.

$$\iint_A \nabla.(\kappa \nabla q)\, dA = \oint \kappa \nabla q.\hat{\mathbf{n}}\, dl = 0$$

↳ This result means that as we integrate around this flow contour, the gradient perpendicular to this contour must integrate to zero in a steady unforced flow.



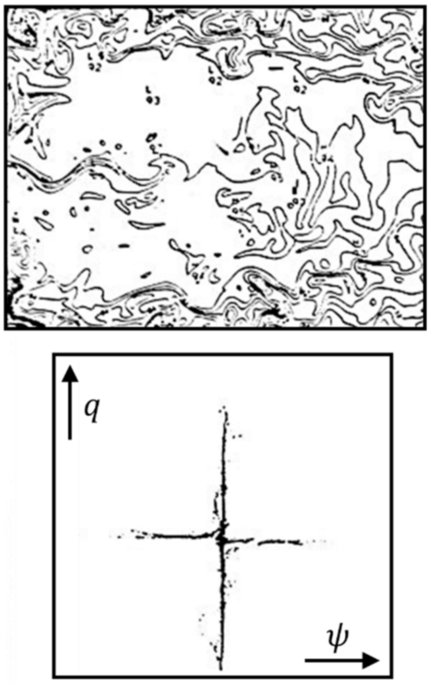

As illustrated on the schematic, **this cannot be true if the contour encloses an extremum of q**. In this case, the gradient of $q$ perpendicular to a flow-contour is not going to integrate to zero. This means that the eddies are going to transfer properties to the mean flow until such a point is that it does become zero. The extremum in $q$ is going to get eroded and eliminated, until a state is achieved in which the potential vorticity is uniform (one constant value) throughout the region

Talking of an extremum in potential vorticity reminds us of the **Rayleigh criterion** for barotropic instability (see **#GFD3.3e**). An extremum in PV is a necessary condition to create instabilities and generate transient flow. In turn, the transient flow will act to eliminate the source of the instability. The final result is that all gradients of potential vorticity will be eradicated, resulting in the **homogenization of the potential vorticity** to a uniform value (in regions remote from sources of $q$).

### *5.3.b) Examples in models and observations*

⇨ Here are a couple of examples.

**QG model - mid-level PV**

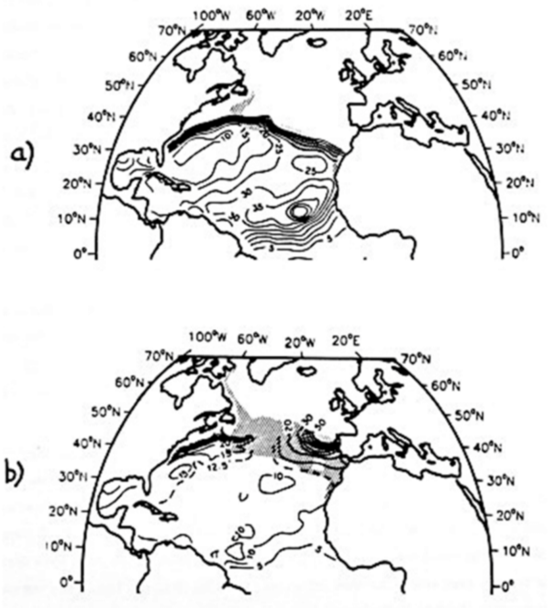



• On the right is the potential vorticity from an ocean model. It is not the top layer of the ocean. It is at a depth where the flow is isolated from forcing. On the top panel, we observe a large region of uniform potential vorticity (no horizontal gradients) where the gyre is active. The gradients are pushed out to the edge, where there is no flow.

This is illustrated on the bottom panel, in which values of $q$ (vertical axis) are plotted as a function of $\psi$ (horizontal axis). It comes down to this ultimate state where

1) either there are variations in $q$ but in that case $\psi = 0$, i.e. there is no flow, i.e. the $\beta$-effect outside the flow region,
2) or $\psi$ is varying (there is a mean-flow), in which case $q$ is uniform (within the gyre).

**Observed PV on isopycnal surfaces**



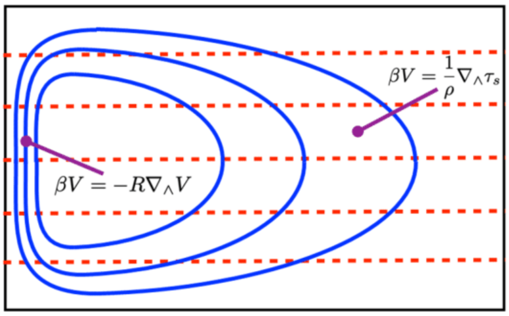

• On the right is an example of ocean gyres from observations. The lower figure shows a deeper layer and uniform values of potential vorticity can be observed within the gyre.

↳ The upper figure shows a layer nearer the surface where the flow is not isolated from the surface forcing. $q$ values are not uniform but portray closed contours around the gyre. This is different from the classical large-scale ocean circulation theories (see **#GFD5.3c**).

### 5.3.c) Stommel vs. Fofonoff

⇨ In this section, we focus on the large-scale ocean circulation and we contrast two paradigms of large-scale ocean circulation theory.

• **The first one is the Stommel solution**. The Sverdrup term, i.e. the advection of planetary vorticity ($\beta V$), is balanced by forcing and friction, so:




$$\beta V = \frac{1}{\rho}\nabla_\wedge \tau_s - R\nabla_\wedge V$$

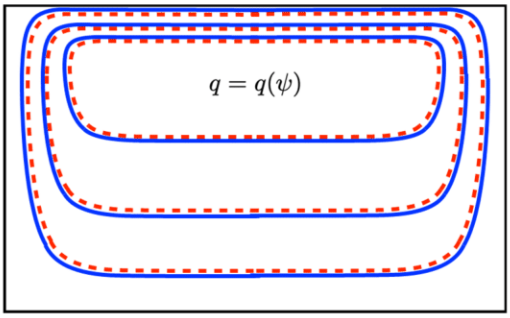
Contours of potential vorticity are parallel to latitude lines.

➢ As the flow goes south, the wind stress forces it to cross these contours. The flow is forced to change its potential vorticity.

➢ As the flow goes back north, it has to become an intense jet, so the friction term can be large enough to remove the vorticity that was injected by the wind stress.

↳ The **Stommel gyre is forced and dissipated** and the absolute vorticity is always changing.

• **There is another solution in an unforced context**.




➢ Imagine an unforced system, an Ocean gyre in which the **potential vorticity is conserved** – it never changes. This means that the flow goes around a flow-contour, the PV-contour remains parallel to that flow contour. An unforced barotropic system follows:

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \boldsymbol{v}\nabla q = 0 \quad \text{with} \quad q = \xi + f$$

$$v.\nabla\xi + \beta v = v.\nabla q = 0$$

➢ There is a cancellation between the advection of relative vorticity and the advection of planetary vorticity. The sum of the two is conserved, i.e. PV is conserved and $J(\psi, q) = 0$ (see **#GFD5.1c**). This means that $q$ is strictly a function of the stream function ($q = q(\psi)$). In this barotropic case, the potential vorticity is the relative vorticity plus $\beta y$ and is a function of $\psi$, so that:

$$\nabla^2\psi + \beta y = fn(\psi)$$

*We do not know this function*

↳ This is a kind of opposite extreme view of the ocean circulation compared to the forced dissipative Stommel gyre. It is called a Fofonoff gyre. We imagine that the ocean circulation gets into this state due to the action of transient eddies modifying the potential vorticity field.

### 5.3.d) Diffusion and the strength of the gyre

⇨ We do not know the relationship between $q$ and $\psi$. The simplest relationship we can consider is a linear relationship, meaning that the gradient of $q$ is proportional to the gradient of $\psi$:

$$\nabla q \approx \frac{dq}{d\psi}\nabla\psi$$

⇨ The budget of (steady) $q$ in the upper layer involves some forcing and dissipation:

$$J(\psi, q) = \nabla.(\kappa\nabla q) + \mathcal{S}$$

⇨ To estimate the value of the linear coefficient, we can integrate this equation within a streamline $\psi$, i.e. around a closed contour of $q$.

↳ For a ***non-divergent*** flow, the LHS is zero (see **#GFD5.3a**), which reveals a balance between a dissipative term (from the transients) and a forcing term:

$$0 = \iint_A \nabla.(\kappa\nabla q)\, dA + \iint_A \mathcal{S}\, dA$$

⇨ Using the divergence theorem (see **#GFD1.3a**), we can eliminate the divergence of the diffusion by transforming the area integral into the line integral of $\kappa\nabla q$. $\nabla q$ is expressed in terms of $\nabla\psi$ using the linear relation we hypothesized. It follows:

$$\Rightarrow \iint_A \mathcal{S}\,dA = -\oint_\psi \kappa\nabla q.\hat{\mathbf{n}}\,dl = -\oint_\psi \kappa\frac{dq}{d\psi}\nabla\psi.\hat{\mathbf{n}}\,dl$$

↳ As $\frac{dq}{d\psi}$ is a constant, it can be moved outside the integral and we get: $\frac{dq}{d\psi} = -\frac{\iint_A \mathcal{S}\,dA}{\oint_\psi \kappa\,\mathbf{v}.dl}$

▪ The linear relationship between $q$ and $\psi$ is determined by integrals of forcing and dissipation around the closed gyre circulation.

▪ Integrated eddy diffusion provides the link between the $q$ / $\psi$ relationship and **the strength of the circulation**.

▪ In regions isolated from forcing, the numerator is zero but the denominator is non-zero, so the field of $q$ must be uniform. $q$ is homogenized as seen in **#GFD5.3a**.

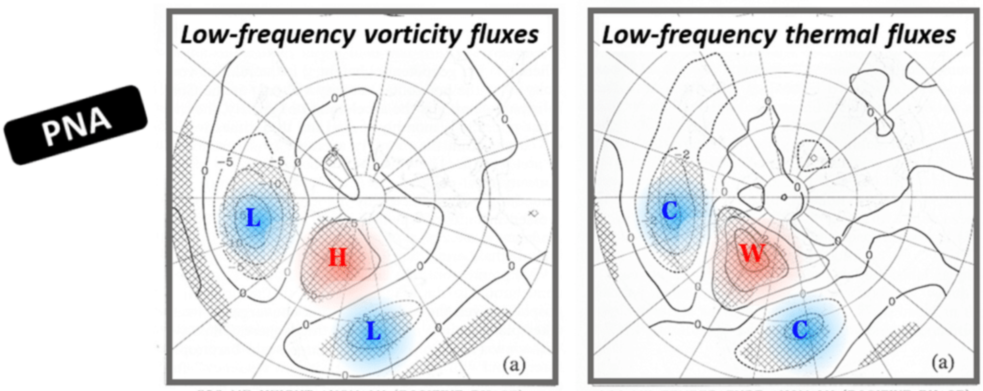
↳ This is the other extreme view of the ocean circulation.

## GFD5.4: Examples of Scale Interactions in the Atmosphere

### *5.4.a) Long-lived atmospheric flow anomalies*

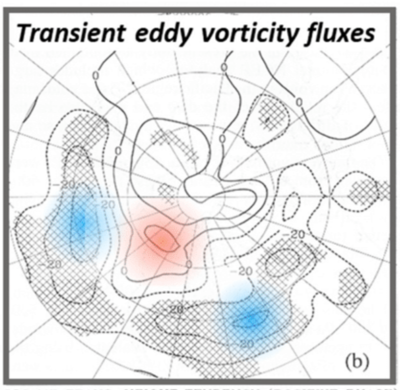
⇨ In this section, we show examples from the atmosphere and focus on the maintenance of low-frequency variability. We ask the questions:

- How does the atmosphere stay in a particular configuration over long periods of time?
- What is the relationship between low-frequency variations and the fast-transient eddies?



250 MB HEIGHT ANOMALY (POSITIVE PHASE) — 700 MB TEMPERATURE ANOMALY (POSITIVE PHASE)

⇨ Here is a **first example** (from Sheng et al., 1998) of a very important feature of the low-frequency variability of the Atmosphere. The figure above shows the result of a composite analysis of the northern hemisphere (bottom is North America) emphasizing a typical Low-High-Low Cold-Warm-Cold configuration associated with the Pacific North American (PNA) pattern. The atmosphere very often finds itself in this pattern, either in its positive or negative ***phase***.



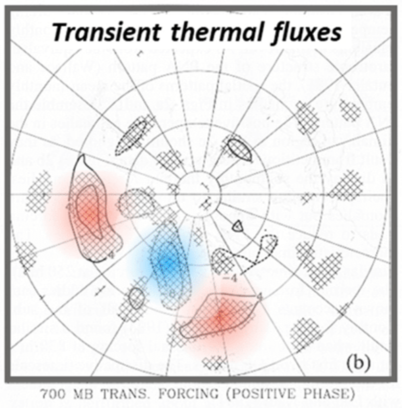
250 MB TRANS. HEIGHT TENDENCY (POSITIVE PHASE)

↳ **The question here is**: Does this pattern get ***dissipated*** by the transient eddies - their systematic effect - or is it ***reinforced***?

• On the left is the geopotential height tendency ***due to the transient eddy fluxes*** during these episodes of positive PNA. The pattern is in phase with the low-frequency pattern, i.e. Negative-Positive-Negative configuration, thus reinforcing the low-frequency pattern during episodes of positive PNA. The transient eddy momentum fluxes act to maintain the pattern in the geopotential height.

• **Conversely**, the transient fluxes of ***temperature*** show a Positive-Negative-Positive configuration, which tends to warm up cold regions and cool down where it is warm. Transient fluxes of temperature are thus dissipating the temperature signature.

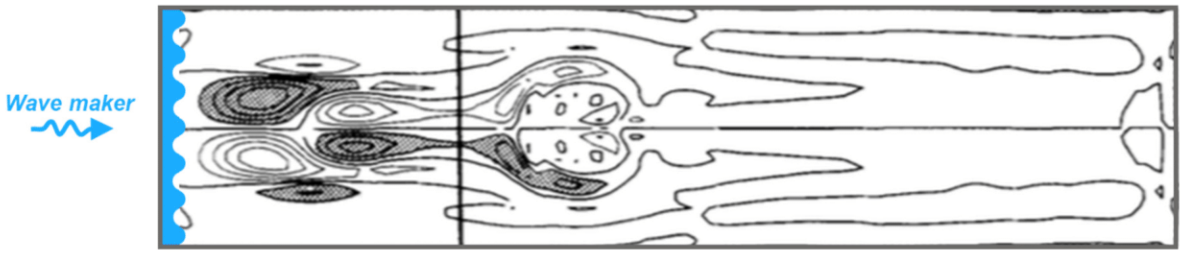
⇨ In conclusion, observational analyses consistently show that high-frequency transient eddy vorticity fluxes reinforce the low-frequency patterns, while transient thermal fluxes dissipate them.




⇨ Here is a **second example** (from Haynes and Marshall, 1986) of a long-lived atmospheric feature called **blocking** that can be observed over Europe. It manifests as a High to the north and a Low to the south. In the wintertime, it brings very cold air from Russia to western Europe. This configuration remained there for a long time in February 2012.

↳ It is interesting to analyze what transient systems coming across the Atlantic do to this pattern: do they sweep it away or do they act to maintain it?

• Here are the results from an idealized model experiment in which a wavemaker is put upstream to generate high-frequency disturbances. The potential vorticity flux divergence shows that these transient eddies impinge upon this reversed dipole downstream. The transfer of potential vorticity is such as to maintain the stable block against dissipation.




⇨ In conclusion, there is evidence that high-frequency transient eddy vorticity fluxes maintain this blocking configuration and this explains why it is such a long-lived feature.

### *5.4.b) Transient feedback on a forced response*

⇨ In this section, we study how transient eddies modify the atmospheric response to some other external forcing (from Hall et al., 2001). We recall the potential vorticity development equation:

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{F} - \mathcal{D}$$

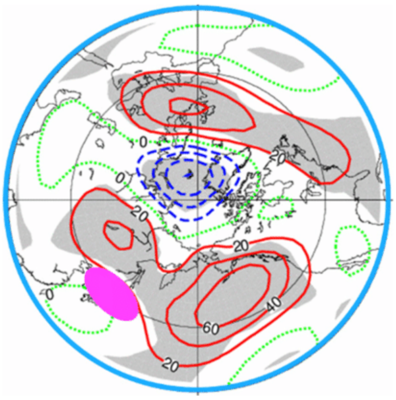
↳ The time average of the potential vorticity flux is the average forcing minus the average dissipation called $\mathcal{G}$:

$$\overline{\mathbf{v}.\nabla q} = \overline{\mathcal{F}} - \overline{\mathcal{D}} = \mathcal{G}$$

↳ We then split the time-averaged potential vorticity flux $\overline{\boldsymbol{v}.\nabla q}$ into two components: the flux by the time means ($\overline{\boldsymbol{v}}.\nabla\overline{q}$) and the transient term ($\overline{\boldsymbol{v}'.\nabla q'}$). The latter is put on the RHS to be considered as a forcing (as in **#GFD1.1e** and **#GFD5.1c**) and the sum of the forcing is then called $\mathcal{H}$:

$$\overline{\mathbf{v}}.\nabla\overline{q} = \overline{\mathcal{F}} - \overline{\mathcal{D}} - \overline{\mathbf{v}'.\nabla q'} = \mathcal{H}$$

⇨ Two forcings: one is the real forcing ($\mathcal{G}$), and another is a forcing that includes the transient eddy fluxes ($\mathcal{H}$). These two forcing terms can be used to drive a model.




SET1 $\mathcal{G}$ is diagnosed from data and is used to drive a simple atmospheric General Circulation Model (GCM):

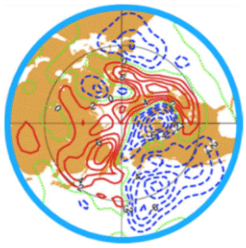
$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{G}$$

↳ In a second experiment, a small perturbation $f'$ is prescribed. Here, we add a perturbation to the sea surface temperature in the western Pacific. We run the GCM again with this extra bit of forcing associated with this perturbation:

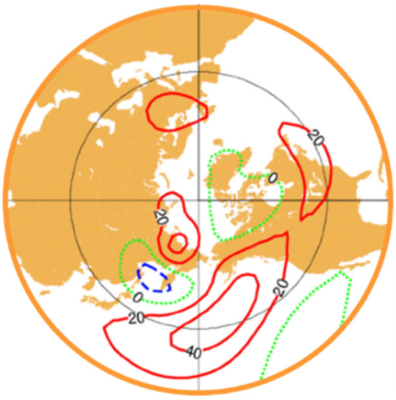
$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{G} + f'$$

⇨ The difference between the two long runs (see figure above) shows a ***global response***, characterized by a large high downstream in the Pacific, a low over the north pole, and another high over the Atlantic-European sector.

⇨ We ask now what the contribution of the transients in this response is. As these two experiments will not necessarily have the same value of the transient component to the forcing ($\overline{\boldsymbol{v}'.\nabla q'}$), we diagnose the difference in transient forcing between the two experiments ($\Delta(v'.\nabla q')$, see figure on the right).



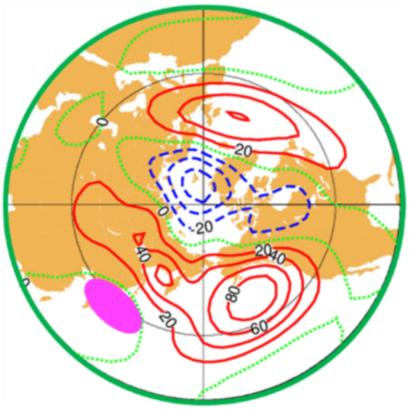
SET2 We now force the GCM with the forcing $\mathcal{H}$ (instead of $\mathcal{G}$) : $\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{H}$




↳ If we initialize the model with its time-mean $\overline{q}$, there is no development because $\mathcal{H}$ is what is needed to stop any development. Therefore, the initial conditions perpetuate.

Then, we add the same small perturbation $f'$ to this forcing and perform another simulation. We have a model in which the transient part is already taken into account in the forcing and we apply a small perturbation. We have a linear perturbation model. In response to the Pacific SST anomaly, we get a response which is not as global as in SET1 (which was the fully nonlinear response with modified transient eddy feedback). The linear response is basically just a Pacific response.

SET3 So then the question is: ***Is the difference between these two sets of experiments due to the change in the transient eddy forcing?*** Can we prove that we can represent the aggregate transient eddy effect in a linear model?




↳ To test this hypothesis, we take $\Delta(v'.\nabla q')$, scale it appropriately, and add it to the linear model as an extra transient forcing, giving:

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \mathcal{H} + f' - \Delta(\overline{v'.\nabla q'})$$

⇨ The resulting pattern resembles closely the difference between the runs in SET1 and yet it is not the same kind of experiment at all. In SET1, it was the ***difference between two fully non-linear turbulent experiments***, while SET3 is ***a linear model response*** in which the turbulence has been added as a constant forcing.

***We have thus proved that in a linear framework we can reproduce the effect of nonlinear transient eddies in the context of the response to a heating perturbation.***

### *5.4.c) The importance of nonlinearity*

⇨ And we arrive at fundamental considerations about the ***importance of non-linearity in low-frequency variability***.

• There is absolutely no question that ***the dynamics of the Atmosphere and Ocean are fundamentally nonlinear***. For example, we mentioned that the difference between cyclones and anticyclones is associated with non-linear dynamics (see **#GFD2.1a**).

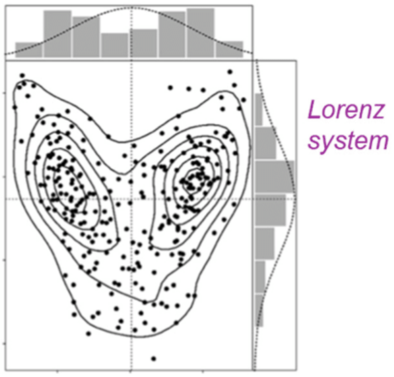
↳ Does this mean that the contribution of transient fluxes to low-frequency variability is automatically a nonlinear phenomenon? Or can it be thought of as a linear phenomenon? Changing something, the transients change one way, then changing it in the opposite way and the transients change in the opposite way? (*That would be linear*)

⇨ We are asking two different questions related to different time-scales:

▪ The first question is "are these eddies nonlinear?" and the answer is "yes, definitely!"

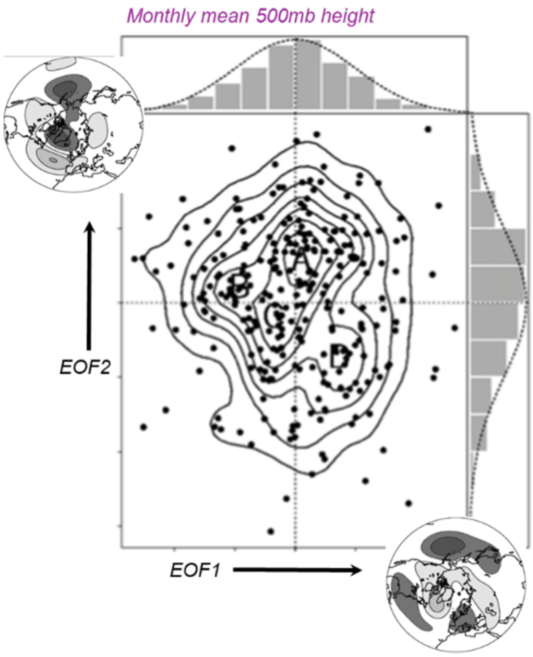
▪ The second question is "is the aggregate systematic effect of these eddies in modifying the atmospheric response to other types of forcing nonlinear?" and the answer is "yes, maybe".

↳ This is not the same question and there is no universal accord in the research community. There is a spectrum of opinions.




Supporters of nonlinear systems identify the Lorenz attractor system, the famous butterfly attractor, as a good model for the Atmospheric variability on low frequencies.

The figure to the left shows the Lorenz system mapped out in phase space. The many points show the instantaneous state of the system throughout a long integration of the simple Lorenz equations. They cluster very clearly onto two nodes with a bimodal distribution in one of the variables. Here we can identify two "regimes". The state goes from one regime to another and the time spent between the regimes remains quite small compared to the time spent in either one regime or the other.




This might be a useful way to think about the Atmosphere. On the right is an example of the (monthly mean) atmospheric variability represented in terms of the occurrence of two important patterns of low-frequency variability Pacific North America (PNA, horizontal axis) vs. the North Atlantic Oscillation (NAO, vertical axis). For each mode of variability, the associated PDF is also shown.

The question is "are the points clustering in specific regimes?" This is something that not everybody agrees about.

▪ It is possible that they are clustering in two regimes and we can think of transitions between regimes.

▪ Or it is possible that this impression of clusters is due to the sample of data that is finite (limited). In a finite sample of statistically random variables, you are always going to find some sort of clustering. So, it may also be appropriate to explain all this in a linear framework.

In linear dynamics, you will generally have Gaussian statistics and not the bimodal statistics associated with the Lorenz attractor.

Below is a linear equation that can be used, in which $x$ is a state vector that represents the entire state of the atmosphere. Its development $\frac{dx}{dt}$ is determined by a linear operator, some external forcing, and some Gaussian noise:

$$\frac{dx}{dt} = Lx + f + B\eta$$

linear operators — state vector — external forcing — Gaussian noise

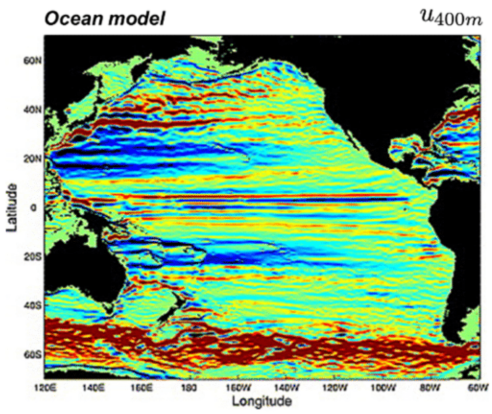
The Gaussian noise can be modified by another linear operator while remaining a linear system. If that second linear operator is independent of the flow $x$, then we still have Gaussian statistics everywhere. It is also possible to have non-Gaussian statistics with this linear system - skewed PDFs - provided that $B$ is a function of the flow $x$. So, we can go a long way with such linear model to analyze the low-frequency variability.

# GFD5.5: Zonal Jets and Turbulence

### *5.5.a) Zonal jets revisited: Ocean currents*

⇨ The characteristics of the ocean circulation depend on the time scales considered:

- An instantaneous snapshot resembles a sea of eddies, i.e. little round blobs everywhere.
- A very long-time average primarily captures the anticyclonic gyres, Gulfstream, Kuroshio.
- The large-scale ocean circulation on a time-scale of a few months is characterized by zonal jets of alternating sign, eastward and westward jets separated by a typical length scale in the meridional direction.




This is illustrated in the picture on the left (from Richards et al., 2006) showing the zonal flow at 400 meters depth from a long simulation with a numerical model of the Ocean.

The situation is a little bit noisier in the observations, but the surface geostrophic flow and geostrophic vorticity, estimated from altimetric observations, also reveals these zonal jets.

### *5.5.b) Wave-Turbulence crossover*

In this section, we look at the theory of turbulence at zonal jet length scales.

⇨ The **Rossby radius** is the length scale on which ***relative vorticity and vortex stretching*** make equal contributions to **potential vorticity** (see **#GFD1.2a**, and **#GFD3.4c**):

$$\nabla^2\psi \sim \frac{f^2}{gH}\psi \quad \Rightarrow \quad L \sim \frac{\sqrt{gH}}{f}$$

↳ The Rossby radius is the gravity wave speed divided by the Coriolis parameter.

⇨ Let's now consider larger length scales. We use the ***vorticity equation*** (see **#GFD1.3b**):

$$\frac{\partial \xi}{\partial t} + \mathbf{v}.\nabla\xi + \beta v = 0 \quad \rightarrow \quad \mathbf{v}.\nabla\xi \sim \beta v$$

⇨ The development of relative vorticity is balanced by the advection of relative vorticity and the advection of planetary vorticity. If the last two terms are of similar magnitude, a scale analysis yields to a length-scale on which it is true:

$$U\frac{U}{L^2} \sim \beta U \quad \rightarrow \quad L \sim \sqrt{\frac{u}{\beta}}$$

↳ $L$ is the length scale on which **advection of planetary and relative vorticity compare**. Note that this very similar to the ***equatorial radius*** (see **#GFD4.3a**), except that in the square root we now have the actual flow speed and not the gravity wave speed. This length is called the **Rhines scale**, where Rossby waves give way to turbulence.

⇨ Let's now focus on what happens in the ***transition between Rossby waves and closed geostrophic eddies*** (turbulence). We compare the frequencies associated with these two processes: i.e., the frequency associated with (barotropic) Rossby waves (see **#GFD3.1c**) and a typical turbulence inverse timescale:

$$\omega = \frac{\beta l}{k^2} \sim u^* k$$

$\boldsymbol{k} = (l, m)$ is the horizontal wavenumber and $l$ is the zonal wavenumber ($k^2 = l^2 + m^2$)

↳ The frequency associated with turbulence is the length scale of that turbulence (the equivalent wavenumber) multiplied by a typical turbulent flow velocity scale.

⇨ Equating these two frequencies gives the spatial scales at which the two processes are of the same order:

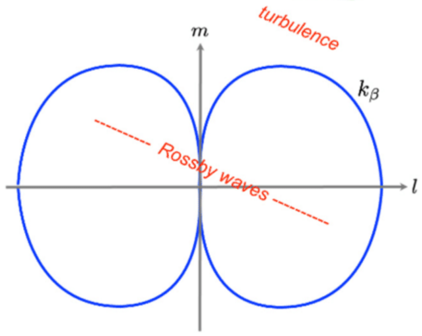
$$k^2 = \frac{\beta}{u^*} \cos\theta$$

$\boldsymbol{k} = (l, m)$ forms an angle $\theta$ with the horizontal wavenumber $l$

↳ This equation is plotted in wavenumber space on the figure on the right. It looks like a dumbbell. The blue curve is the boundary between where the turbulence takes over and where Rossby waves dominate. It is ***anisotropic*** (≡ not isotropic).

• For larger scales, inside the blue dumbbell, there are Rossby waves (propagating westwards and mainly zonally).

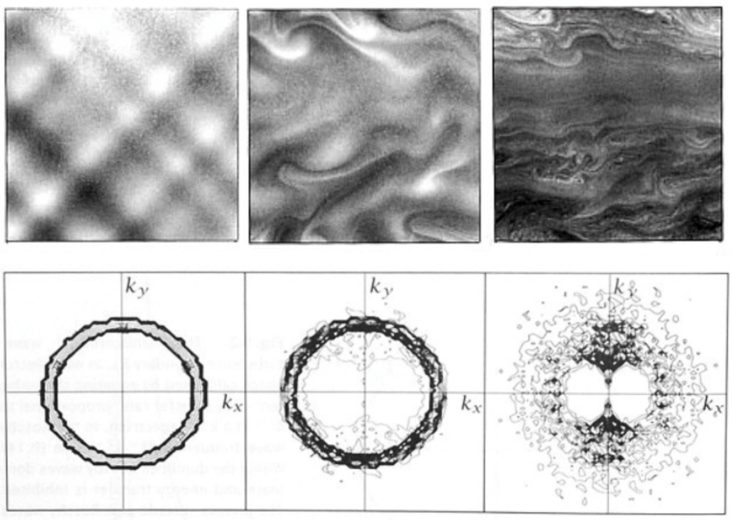
• For scales outside the dumbbell contour, geostrophic turbulence prevails.




⇨ Of particular interest are the points (positive and negative) where there the zonal wavenumber is small (large zonal scales) and there is a typical meridional length scale. Does this particular meridional length scale emerge from an analysis of the variability?. Yes, it does and it is the **Rhines scale** (see **#GFD5.5c**).

### *5.5.c) Collapse to zonal jets*

⇨ Here are the results of an idealized numerical experiment in which variability naturally collapses into zonal jets.




• A turbulent model is initialized with only one single length-scale. The initial condition resembles a sort of grid lattice where the length equals the size of the grid. In $(k_x, k_y)$ space, it is a circle ($k$=constant).

• Then, the flow gradually develops into turbulence and there will be scale interactions because the dynamics is nonlinear. The variability spreads across scales and the original circle in the wavenumber representation starts to spread out to other length scales.

• As the flow continues to develop, the variability spreads into a shape where turbulence is everywhere except inside the dumbbell associated with the Rossby wave regime.

↳ Most of the energy congregates to long zonal scales and a particular meridional length scale. This scale is the distance between zonal Jets that naturally emerges.

⇨ This is a neat theoretical account of the zonal jets observed in the Ocean.

# EPILOGUE
# Dynamics on other planets …

## Scale separation and boundary condition

In this course, most of the work is based on idealized theories that we tried to apply to real-world situations, in the Earth's Atmosphere and Ocean.

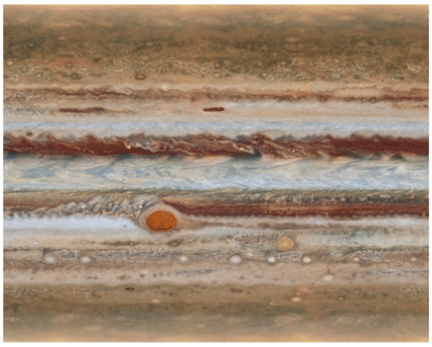
But, our atmosphere or ocean do not always comply with the theoretical framework because they are actually more complicated!

For instance, in the ocean, there are complicated coastlines, which are a bit annoying because they prevent the flow from going all around the world. These coastlines make the theory much more difficult to apply to the real world. And often, we put a rigid lid on the ocean as well, but in reality, there is none.

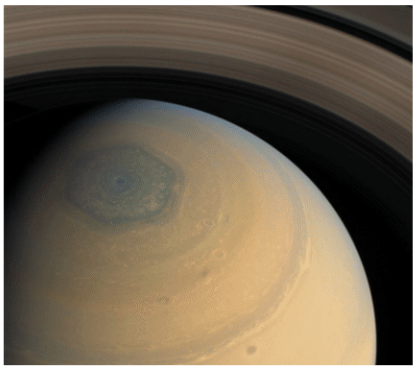
Scale separation is a big problem in the Atmosphere. In the Ocean, it is fairly easy because turbulence is very small-scale compared to the general circulation. The ocean Rossby radius is just a few hundred kilometers or less. But in the atmosphere, the Rossby radius is of the order of a thousand kilometers, which is pretty much the same scale as the low-frequency variability. The scale separation is really on the limit of being applicable. The atmosphere also has a boundary condition where you have mountains so that makes things more complicated.

### *To Jupiter and beyond*

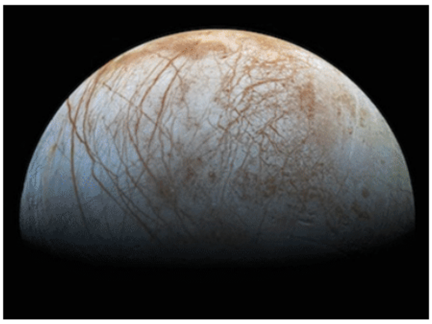
All these complications get in the way of trying to apply our beautiful theories. One way to cope with this is to leave the Earth and apply these theories to planets in which there appear to be fewer complications.



On the left is a picture of a planet where there is a beautiful scale separation and the fluid can go all the way around unimpeded. We observe zonal jets along with some small-scale eddies interacting with the large-scale flow. We even observe a beautiful example of a long-lived phenomenon that seems to be fueled upstream by a train of eddies, allowing it to maintain itself against dissipation. Of course, this planet is **Jupiter**.



On the right is **Saturn** with its banded cloud structures. It even has a marvelous perfect hexagon shape on its North Pole. This can be reproduced in theory and in experiments. With a laboratory tank, one can generate perfect symmetric patterns of various wavenumbers depending on the chosen parameters (rotation rate, temperature gradients, etc.)



On the left, it is one of the moons of Jupiter, called **Europa**. The whole moon is an ocean covered by a thick layer of ice - a rigid lid! So here, we have an ocean that goes all the way around the world and it has more water than we have here on Earth!

Well, it’s easy to give a lecture course where you grab pictures from NASA and say how cool fluid dynamics is. But you can appreciate this directly from your own garden or even your balcony. Below is a final picture of Jupiter, taken by Nick Hall through a 20cm Newtonian reflector. You can marvel at the cloud bands and the zonal jets, along with the four moons all in a row.